1 ☐ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1,-2)$ is on the graph of f, so $f(-1)=-2$.

(b) When $x=2$, y is about 2.8, so $f(2)\approx 2.8$.

(c) $f(x)=2$ is equivalent to $y=2$. When $y=2$, we have $x=-3$ and $x=1$.

(d) Reasonable estimates for x when $y=0$ are $x=-2.5$ and $x=0.3$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3\le x\le 3$, or $[-3,3]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2\le y\le 3$, or $[-2,3]$.

(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1,3]$.

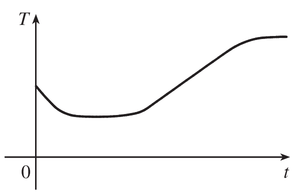
3. From Figure 1 in the text, the lowest point occurs at about $(t,a)=(12,-85)$. The highest point occurs at about $(17,115)$. Thus, the range of the vertical ground acceleration is $-85\le a\le 115$. Written in interval notation, we get $[-85,115]$.

5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

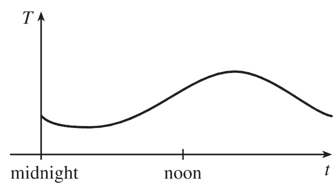
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3,2]$ and the range is $[-3,-2)\cup[-1,3]$.

9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

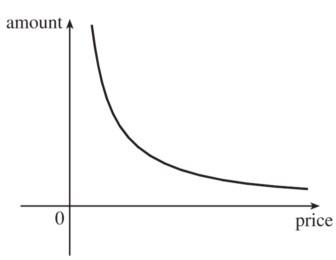
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



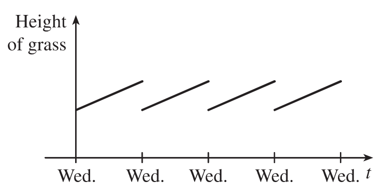
13. Of course, this graph depends strongly on the geographical location!



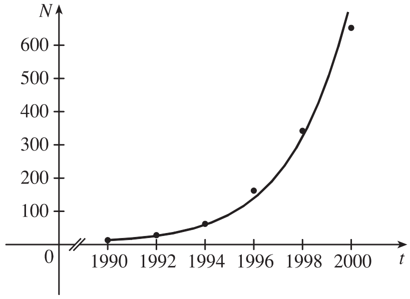
15. As the price increases, the amount sold decreases.



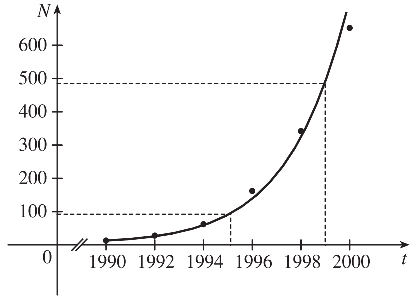
17.



19. (a)



(b) From the graph, we estimate the number of cell-phone subscribers worldwide to be about 92 million in 1995 and 485 million in 1999.



21. $f(x) = 3x^2 - x + 2$.

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12$.

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16$.

$f(a) = 3a^2 - a + 2$.

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2$.

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4$.

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4$.

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2$.

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2$.

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2$.

23. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$,

and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h$.

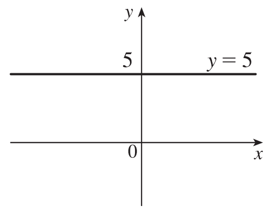
25. $$\frac{f(x) - f(a)}{x - a} = \frac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \frac{\dfrac{a - x}{xa}}{x - a} = \frac{a - x}{xa(x - a)} = \frac{-1(x - a)}{xa(x - a)} = -\frac{1}{ax}$$

27. $f(x) = x/(3x - 1)$ is defined for all x except when $0 = 3x - 1 \quad \Leftrightarrow \quad x = \frac{1}{3}$, so the domain is $\left\{x \in \mathbb{R} \mid x \neq \frac{1}{3}\right\} = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

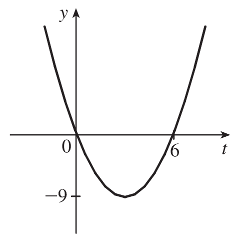
29. $f(t) = \sqrt{t} + \sqrt[3]{t}$ is defined when $t \geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0, \infty)$.

31. $h(x) = 1 \big/ \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x - 5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x - 5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

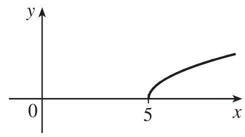
33. $f(x) = 5$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a horizontal line with y-intercept 5.



35. $f(t) = t^2 - 6t$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = t^2 - 6t = t(t-6) \quad \Rightarrow \quad t = 0$ and $t = 6$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = 3$. Since $f(3) = 3^2 - 6 \cdot 3 = -9$, the vertex is $(3, -9)$.

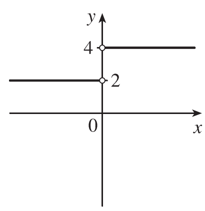


37. $g(x) = \sqrt{x-5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \quad \Rightarrow \quad y^2 = x - 5 \quad \Rightarrow \quad x = y^2 + 5$, we see that g is the top half of a parabola.



39. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

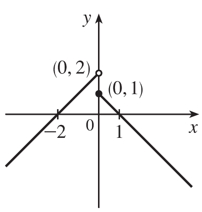
$$G(x) = \begin{cases} \dfrac{3x + x}{x} & \text{if } x > 0 \\ \dfrac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$



Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.

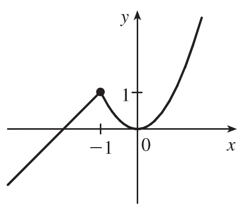
41. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

The domain is $\mathbb{R}$.



43. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x + 2$ and x^2 are equal to 1. The domain is $\mathbb{R}$.



45. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\dfrac{7 - (-3)}{5 - 1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x - 1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \leq x \leq 5$.

47. We need to solve the given equation for y. $x + (y-1)^2 = 0 \quad \Leftrightarrow \quad (y-1)^2 = -x \quad \Leftrightarrow \quad y - 1 = \pm\sqrt{-x} \quad \Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \leq 0$.

49. For $0 \leq x \leq 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \leq 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x - 3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x + 3 & \text{if } 0 \leq x \leq 3 \\ 2x - 6 & \text{if } 3 < x \leq 5 \end{cases}$$

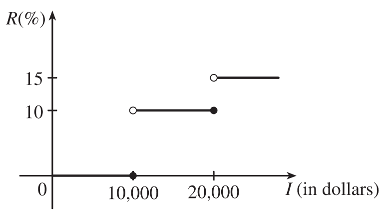
51. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

53. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

55. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

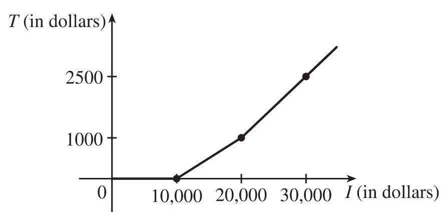
57. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$; $W > 0 \quad \Leftrightarrow \quad 12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

59. (a)



(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.
On \$26,000, tax is assessed on \$16,000, and
$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



61. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

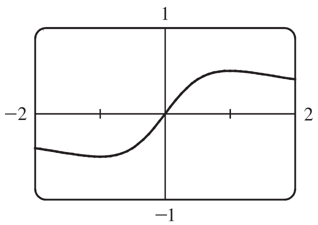
63. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5,3)$ is on the graph of this even function, the point $(-5,3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5,3)$ is on the graph of this odd function, the point $(-5,-3)$ must also be on its graph.

65. $f(x) = \dfrac{x}{x^2+1}$.

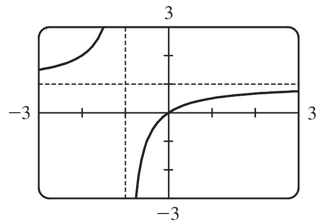
$$f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -\frac{x}{x^2+1} = -f(x).$$

So f is an odd function.



67. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

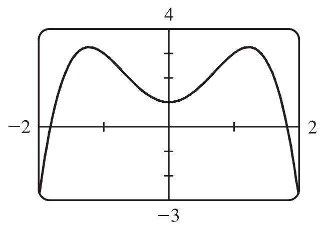
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



69. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1 + 3(-x)^2 - (-x)^4 = 1 + 3x^2 - x^4 = f(x)$.

So f is an even function.



1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \sqrt[5]{x}$ is a root function with $n = 5$.

(b) $g(x) = \sqrt{1-x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x) = x^9 + x^4$ is a polynomial of degree 9.

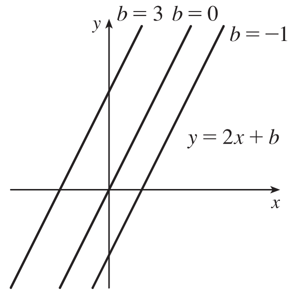
(d) $r(x) = \dfrac{x^2+1}{x^3+x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x) = \tan 2x$ is a trigonometric function.

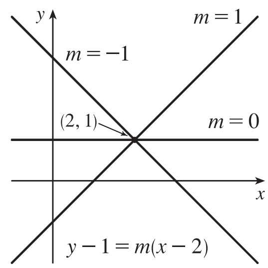
(f) $t(x) = \log_{10} x$ is a logarithmic function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

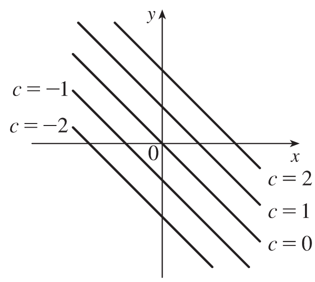


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.



(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.

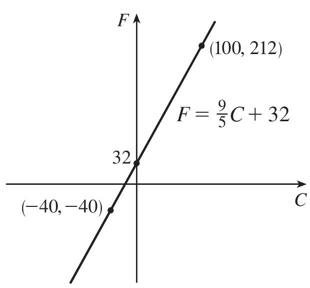


9. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x - (-1)](x - 0)(x - 2)$, or $f(x) = ax(x + 1)(x - 2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.
$6 = a(1)(2)(-1) \quad \Rightarrow \quad -2a = 6 \quad \Rightarrow \quad a = -3$, so an equation for f is $f(x) = -3x(x + 1)(x - 2)$.

11. (a) $D = 200$, so $c = 0.0417D(a + 1) = 0.0417(200)(a + 1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

13. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of $1°$C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

15. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \quad \Leftrightarrow \quad T - 80 = \frac{1}{6}N - \frac{173}{6} \quad \Leftrightarrow \quad T = \frac{1}{6}N + \frac{307}{6} \quad \left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\text{F} \approx 76\,^\circ\text{F}$.

17. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

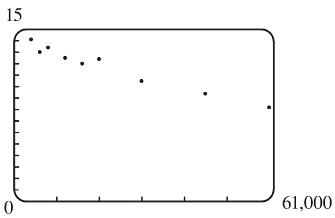
(b) When $P = 100$, then $100 = 0.434d + 15 \quad \Leftrightarrow \quad 0.434d = 85 \quad \Leftrightarrow \quad d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \text{lb/in}^2$ at a depth of approximately 196 feet.

19. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

21. (a)

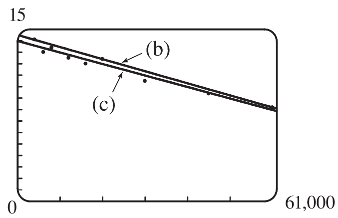


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

$$y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000) \text{ or, equivalently,}$$

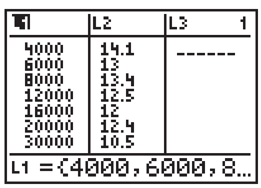
$y \approx -0.000105357x + 14.521429$.

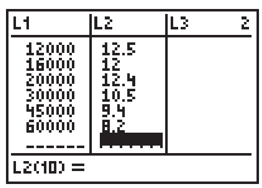


(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

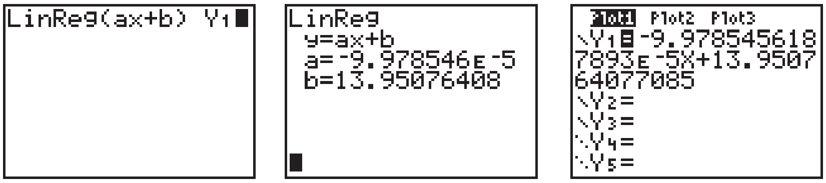
The following commands and screens illustrate how to find the least squares regression line on a TI-83 Plus.

Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.



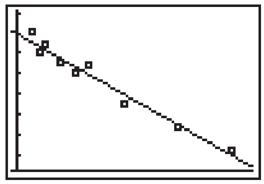


Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER .



Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER .

Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

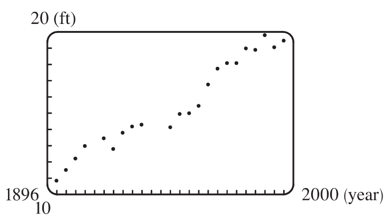


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

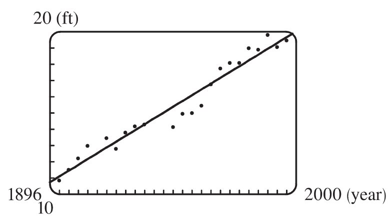
(f) When $x = 200{,}000$, y is negative, so the model does not apply.

23. (a)



A linear model does seem appropriate.

(b)

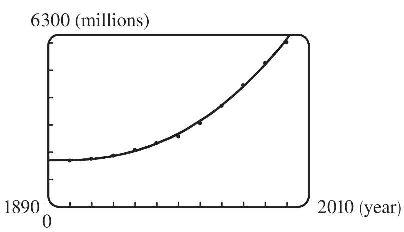


Using a computing device, we obtain the least squares regression line $y = 0.089119747x - 158.2403249$, where x is the year and y is the height in feet.

(c) When $x = 2000$, the model gives $y \approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996—so much for that prediction.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

25.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12{,}823$, and $d = -7{,}743{,}770$. When $x = 1925$, $y \approx 1914$ (million).

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

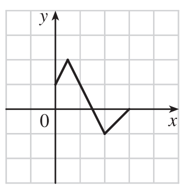
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

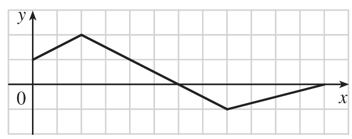
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



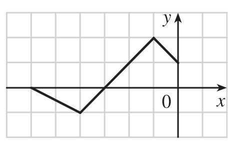
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



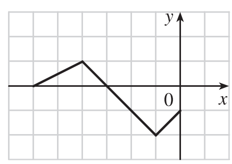
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

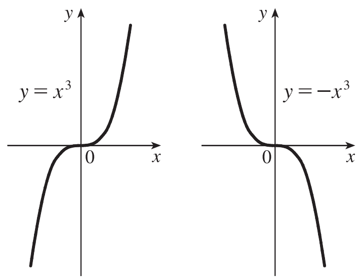
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f \underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

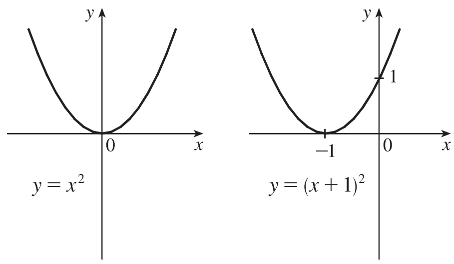
This function can be written as

$$y = -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1$$

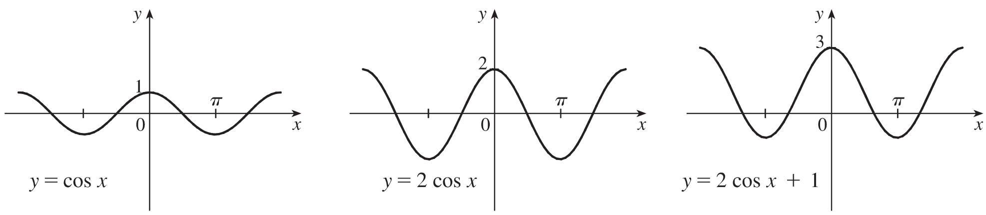
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



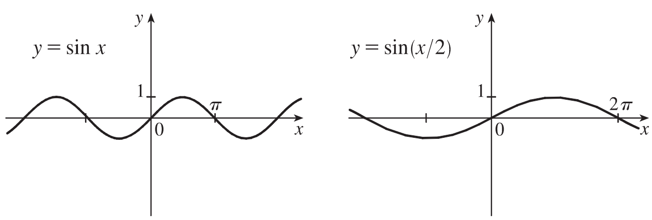
11. $y = (x+1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



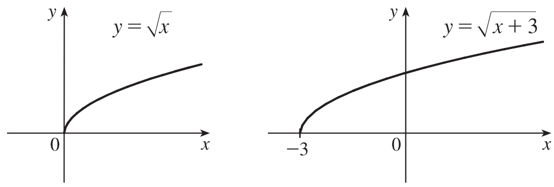
13. $y = 1 + 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



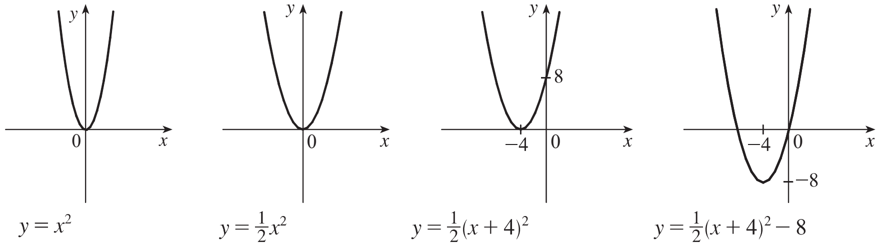
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



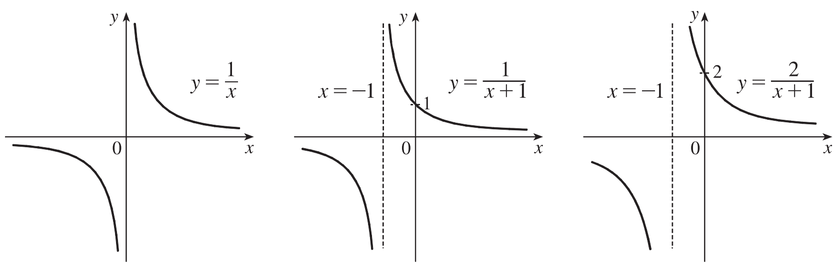
17. $y = \sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.



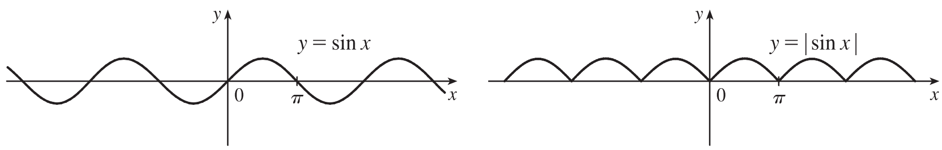
19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x+4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.



21. $y = 2/(x+1)$: Start with the graph of $y = 1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



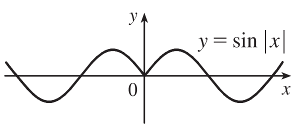
23. $y = |\sin x|$: Start with the graph of $y = \sin x$ and reflect all the parts of the graph below the x-axis about the x-axis.



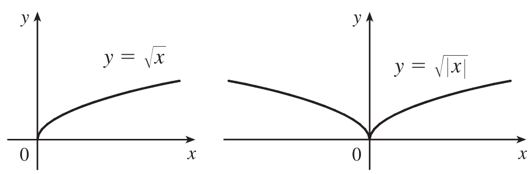
25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



29. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

$(f+g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1,\ D = \mathbb{R}.$

$(f-g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1,\ D = \mathbb{R}.$

$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2,\ D = \mathbb{R}.$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1},\ D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

31. $f(x) = x^2 - 1,\ D = \mathbb{R};\quad g(x) = 2x + 1,\ D = \mathbb{R}.$

(a) $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 - 1 = (4x^2 + 4x + 1) - 1 = 4x^2 + 4x,\ D = \mathbb{R}.$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1 = (2x^2 - 2) + 1 = 2x^2 - 1,\ D = \mathbb{R}.$

(c) $(f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2,\ D = \mathbb{R}.$

(d) $(g \circ g)(x) = g(g(x)) = g(2x+1) = 2(2x+1) + 1 = (4x+2) + 1 = 4x + 3,\ D = \mathbb{R}.$

33. $f(x) = 1 - 3x$; $g(x) = \cos x$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(\cos x) = 1 - 3\cos x.$

(b) $(g \circ f)(x) = g(f(x)) = g(1 - 3x) = \cos(1 - 3x).$

(c) $(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2.$

(d) $(g \circ g)(x) = g(g(x)) = g(\cos x) = \cos(\cos x)$ [Note that this is *not* $\cos x \cdot \cos x$.]

35. $f(x) = x + \dfrac{1}{x},\ D = \{x \mid x \neq 0\};\quad g(x) = \dfrac{x+1}{x+2},\ D = \{x \mid x \neq -2\}$

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x+1}{x+2}\right) = \dfrac{x+1}{x+2} + \dfrac{1}{\dfrac{x+1}{x+2}} = \dfrac{x+1}{x+2} + \dfrac{x+2}{x+1}$

$$= \frac{(x+1)(x+1) + (x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2 + 2x + 1) + (x^2 + 4x + 4)}{(x+2)(x+1)} = \frac{2x^2 + 6x + 5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(x + \dfrac{1}{x}\right) = \dfrac{\left(x + \dfrac{1}{x}\right) + 1}{\left(x + \dfrac{1}{x}\right) + 2} = \dfrac{\dfrac{x^2 + 1 + x}{x}}{\dfrac{x^2 + 1 + 2x}{x}} = \dfrac{x^2 + x + 1}{x^2 + 2x + 1} = \dfrac{x^2 + x + 1}{(x+1)^2}$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$,

the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(x + \dfrac{1}{x}\right) = \left(x + \dfrac{1}{x}\right) + \dfrac{1}{x + \frac{1}{x}} = x + \dfrac{1}{x} + \dfrac{1}{\frac{x^2+1}{x}} = x + \dfrac{1}{x} + \dfrac{x}{x^2 + 1}$

$$= \frac{x(x)(x^2 + 1) + 1(x^2 + 1) + x(x)}{x(x^2 + 1)} = \frac{x^4 + x^2 + x^2 + 1 + x^2}{x(x^2 + 1)}$$

$$= \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}, \quad D = \{x \mid x \neq 0\}$$

(d) $(g\circ g)(x)=g(g(x))=g\left(\dfrac{x+1}{x+2}\right)=\dfrac{\dfrac{x+1}{x+2}+1}{\dfrac{x+1}{x+2}+2}=\dfrac{\dfrac{x+1+1(x+2)}{x+2}}{\dfrac{x+1+2(x+2)}{x+2}}=\dfrac{x+1+x+2}{x+1+2x+4}=\dfrac{2x+3}{3x+5}$

Since $g(x)$ is not defined for $x=-2$ and $g(g(x))$ is not defined for $x=-\frac{5}{3}$,

the domain of $(g\circ g)(x)$ is $D=\left\{x \mid x\neq -2,-\frac{5}{3}\right\}$.

37. $(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x-1))=f(2(x-1))=2(x-1)+1=2x-1$

39. $(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x^3+2))=f[(x^3+2)^2]$
$=f(x^6+4x^3+4)=\sqrt{(x^6+4x^3+4)-3}=\sqrt{x^6+4x^3+1}$

41. Let $g(x)=x^2+1$ and $f(x)=x^{10}$. Then $(f\circ g)(x)=f(g(x))=f(x^2+1)=(x^2+1)^{10}=F(x)$.

43. Let $g(x)=\sqrt[3]{x}$ and $f(x)=\dfrac{x}{1+x}$. Then $(f\circ g)(x)=f(g(x))=f(\sqrt[3]{x})=\dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}}=F(x)$.

45. Let $g(t)=\cos t$ and $f(t)=\sqrt{t}$. Then $(f\circ g)(t)=f(g(t))=f(\cos t)=\sqrt{\cos t}=u(t)$.

47. Let $h(x)=x^2$, $g(x)=3^x$, and $f(x)=1-x$. Then

$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x^2))=f\left(3^{x^2}\right)=1-3^{x^2}=H(x)$.

49. Let $h(x)=\sqrt{x}$, $g(x)=\sec x$, and $f(x)=x^4$. Then

$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(\sqrt{x}))=f(\sec\sqrt{x})=(\sec\sqrt{x})^4=\sec^4(\sqrt{x})=H(x)$.

51. (a) $g(2)=5$, because the point $(2,5)$ is on the graph of g. Thus, $f(g(2))=f(5)=4$, because the point $(5,4)$ is on the graph of f.

(b) $g(f(0))=g(0)=3$

(c) $(f\circ g)(0)=f(g(0))=f(3)=0$

(d) $(g\circ f)(6)=g(f(6))=g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

(e) $(g\circ g)(-2)=g(g(-2))=g(1)=4$

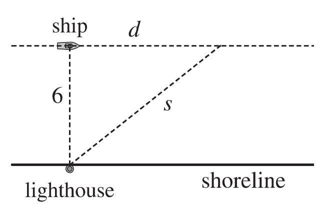
(f) $(f\circ f)(4)=f(f(4))=f(2)=-2$

53. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t)=60t$.

(b) $A=\pi r^2 \quad\Rightarrow\quad (A\circ r)(t)=A(r(t))=\pi(60t)^2=3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

55. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s.

By the Pythagorean Theorem, $d^2+6^2=s^2 \quad\Rightarrow\quad s=f(d)=\sqrt{d^2+36}$.

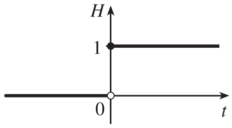


(b) Using $d=rt$, we get $d=(30\text{ km/hr})(t\text{ hr})=30t$ (in km). Thus,

$d=g(t)=30t$.

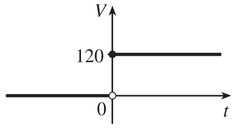
(c) $(f\circ g)(t)=f(g(t))=f(30t)=\sqrt{(30t)^2+36}=\sqrt{900t^2+36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

57. (a)



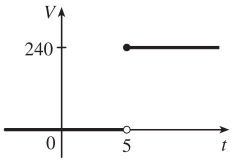
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

59. If $f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2 x + b_2) = m_1(m_2 x + b_2) + b_1 = m_1 m_2 x + m_1 b_2 + b_1$.

So $f \circ g$ is a linear function with slope $m_1 m_2$.

61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let

$f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since

$h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But

$h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

63. (a) If f and g are even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$.

(i) $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$, so $f + g$ is an *even* function.

(ii) $(fg)(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = (fg)(x)$, so fg is an *even* function.

(b) If f and g are odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$.

(i) $(f + g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f + g)(x)$, so $f + g$ is an *odd* function.

(ii) $(fg)(-x) = f(-x) \cdot g(-x) = -f(x) \cdot [-g(x)] = f(x) \cdot g(x) = (fg)(x)$, so fg is an *even* function.

65. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad [\text{because } g \text{ is even}] \quad = h(x)$$

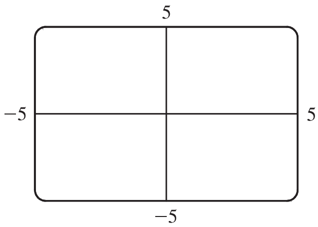
Because $h(-x) = h(x)$, h is an even function.

1.4 Graphing Calculators and Computers

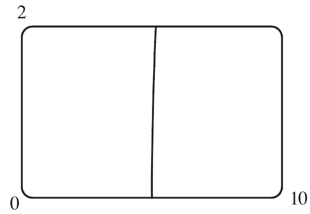
1. $f(x) = \sqrt{x^3 - 5x^2}$

(a) $[-5, 5]$ by $[-5, 5]$

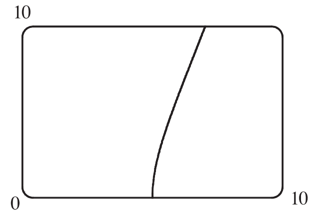
(There is no graph shown.)



(b) $[0, 10]$ by $[0, 2]$

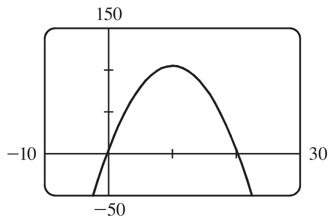


(c) $[0, 10]$ by $[0, 10]$

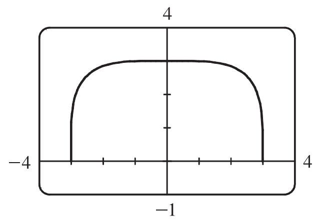


The most appropriate graph is produced in viewing rectangle (c).

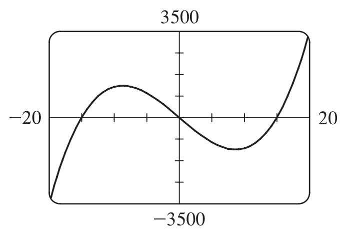
3. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



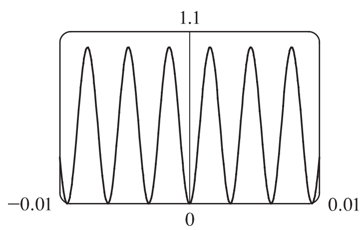
5. $f(x) = \sqrt[4]{81 - x^4}$ is defined when $81 - x^4 \geq 0 \quad \Leftrightarrow \quad x^4 \leq 81 \quad \Leftrightarrow \quad |x| \leq 3$, so the domain of f is $[-3, 3]$. Also $0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3$, so the range is $[0, 3]$.



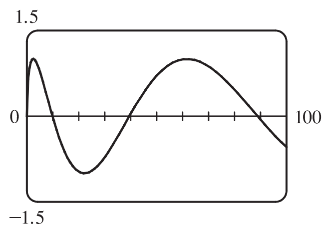
7. The graph of $f(x) = x^3 - 225x$ is symmetric with respect to the origin. Since $f(x) = x^3 - 225x = x(x^2 - 225) = x(x + 15)(x - 15)$, there are x-intercepts at 0, -15, and 15. $f(20) = 3500$.



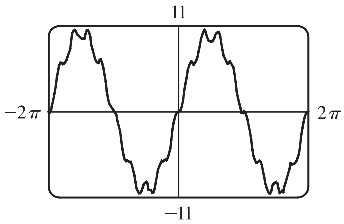
9. The period of $g(x) = \sin(1000x)$ is $\frac{2\pi}{1000} \approx 0.0063$ and its range is $[-1, 1]$. Since $f(x) = \sin^2(1000x)$ is the square of g, its range is $[0, 1]$ and a viewing rectangle of $[-0.01, 0.01]$ by $[0, 1.1]$ seems appropriate.

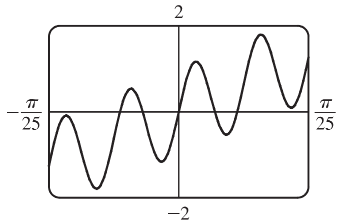


11. The domain of $y = \sqrt{x}$ is $x \geq 0$, so the domain of $f(x) = \sin\sqrt{x}$ is $[0, \infty)$ and the range is $[-1, 1]$. With a little trial-and-error experimentation, we find that an Xmax of 100 illustrates the general shape of f, so an appropriate viewing rectangle is $[0, 100]$ by $[-1.5, 1.5]$.



13. The first term, $10\sin x$, has period 2π and range $[-10, 10]$. It will be the dominant term in any "large" graph of $y = 10\sin x + \sin 100x$, as shown in the first figure. The second term, $\sin 100x$, has period $\frac{2\pi}{100} = \frac{\pi}{50}$ and range $[-1, 1]$. It causes the bumps in the first figure and will be the dominant term in any "small" graph, as shown in the view near the origin in the second figure.

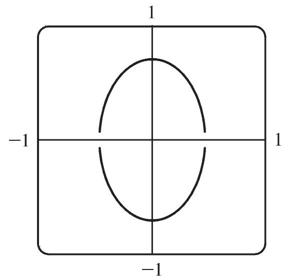




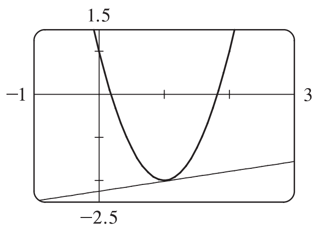
15. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

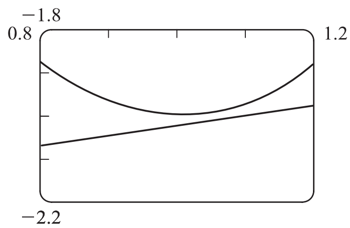
$4x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad 2y^2 = 1 - 4x^2 \quad \Leftrightarrow \quad y^2 = \dfrac{1 - 4x^2}{2} \quad \Leftrightarrow$

$y = \pm\sqrt{\dfrac{1 - 4x^2}{2}}$

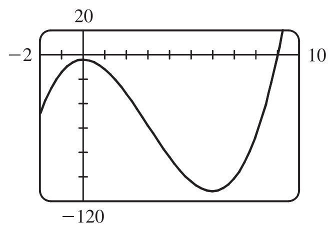


17. From the graph of $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$, it is difficult to see if the graphs intersect. If we zoom in on the fourth quadrant, we see the graphs do not intersect.

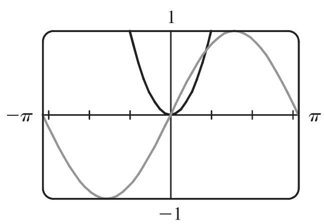




19. From the graph of $f(x) = x^3 - 9x^2 - 4$, we see that there is one solution of the equation $f(x) = 0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x \approx 9.05$.

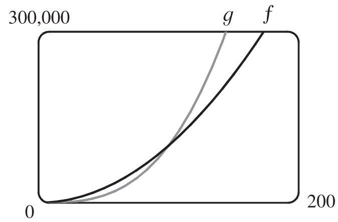


21. We see that the graphs of $f(x) = x^2$ and $g(x) = \sin x$ intersect twice. One solution is $x = 0$. The other solution of $f = g$ is the x-coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x) = x^2 - \sin x$.

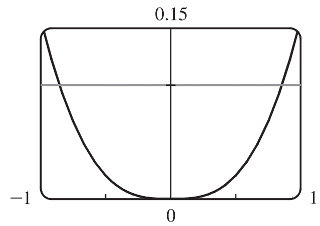


Note: After producing the graph on a TI-83 Plus, we can find the approximate value 0.88 by using the following keystrokes: 2nd CALC 5 ENTER ENTER 1 ENTER . The "1" is just a guess for 0.88.

23. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.

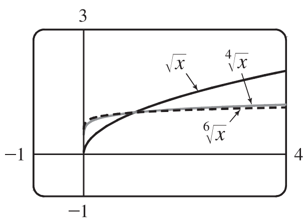


25.

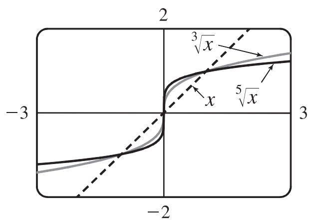


We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values, that is, $-0.85 < x < 0.85$.

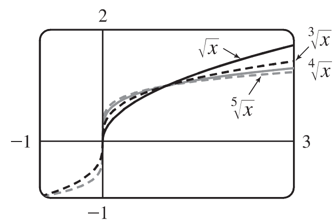
27. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$



(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$



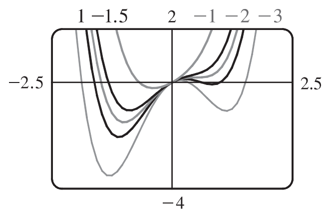
(d) • For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.

• For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.

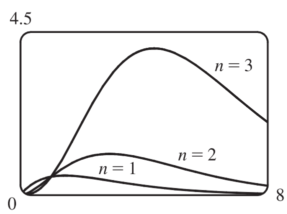
• Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.

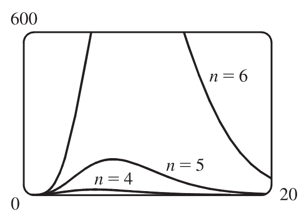
• As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

29. $f(x) = x^4 + cx^2 + x$. If $c < -1.5$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = -1.5$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

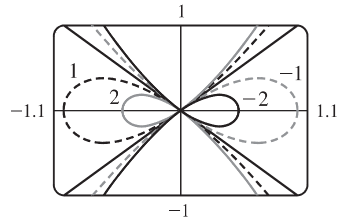


31. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.





33. $y^2 = cx^3 + x^2$. If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite," that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



35. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 "gaps" between pixels, the distance between pixels is $\frac{2\pi - 0}{94}$. Thus, the x-values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \ldots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94} \cdot n, \sin\left(2 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. For $y = \sin 96x$, the points plotted are $\left(\frac{2\pi}{94} \cdot n, \sin\left(96 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(96 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(94 \cdot \tfrac{2\pi}{94} \cdot n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(2\pi n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[by periodicity of sine]}, \quad n = 0, 1, \ldots, 94\end{aligned}$$

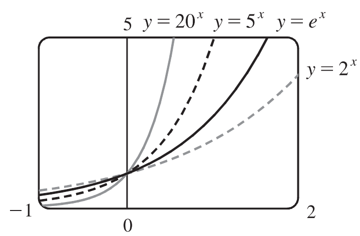
So the y-values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.
Note: Try graphing $y = \sin 94x$. Can you see why all the y-values are zero?

1.5 Exponential Functions

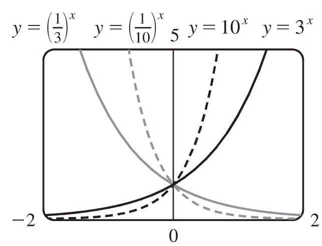
1. (a) $f(x) = a^x,\ a > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.

3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.

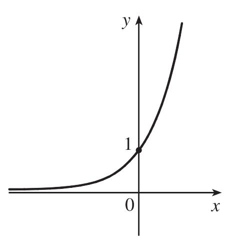
Note: The notation "$x \to \infty$" can be thought of as "x becomes large" at this point. More details on this notation are given in Chapter 2.



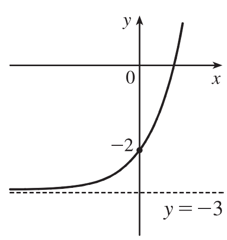
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.

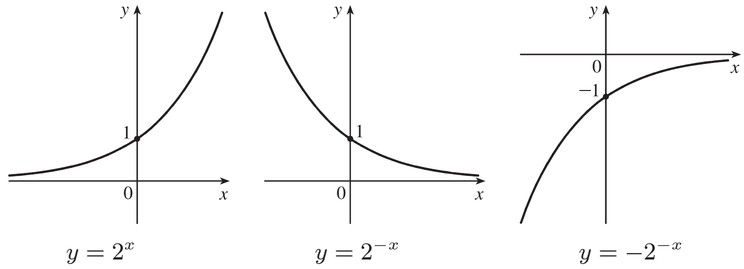


$y = 4^x$

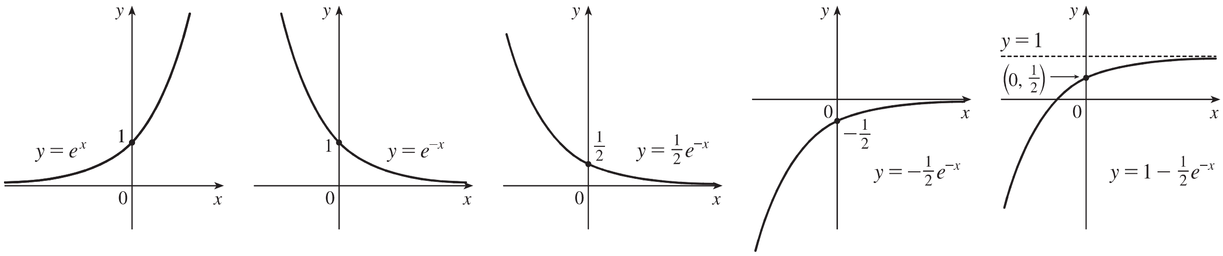


$y = 4^x - 3$

9. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis, and then about the x-axis (or just rotate $180°$ to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



11. We start with the graph of $y = e^x$ (Figure 13) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph upward one unit to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.

(b) $1 - e^x = 0 \quad \Leftrightarrow \quad e^x = 1 \quad \Leftrightarrow \quad x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.

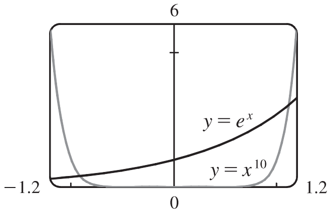
17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $\;6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{a}\right)a^3 \quad \Rightarrow$

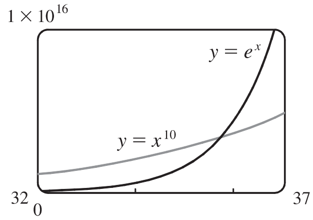
$4 = a^2 \quad \Rightarrow \quad a = 2 \quad [\text{since } a > 0]$ and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

19. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x\left(5^h - 1\right)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

21. 2 ft = 24 in, $f(24) = 24^2$ in = 576 in = 48 ft. $g(24) = 2^{24}$ in = $2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

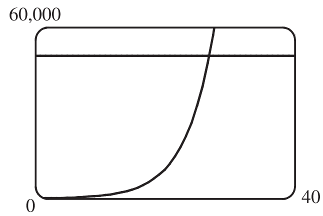
23. The graph of g finally surpasses that of f at $x \approx 35.8$.





25. (a) Fifteen hours represents 5 doubling periods (one doubling period is three hours). $100 \cdot 2^5 = 3200$

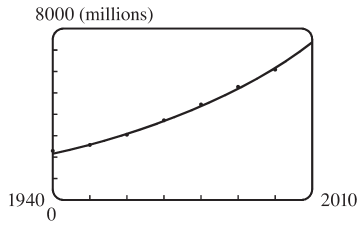
(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100, so the population y at time t is $y = 100 \cdot 2^{t/3}$.



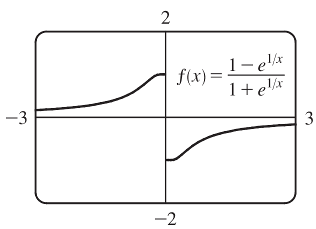
(c) $t = 20 \quad \Rightarrow \quad y = 100 \cdot 2^{20/3} \approx 10{,}159$

(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50{,}000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50,000 in about 26.9 hours.

27. An exponential model is $y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



29.



From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

so f is an odd function.

1.6 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

9. The graph of $f(x) = x^2 - 2x$ is a parabola with axis of symmetry $x = -\dfrac{b}{2a} = -\dfrac{-2}{2(1)} = 1$. Pick any x-values equidistant from 1 to find two equal function values. For example, $f(0) = 0$ and $f(2) = 0$, so f is not one-to-one.

11. $g(x) = 1/x$. $\quad x_1 \neq x_2 \quad\Rightarrow\quad 1/x_1 \neq 1/x_2 \quad\Rightarrow\quad g(x_1) \neq g(x_2)$, so g is one-to-one.
Geometric solution: The graph of g is the hyperbola shown in Figure 14 in Section 1.2. It passes the Horizontal Line Test, so g is one-to-one.

13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

15. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

17. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

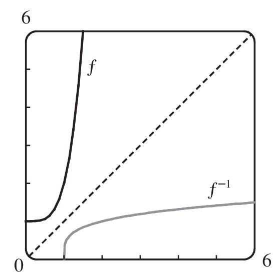
19. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad\Rightarrow\quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad\Rightarrow\quad \frac{9}{5}C + 32 \geq -459.67 \quad\Rightarrow\quad \frac{9}{5}C \geq -491.67 \quad\Rightarrow\quad C \geq -273.15$, the domain of the inverse function.

21. $f(x) = \sqrt{10 - 3x} \quad\Rightarrow\quad y = \sqrt{10 - 3x} \quad (y \geq 0) \quad\Rightarrow\quad y^2 = 10 - 3x \quad\Rightarrow\quad 3x = 10 - y^2 \quad\Rightarrow\quad x = -\frac{1}{3}y^2 + \frac{10}{3}$.
Interchange x and y: $\quad y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

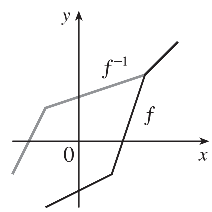
23. $y = f(x) = e^{x^3} \quad\Rightarrow\quad \ln y = x^3 \quad\Rightarrow\quad x = \sqrt[3]{\ln y}$. Interchange x and y: $\quad y = \sqrt[3]{\ln x}$. So $f^{-1}(x) = \sqrt[3]{\ln x}$.

25. $y = f(x) = \ln(x + 3) \quad\Rightarrow\quad x + 3 = e^y \quad\Rightarrow\quad x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

27. $y = f(x) = x^4 + 1 \quad\Rightarrow\quad y - 1 = x^4 \quad\Rightarrow\quad x = \sqrt[4]{y - 1}$ (not $\pm$ since $x \geq 0$). Interchange x and y: $y = \sqrt[4]{x - 1}$. So $f^{-1}(x) = \sqrt[4]{x - 1}$. The graph of $y = \sqrt[4]{x - 1}$ is just the graph of $y = \sqrt[4]{x}$ shifted right one unit. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



29. Reflect the graph of f about the line $y = x$. The points $(-1, -2)$, $(1, -1)$, $(2, 2)$, and $(3, 3)$ on f are reflected to $(-2, -1)$, $(-1, 1)$, $(2, 2)$, and $(3, 3)$ on f^{-1}.



31. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad\Leftrightarrow\quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

33. (a) $\log_5 125 = 3$ since $5^3 = 125$. (b) $\log_3 \dfrac{1}{27} = -3$ since $3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$.

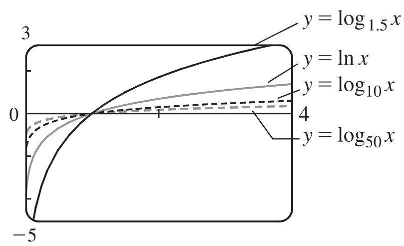
35. (a)
$$\begin{aligned} \log_2 6 - \log_2 15 + \log_2 20 &= \log_2\left(\tfrac{6}{15}\right) + \log_2 20 && \text{[by Law 2]} \\ &= \log_2\left(\tfrac{6}{15} \cdot 20\right) && \text{[by Law 1]} \\ &= \log_2 8, \text{ and } \log_2 8 = 3 \text{ since } 2^3 = 8. \end{aligned}$$

(b)
$$\begin{aligned} \log_3 100 - \log_3 18 - \log_3 50 &= \log_3\left(\tfrac{100}{18}\right) - \log_3 50 = \log_3\left(\tfrac{100}{18 \cdot 50}\right) \\ &= \log_3\left(\tfrac{1}{9}\right), \text{ and } \log_3\left(\tfrac{1}{9}\right) = -2 \text{ since } 3^{-2} = \tfrac{1}{9}. \end{aligned}$$

37.
$$\begin{aligned} \ln 5 + 5\ln 3 &= \ln 5 + \ln 3^5 && \text{[by Law 3]} \\ &= \ln(5 \cdot 3^5) && \text{[by Law 1]} \\ &= \ln 1215 \end{aligned}$$

39. $\ln(1 + x^2) + \frac{1}{2}\ln x - \ln \sin x = \ln(1 + x^2) + \ln x^{1/2} - \ln \sin x = \ln[(1 + x^2)\sqrt{x}\,] - \ln \sin x = \ln \dfrac{(1 + x^2)\sqrt{x}}{\sin x}$

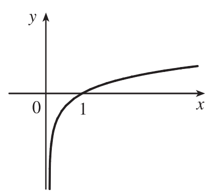
41. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



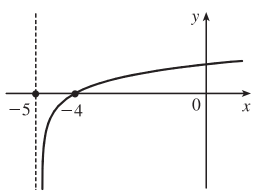
43. $3 \text{ ft} = 36$ in, so we need x such that $\log_2 x = 36 \quad\Leftrightarrow\quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

45. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.

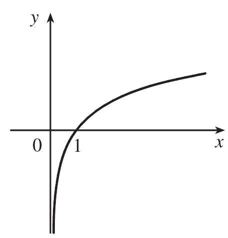


$y = \log_{10} x$

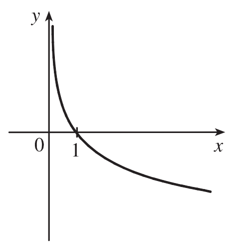


$y = \log_{10}(x + 5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.



$y = \ln x$



$y = -\ln x$

47. (a) $2\ln x = 1 \quad\Rightarrow\quad \ln x = \frac{1}{2} \quad\Rightarrow\quad x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \quad\Rightarrow\quad -x = \ln 5 \quad\Rightarrow\quad x = -\ln 5$

49. (a) $2^{x-5} = 3 \quad\Leftrightarrow\quad \log_2 3 = x - 5 \quad\Leftrightarrow\quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad\Leftrightarrow\quad \ln\left(2^{x-5}\right) = \ln 3 \quad\Leftrightarrow\quad (x - 5)\ln 2 = \ln 3 \quad\Leftrightarrow\quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad\Leftrightarrow\quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \quad \Leftrightarrow \quad x(x-1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

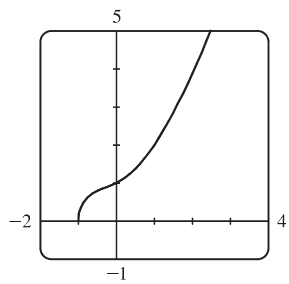
51. (a) $e^x < 10 \quad \Rightarrow \quad \ln e^x < \ln 10 \quad \Rightarrow \quad x < \ln 10 \quad \Rightarrow \quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad \Rightarrow \quad e^{\ln x} > e^{-1} \quad \Rightarrow \quad x > e^{-1} \quad \Rightarrow \quad x \in (1/e, \infty)$

53. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \quad \Rightarrow \quad e^{2x} \leq 3 \quad \Rightarrow \quad 2x \leq \ln 3 \quad \Rightarrow \quad x \leq \frac{1}{2}\ln 3$. Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow \quad y^2 = 3 - e^{2x} \quad \Rightarrow \quad e^{2x} = 3 - y^2 \quad \Rightarrow \quad 2x = \ln(3 - y^2) \quad \Rightarrow$ $x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \quad \Rightarrow \quad x^2 < 3 \quad \Rightarrow \quad |x| < \sqrt{3} \quad \Rightarrow \quad -\sqrt{3} < x < \sqrt{3} \quad \Rightarrow \quad 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3}\,)$, equals the range of f.

55. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$.

Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

57. (a) $n = 100 \cdot 2^{t/3} \quad \Rightarrow \quad \dfrac{n}{100} = 2^{t/3} \quad \Rightarrow \quad \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \quad \Rightarrow \quad t = 3\log_2\left(\dfrac{n}{100}\right)$. Using formula (10), we can write this as $t = f^{-1}(n) = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \quad \Rightarrow \quad t = f^{-1}(50{,}000) = 3 \cdot \dfrac{\ln\left(\frac{50{,}000}{100}\right)}{\ln 2} = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

59. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

61. (a) $\arctan 1 = \frac{\pi}{4}$ since $\tan\frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ since $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

63. (a) In general, $\tan(\arctan x) = x$ for any real number x. Thus, $\tan(\arctan 10) = 10$.

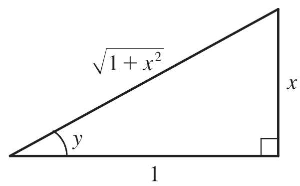
(b) $\sin^{-1}\left(\sin \frac{7\pi}{3}\right) = \sin^{-1}\left(\sin \frac{\pi}{3}\right) = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

[Recall that $\frac{7\pi}{3} = \frac{\pi}{3} + 2\pi$ and the sine function is periodic with period 2π.]

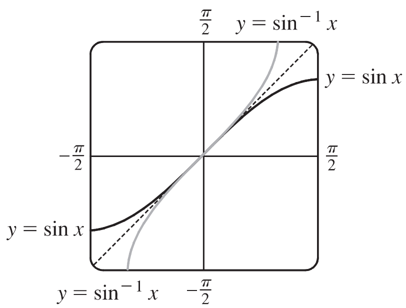
65. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \;\Rightarrow\; \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

67. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1 + x^2}}$.



69.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

71. $g(x) = \sin^{-1}(3x + 1)$.

Domain $(g) = \{x \mid -1 \le 3x + 1 \le 1\} = \{x \mid -2 \le 3x \le 0\} = \left\{x \mid -\frac{2}{3} \le x \le 0\right\} = \left[-\frac{2}{3}, 0\right]$.

Range $(g) = \left\{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

73. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 16.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

(b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^a$

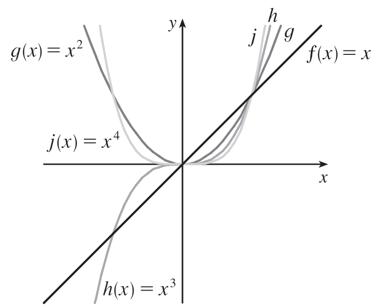
(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

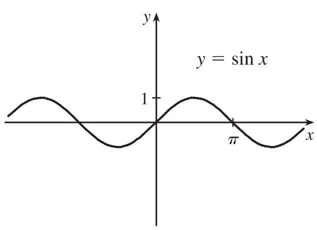
(e) Polynomial of degree 5: $f(x) = x^5 + 2$

(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

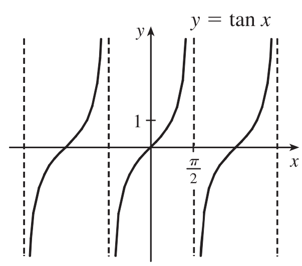
7.



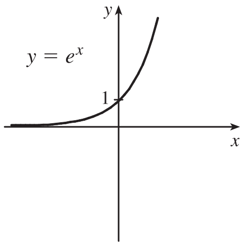
8. (a)



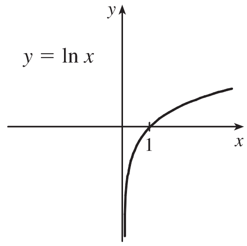
(b)



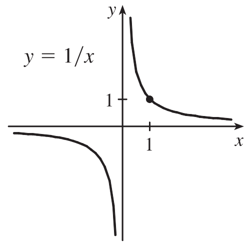
(c)



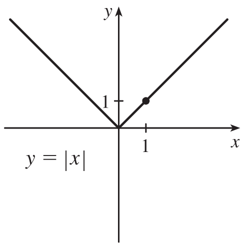
(d)



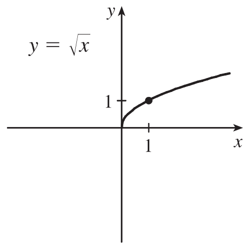
(e)



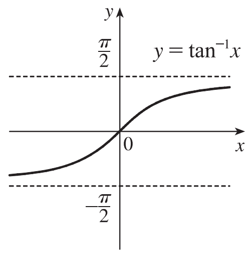
(f)



(g)



(h)



9. (a) The domain of $f+g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f\left(\frac{1}{2}x\right)$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

12. (a) A function f is called a *one-to-one function* if it never takes on the same value twice; that is, if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Or, f is 1-1 if each output corresponds to only one input.)

Use the Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

(b) If f is a one-to-one function with domain A and range B, then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

13. (a) The inverse sine function $f(x) = \sin^{-1} x$ is defined as follows:

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x \qquad \text{and} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Its domain is $-1 \le x \le 1$ and its range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(b) The inverse cosine function $f(x) = \cos^{-1} x$ is defined as follows:

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \qquad \text{and} \qquad 0 \le y \le \pi$$

Its domain is $-1 \le x \le 1$ and its range is $0 \le y \le \pi$.

(c) The inverse tangent function $f(x) = \tan^{-1} x$ is defined as follows:

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \qquad \text{and} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Its domain is $\mathbb{R}$ and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \ne 0 = f(s+t)$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \ne 3f(x)$.

5. True. See the Vertical Line Test.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

13. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \le x \le 6$, or $[-6, 6]$.

(d) The range of f is $-4 \le y \le 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \le x \le 4$.

(f) f is not one-to-one since it fails the Horizontal Line Test.

(g) f is odd since its graph is symmetric about the origin.

3. $f(x) = x^2 - 2x + 3$, so $f(a+h) = (a+h)^2 - 2(a+h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and

$$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

5. $f(x) = 2/(3x - 1)$. Domain: $3x - 1 \neq 0 \;\Rightarrow\; 3x \neq 1 \;\Rightarrow\; x \neq \frac{1}{3}$. $D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.) $R = (-\infty, 0) \cup (0, \infty)$

7. $h(x) = \ln(x + 6)$. Domain: $x + 6 > 0 \;\Rightarrow\; x > -6$. $D = (-6, \infty)$

Range: $x + 6 > 0$, so $\ln(x + 6)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.
$R = (-\infty, \infty)$

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

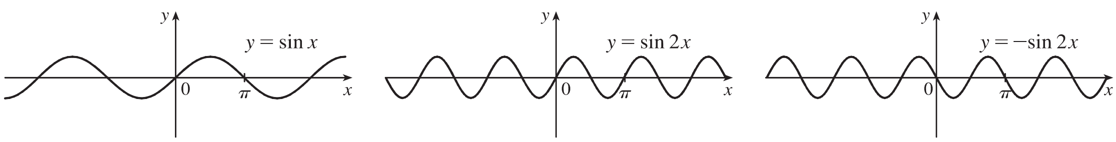
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

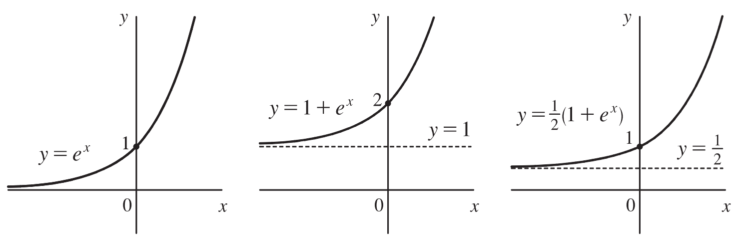
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

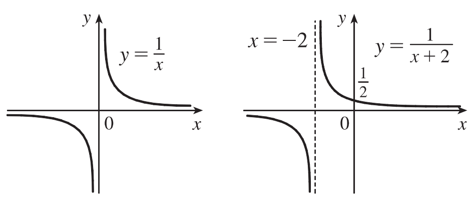
11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



13. $y = \frac{1}{2}(1 + e^x)$:
Start with the graph of $y = e^x$, shift 1 unit upward, and compress vertically by a factor of 2.



15. $f(x) = \dfrac{1}{x + 2}$:
Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

19. $f(x) = \ln x, \quad D = (0, \infty); \quad g(x) = x^2 - 9, \quad D = \mathbb{R}$.

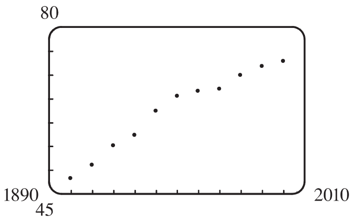
(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

Domain: $x^2 - 9 > 0 \quad \Rightarrow \quad x^2 > 9 \quad \Rightarrow \quad |x| > 3 \quad \Rightarrow \quad x \in (-\infty, -3) \cup (3, \infty)$

(b) $(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

(c) $(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \quad \Rightarrow \quad x > e^0 = 1$, or $(1, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

21. 

Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$, gives us an estimate of 77.6 years for the year 2010.

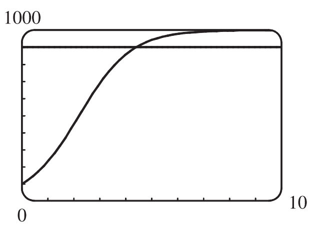
23. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

25. (a) $e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

(c) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(d) Let $\theta = \cos^{-1} \frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin\left(\cos^{-1} \frac{4}{5}\right) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

27. (a) 

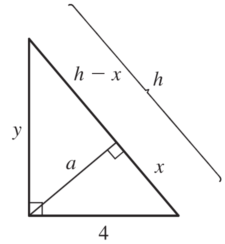
The population would reach 900 in about 4.4 years.

(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \quad \Rightarrow \quad 100P + 900Pe^{-t} = 100{,}000 \Rightarrow \quad 900Pe^{-t} = 100{,}000 - 100P \quad \Rightarrow$

$e^{-t} = \dfrac{100{,}000 - 100P}{900P} \Rightarrow \quad -t = \ln\left(\dfrac{1000 - P}{9P}\right) \quad \Rightarrow \quad t = -\ln\left(\dfrac{1000 - P}{9P}\right)$, or $\ln\left(\dfrac{9P}{1000 - P}\right)$; this is the time required for the population to reach a given number P.

(c) $P = 900 \quad \Rightarrow \quad t = \ln\left(\dfrac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

PRINCIPLES OF PROBLEM SOLVING

1. 

By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2-16}}{h}$.

3. $|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x+5| = \begin{cases} x+5 & \text{if } x \geq -5 \\ -x-5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x-5) = 3 \quad \Leftrightarrow \quad x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x+5) = 3 \quad \Leftrightarrow \quad x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x+5) = 3 \quad \Leftrightarrow \quad x = 9$.

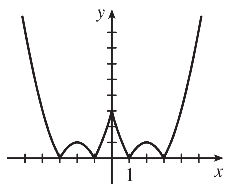
So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

5. $f(x) = \left|x^2 - 4|x| + 3\right|$. If $x \geq 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.



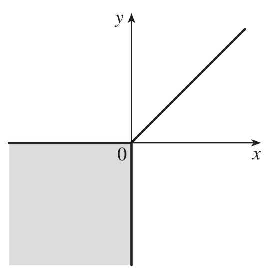
This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.

7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	



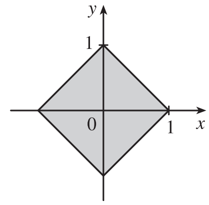
Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

Case 4 gives us the entire third quadrant.

9. $|x| + |y| \le 1$. The boundary of the region has equation $|x| + |y| = 1$. In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



11. $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32) = \left(\dfrac{\ln 3}{\ln 2}\right)\left(\dfrac{\ln 4}{\ln 3}\right)\left(\dfrac{\ln 5}{\ln 4}\right)\cdots\left(\dfrac{\ln 32}{\ln 31}\right) = \dfrac{\ln 32}{\ln 2} = \dfrac{\ln 2^5}{\ln 2} = \dfrac{5\ln 2}{\ln 2} = 5$

13. $\ln\left(x^2 - 2x - 2\right) \le 0 \;\Rightarrow\; x^2 - 2x - 2 \le e^0 = 1 \;\Rightarrow\; x^2 - 2x - 3 \le 0 \;\Rightarrow\; (x-3)(x+1) \le 0 \;\Rightarrow\; x \in [-1, 3]$.

Since the argument must be positive, $x^2 - 2x - 2 > 0 \;\Rightarrow\; \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0 \;\Rightarrow$

$x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

2 ☐ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

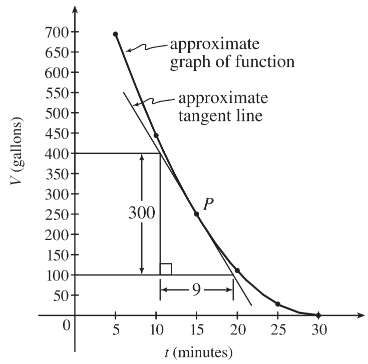
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



3. (a)

	x	Q	m_{PQ}
(i)	0.5	$(0.5, 0.333333)$	0.333333
(ii)	0.9	$(0.9, 0.473684)$	0.263158
(iii)	0.99	$(0.99, 0.497487)$	0.251256
(iv)	0.999	$(0.999, 0.499750)$	0.250125
(v)	1.5	$(1.5, 0.6)$	0.2
(vi)	1.1	$(1.1, 0.523810)$	0.238095
(vii)	1.01	$(1.01, 0.502488)$	0.248756
(viii)	1.001	$(1.001, 0.500250)$	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ or $y = \frac{1}{4}x + \frac{1}{4}$.

5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

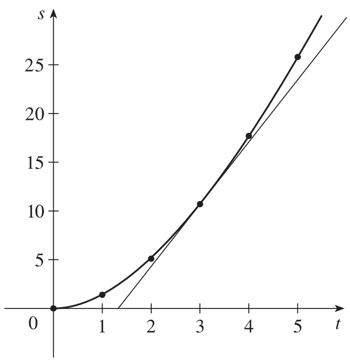
7. (a) (i) On the interval $[1, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(1)}{3 - 1} = \dfrac{10.7 - 1.4}{2} = \dfrac{9.3}{2} = 4.65$ m/s.

(ii) On the interval $[2, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(2)}{3 - 2} = \dfrac{10.7 - 5.1}{1} = 5.6$ m/s.

(iii) On the interval $[3, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(3)}{5 - 3} = \dfrac{25.8 - 10.7}{2} = \dfrac{15.1}{2} = 7.55$ m/s.

(iv) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{17.7 - 10.7}{1} = 7$ m/s.

(b)



Using the points $(2, 4)$ and $(5, 23)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\dfrac{23 - 4}{5 - 2} \approx 6.3$ m/s.

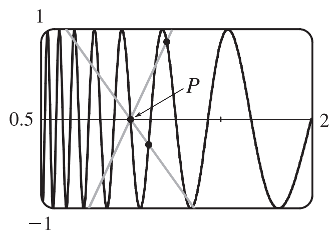
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

5. (a) $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim\limits_{x \to 1^-} f(x) = 2$.

(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim\limits_{x \to 1^+} f(x) = 3$.

(c) $\lim\limits_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim\limits_{x \to 5} f(x) = 4$.

(e) $f(5)$ is not defined, so it doesn't exist.

7. (a) $\lim\limits_{t \to 0^-} g(t) = -1$ (b) $\lim\limits_{t \to 0^+} g(t) = -2$

(c) $\lim\limits_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim\limits_{t \to 2^-} g(t) = 2$ (e) $\lim\limits_{t \to 2^+} g(t) = 0$

(f) $\lim\limits_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$ (h) $\lim\limits_{t \to 4} g(t) = 3$

9. (a) $\lim\limits_{x \to -7} f(x) = -\infty$ (b) $\lim\limits_{x \to -3} f(x) = \infty$ (c) $\lim\limits_{x \to 0} f(x) = \infty$

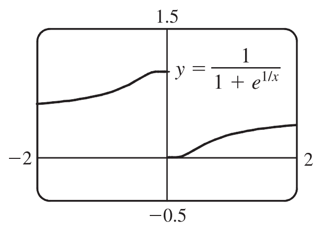
(d) $\lim\limits_{x \to 6^-} f(x) = -\infty$ (e) $\lim\limits_{x \to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

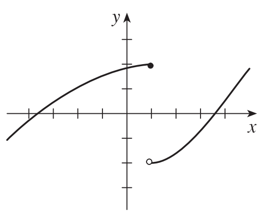
11. (a) $\lim\limits_{x \to 0^-} f(x) = 1$

(b) $\lim\limits_{x \to 0^+} f(x) = 0$

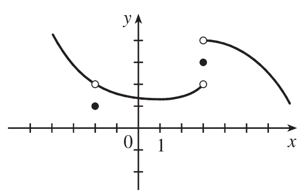
(c) $\lim\limits_{x \to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



13. $\lim\limits_{x \to 1^-} f(x) = 2$, $\lim\limits_{x \to 1^+} f(x) = -2$, $f(1) = 2$



15. $\lim\limits_{x \to 3^+} f(x) = 4$, $\lim\limits_{x \to 3^-} f(x) = 2$, $\lim\limits_{x \to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$



17. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

19. For $f(x) = \dfrac{e^x - 1 - x}{x^2}$:

x	$f(x)$
1	0.718282
0.5	0.594885
0.1	0.517092
0.05	0.508439
0.01	0.501671

x	$f(x)$
-1	0.367879
-0.5	0.426123
-0.1	0.483742
-0.05	0.491770
-0.01	0.498337

It appears that $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2}$.

21. For $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

23. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim\limits_{x \to 1} \dfrac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

25. $\lim\limits_{x \to -3^+} \dfrac{x+2}{x+3} = -\infty$ since the numerator is negative and the denominator approaches 0 from the positive side as $x \to -3^+$.

27. $\lim\limits_{x \to 1} \dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

29. Let $t = x^2 - 9$. Then as $x \to 3^+$, $t \to 0^+$, and $\lim\limits_{x \to 3^+} \ln(x^2 - 9) = \lim\limits_{t \to 0^+} \ln t = -\infty$ by (3).

31. $\lim\limits_{x \to 2\pi^-} x \csc x = \lim\limits_{x \to 2\pi^-} \dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

33. (a) $f(x) = \dfrac{1}{x^3 - 1}$.

From these calculations, it seems that $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$.

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	$-33{,}333.7$

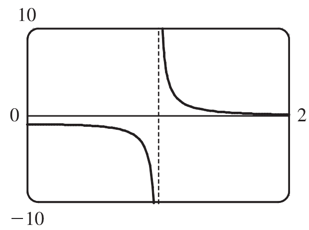
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x\to 1^-} f(x) = -\infty$.

If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that

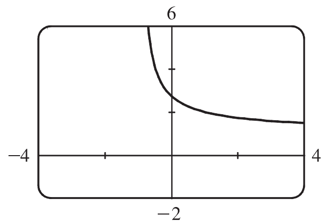
$\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.



35. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
−0.001	2.71964
−0.0001	2.71842
−0.00001	2.71830
−0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

(b)



It appears that $\lim_{x\to 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 3.6 we will see that the value of the limit is exactly e.

37. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

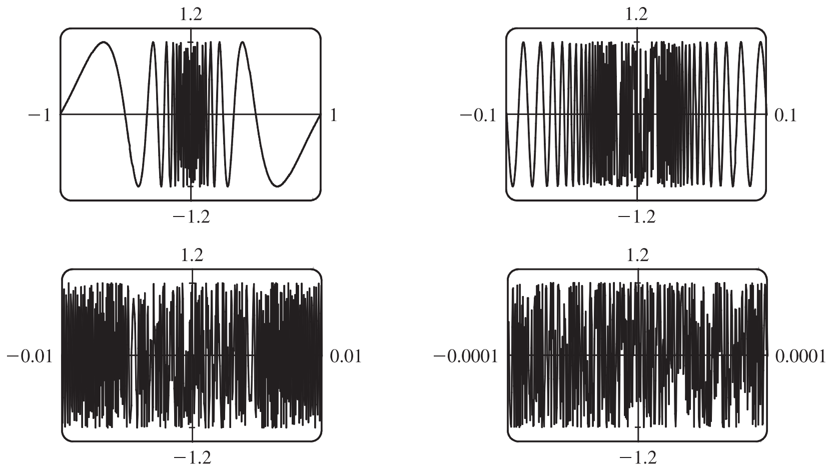
It appears that $\lim_{x\to 0} f(x) = 0$.

(b)

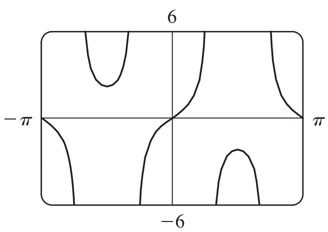
x	$f(x)$
0.04	0.000572
0.02	−0.000614
0.01	−0.000907
0.005	−0.000978
0.003	−0.000993
0.001	−0.001000

It appears that $\lim_{x\to 0} f(x) = -0.001$.

39. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.



41.



There appear to be vertical asymptotes of the curve $y = \tan(2\sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of vertical asymptotes (corresponding to $x \approx \pm 2.24$).

2.3 Calculating Limits Using the Limit Laws

1. (a)
$$\begin{aligned}\lim_{x\to 2}[f(x) + 5g(x)] &= \lim_{x\to 2} f(x) + \lim_{x\to 2}[5g(x)] && \text{[Limit Law 1]}\\ &= \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x) && \text{[Limit Law 3]}\\ &= 4 + 5(-2) = -6\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to 2}[g(x)]^3 &= \left[\lim_{x\to 2} g(x)\right]^3 && \text{[Limit Law 6]}\\ &= (-2)^3 = -8\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x\to 2}\sqrt{f(x)} &= \sqrt{\lim_{x\to 2} f(x)} && \text{[Limit Law 11]}\\ &= \sqrt{4} = 2\end{aligned}$$

(d) $$\lim_{x\to 2}\frac{3f(x)}{g(x)} = \frac{\lim\limits_{x\to 2}[3f(x)]}{\lim\limits_{x\to 2}g(x)} \qquad \text{[Limit Law 5]}$$
$$= \frac{3\lim\limits_{x\to 2}f(x)}{\lim\limits_{x\to 2}g(x)} \qquad \text{[Limit Law 3]}$$
$$= \frac{3(4)}{-2} = -6$$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $$\lim_{x\to 2}\frac{g(x)\,h(x)}{f(x)} = \frac{\lim\limits_{x\to 2}[g(x)\,h(x)]}{\lim\limits_{x\to 2}f(x)} \qquad \text{[Limit Law 5]}$$
$$= \frac{\lim\limits_{x\to 2}g(x)\cdot\lim\limits_{x\to 2}h(x)}{\lim\limits_{x\to 2}f(x)} \qquad \text{[Limit Law 4]}$$
$$= \frac{-2\cdot 0}{4} = 0$$

3. $$\lim_{x\to -2}(3x^4+2x^2-x+1) = \lim_{x\to -2}3x^4 + \lim_{x\to -2}2x^2 - \lim_{x\to -2}x + \lim_{x\to -2}1 \qquad \text{[Limit Laws 1 and 2]}$$
$$= 3\lim_{x\to -2}x^4 + 2\lim_{x\to -2}x^2 - \lim_{x\to -2}x + \lim_{x\to -2}1 \qquad [3]$$
$$= 3(-2)^4 + 2(-2)^2 - (-2) + (1) \qquad \text{[9, 8, and 7]}$$
$$= 48+8+2+1 = 59$$

5. $$\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3) = \lim_{x\to 8}(1+\sqrt[3]{x})\cdot\lim_{x\to 8}(2-6x^2+x^3) \qquad \text{[Limit Law 4]}$$
$$= \left(\lim_{x\to 8}1 + \lim_{x\to 8}\sqrt[3]{x}\right)\cdot\left(\lim_{x\to 8}2 - 6\lim_{x\to 8}x^2 + \lim_{x\to 8}x^3\right) \qquad \text{[1, 2, and 3]}$$
$$= (1+\sqrt[3]{8})\cdot(2-6\cdot 8^2+8^3) \qquad \text{[7, 10, 9]}$$
$$= (3)(130) = 390$$

7. $$\lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3 = \left(\lim_{x\to 1}\frac{1+3x}{1+4x^2+3x^4}\right)^3 \qquad [6]$$
$$= \left[\frac{\lim\limits_{x\to 1}(1+3x)}{\lim\limits_{x\to 1}(1+4x^2+3x^4)}\right]^3 \qquad [5]$$
$$= \left[\frac{\lim\limits_{x\to 1}1 + 3\lim\limits_{x\to 1}x}{\lim\limits_{x\to 1}1 + 4\lim\limits_{x\to 1}x^2 + 3\lim\limits_{x\to 1}x^4}\right]^3 \qquad \text{[2, 1, and 3]}$$
$$= \left[\frac{1+3(1)}{1+4(1)^2+3(1)^4}\right]^3 = \left[\frac{4}{8}\right]^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \qquad \text{[7, 8, and 9]}$$

9. $$\lim_{x\to 4^-}\sqrt{16-x^2} = \sqrt{\lim_{x\to 4^-}(16-x^2)} \qquad [11]$$
$$= \sqrt{\lim_{x\to 4^-}16 - \lim_{x\to 4^-}x^2} \qquad [2]$$
$$= \sqrt{16-(4)^2} = 0 \qquad \text{[7 and 9]}$$

11. $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{x-2}=\lim_{x\to 2}(x+3)=2+3=5$

13. $\displaystyle\lim_{x\to 2}\frac{x^2-x+6}{x-2}$ does not exist since $x-2\to 0$ but $x^2-x+6\to 8$ as $x\to 2$.

15. $\displaystyle\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3}=\lim_{t\to -3}\frac{(t+3)(t-3)}{(2t+1)(t+3)}=\lim_{t\to -3}\frac{t-3}{2t+1}=\frac{-3-3}{2(-3)+1}=\frac{-6}{-5}=\frac{6}{5}$

17. $\displaystyle\lim_{h\to 0}\frac{(4+h)^2-16}{h}=\lim_{h\to 0}\frac{(16+8h+h^2)-16}{h}=\lim_{h\to 0}\frac{8h+h^2}{h}=\lim_{h\to 0}\frac{h(8+h)}{h}=\lim_{h\to 0}(8+h)=8+0=8$

19. By the formula for the sum of cubes, we have

$$\lim_{x\to -2}\frac{x+2}{x^3+8}=\lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)}=\lim_{x\to -2}\frac{1}{x^2-2x+4}=\frac{1}{4+4+4}=\frac{1}{12}.$$

21. $\displaystyle\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to 9}\frac{\left(3+\sqrt{t}\right)\left(3-\sqrt{t}\right)}{3-\sqrt{t}}=\lim_{t\to 9}\left(3+\sqrt{t}\right)=3+\sqrt{9}=6$

23. $\displaystyle\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}+3\right)}$

$$=\lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}=\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

25. $\displaystyle\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim_{x\to -4}\frac{\frac{x+4}{4x}}{4+x}=\lim_{x\to -4}\frac{x+4}{4x(4+x)}=\lim_{x\to -4}\frac{1}{4x}=\frac{1}{4(-4)}=-\frac{1}{16}$

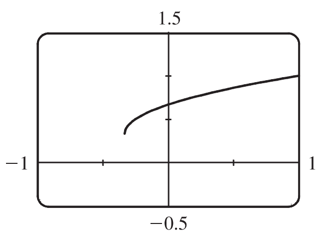
27. $\displaystyle\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$

$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16\left(4+\sqrt{16}\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

29. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

31. (a)



$\displaystyle\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$

(b)

x	$f(x)$
-0.001	0.6661663
-0.0001	0.6666167
-0.00001	0.6666617
-0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$

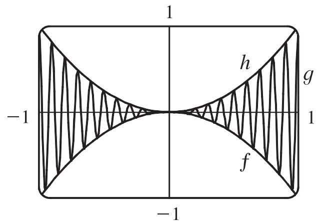
$$\begin{aligned}
&=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right) && \text{[Limit Law 3]}\\
&=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right] && \text{[1 and 11]}\\
&=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right) && \text{[1, 3, and 7]}\\
&=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right) && \text{[7 and 8]}\\
&=\frac{1}{3}(1+1)=\frac{2}{3}
\end{aligned}$$

33. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

$-1\le\cos 20\pi x\le 1 \quad\Rightarrow\quad -x^2\le x^2\cos 20\pi x\le x^2 \quad\Rightarrow\quad f(x)\le g(x)\le h(x)$.

So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have

$\lim\limits_{x\to 0}g(x)=0$.



35. We have $\lim\limits_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim\limits_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$ for $x\ge 0$, $\lim\limits_{x\to 4}f(x)=7$ by the Squeeze Theorem.

37. $-1\le\cos(2/x)\le 1 \quad\Rightarrow\quad -x^4\le x^4\cos(2/x)\le x^4$. Since $\lim\limits_{x\to 0}\left(-x^4\right)=0$ and $\lim\limits_{x\to 0}x^4=0$, we have $\lim\limits_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

39. $|x-3|=\begin{cases}x-3 & \text{if } x-3\ge 0\\ -(x-3) & \text{if } x-3<0\end{cases}=\begin{cases}x-3 & \text{if } x\ge 3\\ 3-x & \text{if } x<3\end{cases}$

Thus, $\lim\limits_{x\to 3^+}(2x+|x-3|)=\lim\limits_{x\to 3^+}(2x+x-3)=\lim\limits_{x\to 3^+}(3x-3)=3(3)-3=6$ and

$\lim\limits_{x\to 3^-}(2x+|x-3|)=\lim\limits_{x\to 3^-}(2x+3-x)=\lim\limits_{x\to 3^-}(x+3)=3+3=6$. Since the left and right limits are equal,

$\lim\limits_{x\to 3}(2x+|x-3|)=6$.

41. $\left|2x^3-x^2\right|=\left|x^2(2x-1)\right|=\left|x^2\right|\cdot|2x-1|=x^2\,|2x-1|$

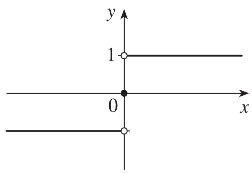
$|2x-1|=\begin{cases}2x-1 & \text{if } 2x-1\ge 0\\ -(2x-1) & \text{if } 2x-1<0\end{cases}=\begin{cases}2x-1 & \text{if } x\ge 0.5\\ -(2x-1) & \text{if } x<0.5\end{cases}$

So $\left|2x^3-x^2\right|=x^2[-(2x-1)]$ for $x<0.5$.

Thus, $\lim\limits_{x\to 0.5^-}\dfrac{2x-1}{|2x^3-x^2|}=\lim\limits_{x\to 0.5^-}\dfrac{2x-1}{x^2[-(2x-1)]}=\lim\limits_{x\to 0.5^-}\dfrac{-1}{x^2}=\dfrac{-1}{(0.5)^2}=\dfrac{-1}{0.25}=-4$.

43. Since $|x| = -x$ for $x < 0$, we have $\displaystyle\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{-x}\right) = \lim_{x\to 0^-}\frac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

45. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\displaystyle\lim_{x\to 0^+}\operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\displaystyle\lim_{x\to 0^-}\operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1$.

(iii) Since $\displaystyle\lim_{x\to 0^-}\operatorname{sgn} x \neq \lim_{x\to 0^+}\operatorname{sgn} x$, $\displaystyle\lim_{x\to 0}\operatorname{sgn} x$ does not exist.

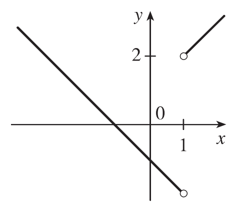
(iv) Since $|\operatorname{sgn} x| = 1$ for $x \neq 0$, $\displaystyle\lim_{x\to 0}|\operatorname{sgn} x| = \lim_{x\to 0} 1 = 1$.

47. (a) (i) $\displaystyle\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+}\frac{x^2-1}{|x-1|} = \lim_{x\to 1^+}\frac{x^2-1}{x-1} = \lim_{x\to 1^+}(x+1) = 2$

(ii) $\displaystyle\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-}\frac{x^2-1}{|x-1|} = \lim_{x\to 1^-}\frac{x^2-1}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$

(b) No, $\displaystyle\lim_{x\to 1} F(x)$ does not exist since $\displaystyle\lim_{x\to 1^+} F(x) \neq \lim_{x\to 1^-} F(x)$.

(c)



49. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\displaystyle\lim_{x\to -2^+}[\![x]\!] = \lim_{x\to -2^+}(-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\displaystyle\lim_{x\to -2^-}[\![x]\!] = \lim_{x\to -2^-}(-3) = -3$.

The right and left limits are different, so $\displaystyle\lim_{x\to -2}[\![x]\!]$ does not exist.

(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\displaystyle\lim_{x\to -2.4}[\![x]\!] = \lim_{x\to -2.4}(-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\displaystyle\lim_{x\to n^-}[\![x]\!] = \lim_{x\to n^-}(n-1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\displaystyle\lim_{x\to n^+}[\![x]\!] = \lim_{x\to n^+} n = n$.

(c) $\displaystyle\lim_{x\to a}[\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

51. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\displaystyle\lim_{x\to 2^-} f(x) = -1$ and $\displaystyle\lim_{x\to 2^+} f(x) = -1$, so $\displaystyle\lim_{x\to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\displaystyle\lim_{x\to 2} f(x) \neq f(2)$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\displaystyle\lim_{x\to a} p(x) = p(a)$.

55. $\lim_{x \to 1} [f(x) - 8] = \lim_{x \to 1} \left[\dfrac{f(x) - 8}{x - 1} \cdot (x - 1) \right] = \lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} \cdot \lim_{x \to 1} (x - 1) = 10 \cdot 0 = 0.$

Thus, $\lim_{x \to 1} f(x) = \lim_{x \to 1} \{[f(x) - 8] + 8\} = \lim_{x \to 1} [f(x) - 8] + \lim_{x \to 1} 8 = 0 + 8 = 8.$

Note: The value of $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1}$ exists.

57. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x \to 0} 0 = 0 = \lim_{x \to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x \to 0} f(x) = 0$.

59. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.57.

Thus, either f or g is 0 for any value of x. Then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but $\lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} 0 = 0$.

61. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim_{x \to -2} \left(3x^2 + ax + a + 3\right) = 0 \quad \Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad \Leftrightarrow \quad 12 - 2a + a + 3 = 0 \quad \Leftrightarrow \quad a = 15$. With $a = 15$, the limit becomes

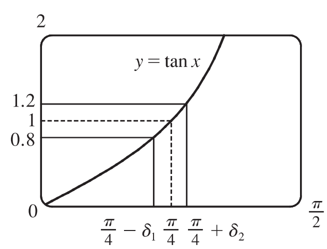
$$\lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \to -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \to -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

2.4 The Precise Definition of a Limit

1. On the left side of $x = 2$, we need $|x - 2| < \left|\frac{10}{7} - 2\right| = \frac{4}{7}$. On the right side, we need $|x - 2| < \left|\frac{10}{3} - 2\right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x - 2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

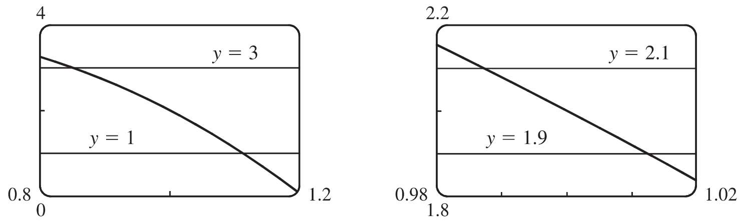
3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

5.

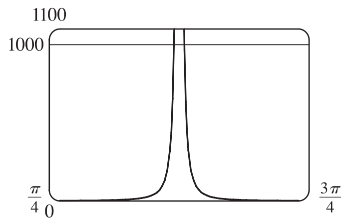


From the graph, we find that $\tan x = 0.8$ when $x \approx 0.675$, so $\frac{\pi}{4} - \delta_1 \approx 0.675 \quad \Rightarrow \quad \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106$. Also, $\tan x = 1.2$ when $x \approx 0.876$, so $\frac{\pi}{4} + \delta_2 \approx 0.876 \quad \Rightarrow \quad \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906$. Thus, we choose $\delta = 0.0906$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

7. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $\left|(4 + x - 3x^3) - 2\right| < 1 \quad \Leftrightarrow \quad 1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \le x \le 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left|(4 + x - 3x^3) - 2\right| < 0.1 \quad \Leftrightarrow \quad 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$. From the graph, we see that we need $0.988 \le x \le 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.

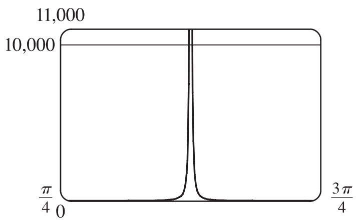


9. (a)



From the graph, we find that $y = \tan^2 x = 1000$ when $x \approx 1.539$ and $x \approx 1.602$ for x near $\frac{\pi}{2}$. Thus, we get $\delta \approx 1.602 - \frac{\pi}{2} \approx 0.031$ for $M = 1000$.

(b)



From the graph, we find that $y = \tan^2 x = 10{,}000$ when $x \approx 1.561$ and $x \approx 1.581$ for x near $\frac{\pi}{2}$. Thus, we get $\delta \approx 1.581 - \frac{\pi}{2} \approx 0.010$ for $M = 10{,}000$.

11. (a) $A = \pi r^2$ and $A = 1000 \text{ cm}^2 \quad \Rightarrow \quad \pi r^2 = 1000 \quad \Rightarrow \quad r^2 = \frac{1000}{\pi} \quad \Rightarrow \quad r = \sqrt{\frac{1000}{\pi}} \quad (r > 0) \quad \approx 17.8412 \text{ cm}$.

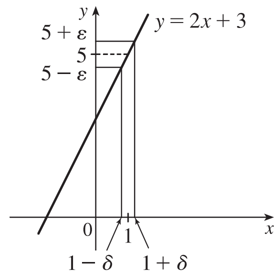
(b) $|A - 1000| \le 5 \quad \Rightarrow \quad -5 \le \pi r^2 - 1000 \le 5 \quad \Rightarrow \quad 1000 - 5 \le \pi r^2 \le 1000 + 5 \quad \Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \quad \Rightarrow \quad 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

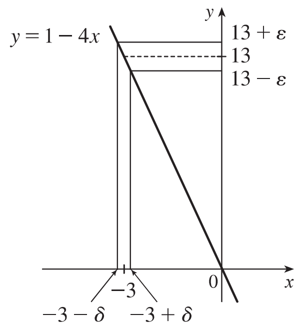
13. (a) $|4x - 8| = 4\,|x - 2| < 0.1 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

(b) $|4x - 8| = 4\,|x - 2| < 0.01 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x+3) - 5| < \varepsilon$. But $|(2x+3) - 5| < \varepsilon \iff |2x - 2| < \varepsilon \iff 2\,|x - 1| < \varepsilon \iff |x - 1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 1| < \delta \Rightarrow |(2x+3) - 5| < \varepsilon$. Thus, $\lim\limits_{x \to 1}(2x+3) = 5$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \iff |-4x - 12| < \varepsilon \iff |-4|\,|x + 3| < \varepsilon \iff |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim\limits_{x \to -3}(1 - 4x) = 13$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon \iff \frac{1}{5}\,|x - 3| < \varepsilon \iff |x - 3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x - 3| < \delta \Rightarrow |x - 3| < 5\varepsilon \Rightarrow \dfrac{|x - 3|}{5} < \varepsilon \Rightarrow \left|\dfrac{x}{5} - \dfrac{3}{5}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 3} \dfrac{x}{5} = \dfrac{3}{5}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\dfrac{x^2 + x - 6}{x - 2} - 5\right| < \varepsilon \iff \left|\dfrac{(x+3)(x-2)}{x - 2} - 5\right| < \varepsilon \iff |x + 3 - 5| < \varepsilon \quad [x \neq 2] \iff |x - 2| < \varepsilon$. So choose $\delta = \varepsilon$. Then $0 < |x - 2| < \delta \Rightarrow |x - 2| < \varepsilon \Rightarrow |x + 3 - 5| < \varepsilon \Rightarrow \left|\dfrac{(x+3)(x-2)}{x - 2} - 5\right| < \varepsilon \quad [x \neq 2] \Rightarrow \left|\dfrac{x^2 + x - 6}{x - 2} - 5\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2} = 5$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \iff x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \Rightarrow \left|x^2 - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 0} x^2 = 0$ by the definition of a limit.

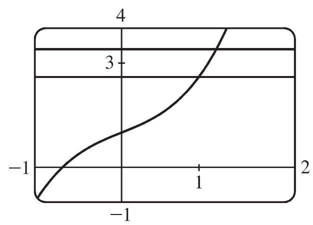
27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left||x| - 0\right| < \varepsilon$. But $\left||x|\right| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x \to 0} |x| = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|\left(x^2-4x+5\right)-1\right| < \varepsilon \Leftrightarrow \left|x^2-4x+4\right| < \varepsilon \Leftrightarrow$ $\left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \Leftrightarrow |x-2| < \sqrt{\varepsilon} \Leftrightarrow \left|(x-2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x\to 2}\left(x^2-4x+5\right) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left|\left(x^2-1\right)-3\right| < \varepsilon$ or upon simplifying we need $\left|x^2-4\right| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x+2 < 1 \Rightarrow -5 < x-2 < -3 \Rightarrow$ $|x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \Rightarrow |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|\left(x^2-1\right)-3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x\to -2}(x^2-1) = 3$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \Rightarrow -2 < x-3 < 2 \Rightarrow$ $4 < x+3 < 8 \Rightarrow |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2-9\right| = |x+3|\,|x-3| < 8\cdot\frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}.\text{ Thus, } \delta = x(\varepsilon) - 1.$$

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$$|\sqrt{x} - \sqrt{a}\,| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon.$$ Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

39. Suppose that $\lim_{x \to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim_{x \to 0} f(x)$ does not exist.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \quad \Leftrightarrow \quad (x+3)^4 < \dfrac{1}{10{,}000} \quad \Leftrightarrow \quad |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \quad \Leftrightarrow \quad |x - (-3)| < \dfrac{1}{10}$

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim_{x \to 0^+} \ln x = -\infty$.

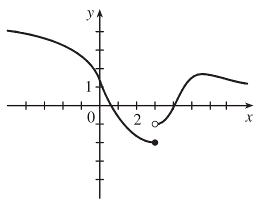
2.5 Continuity

1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

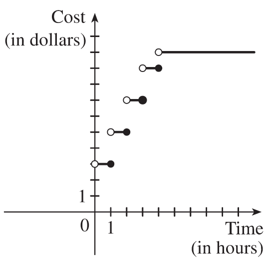
3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -4 (removable), -2 (jump), 2 (jump), 4 (infinite).

(b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

5. The graph of $y = f(x)$ must have a discontinuity at $x = 3$ and must show that $\lim_{x \to 3^-} f(x) = f(3)$.



7. (a)



(b) There are discontinuities at times $t = 1$, 2, 3, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

9. Since f and g are continuous functions,

$$\begin{aligned}
\lim_{x \to 3} [2f(x) - g(x)] &= 2 \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x) && \text{[by Limit Laws 2 and 3]} \\
&= 2f(3) - g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x = 3\text{]} \\
&= 2 \cdot 5 - g(3) = 10 - g(3)
\end{aligned}$$

Since it is given that $\lim_{x \to 3} [2f(x) - g(x)] = 4$, we have $10 - g(3) = 4$, so $g(3) = 6$.

11. $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \left(x + 2x^3\right)^4 = \left(\lim\limits_{x\to -1} x + 2\lim\limits_{x\to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1).$

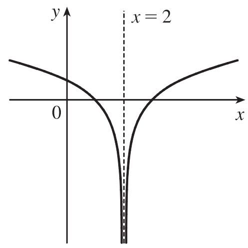
By the definition of continuity, f is continuous at $a = -1$.

13. For $a > 2$, we have

$$\begin{aligned}
\lim_{x\to a} f(x) = \lim_{x\to a} \frac{2x+3}{x-2} &= \frac{\lim\limits_{x\to a}(2x+3)}{\lim\limits_{x\to a}(x-2)} && \text{[Limit Law 5]}\\
&= \frac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2} && \text{[1, 2, and 3]}\\
&= \frac{2a+3}{a-2} && \text{[7 and 8]}\\
&= f(a)
\end{aligned}$$

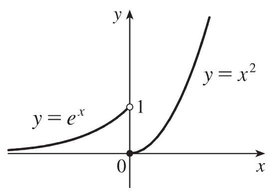
Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

15. $f(x) = \ln|x-2|$ is discontinuous at 2 since $f(2) = \ln 0$ is not defined.

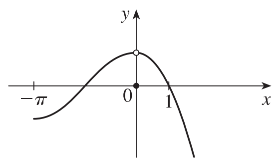


17. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

The left-hand limit of f at $a = 0$ is $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} e^x = 1$. The right-hand limit of f at $a = 0$ is $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0$. Since these limits are not equal, $\lim\limits_{x\to 0} f(x)$ does not exist and f is discontinuous at 0.



19. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim\limits_{x\to 0} f(x) = 1$, but $f(0) = 0 \neq 1$, so f is discontinuous at 0.

21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\left\{x \mid x^2 + 5x + 6 \neq 0\right\} = \{x \mid (x+3)(x+2) \neq 0\} = \{x \mid x \neq -3,\ -2\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

23. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2}, \infty)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x-1}$ is continuous on $[\frac{1}{2}, \infty)$.

25. By Theorem 7, the exponential function e^{-5t} and the trigonometric function $\cos 2\pi t$ are continuous on $(-\infty, \infty)$.

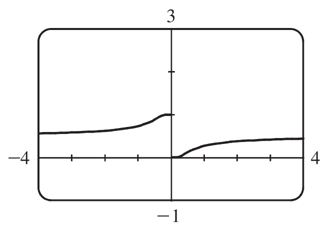
By part 4 of Theorem 4, $L(t) = e^{-5t} \cos 2\pi t$ is continuous on $(-\infty, \infty)$.

27. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$.

By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is

$$\{t \mid t^4 - 1 > 0\} = \{t \mid t^4 > 1\} = \{t \mid |t| > 1\} = (-\infty, -1) \cup (1, \infty)$$

29. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



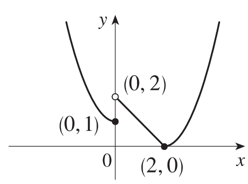
31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x + 5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim\limits_{x \to 4} f(x) = f(4) = \frac{7}{3}$.

33. Because $x^2 - x$ is continuous on $\mathbb{R}$, the composite function $f(x) = e^{x^2 - x}$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to 1} f(x) = f(1) = e^{1-1} = e^0 = 1$.

35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

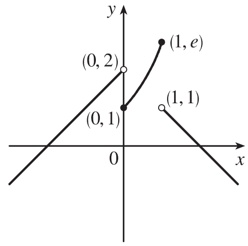
By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x \to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$



f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (1 + x^2) = 1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (2 - x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2 - x) = 0$, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x - 2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

39. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x+2) = 2$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} e^x = e$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2-x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

41. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(x^3 - cx\right) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

43. (a) $f(x) = \dfrac{x^4 - 1}{x - 1} = \dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)(x-1)}{x-1} = (x^2+1)(x+1)$ [or $x^3 + x^2 + x + 1$]

for $x \ne 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \ne 1$ and is continuous on $\mathbb{R}$.

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2} = \dfrac{x(x^2 - x - 2)}{x - 2} = \dfrac{x(x-2)(x+1)}{x-2} = x(x+1)$ [or $x^2 + x$] for $x \ne 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \ne 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} [\![\sin x]\!] = \lim\limits_{x \to \pi^-} 0 = 0$ and $\lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} [\![\sin x]\!] = \lim\limits_{x \to \pi^+} (-1) = -1$, so $\lim\limits_{x \to \pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

45. $f(x) = x^2 + 10 \sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

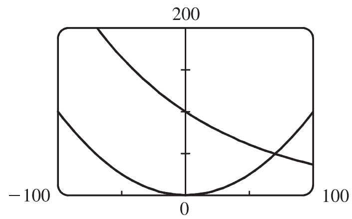
47. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

49. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

51. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

53. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.

55. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\left(\lim_{h\to 0}(a+h)\right) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim\limits_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \;\Rightarrow\; |f(a+h) - f(a)| < \varepsilon$. So if $0 < |x-a| < \delta$, then $|f(x) - f(a)| = |f(a + (x-a)) - f(a)| < \varepsilon$. Thus, $\lim\limits_{x\to a} f(x) = f(a)$ and so f is continuous at a.

57. As in the previous exercise, we must show that $\lim\limits_{h\to 0} \cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h - \sin a\sin h) = \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)\\ &= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a\end{aligned}$$

59. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim\limits_{x\to a} f(x) \neq f(a)$. [In fact, $\lim\limits_{x\to a} f(x)$ does not even exist.]

61. If there is such a number, it satisfies the equation $x^3 + 1 = x \;\Leftrightarrow\; x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

63. $f(x) = x^4\sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \le \sin(1/x) \le 1$, we have $-x^4 \le x^4\sin(1/x) \le x^4$. Because

$\lim\limits_{x\to 0}(-x^4) = 0$ and $\lim\limits_{x\to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim\limits_{x\to 0}(x^4\sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

65. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u-d)(0) = -D$ and $(u-d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u-d)(t_0) = 0 \quad\Leftrightarrow\quad u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

1. (a) As x becomes large, the values of $f(x)$ approach 5.

(b) As x becomes large negative, the values of $f(x)$ approach 3.

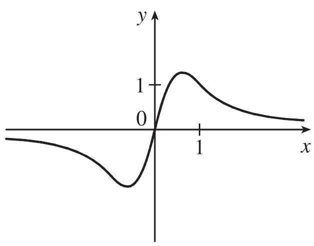
3. (a) $\lim\limits_{x\to 2} f(x) = \infty$ (b) $\lim\limits_{x\to -1^-} f(x) = \infty$ (c) $\lim\limits_{x\to -1^+} f(x) = -\infty$

(d) $\lim\limits_{x\to \infty} f(x) = 1$ (e) $\lim\limits_{x\to -\infty} f(x) = 2$ (f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

5. $f(0) = 0$, $f(1) = 1$,

$\lim\limits_{x\to\infty} f(x) = 0$,

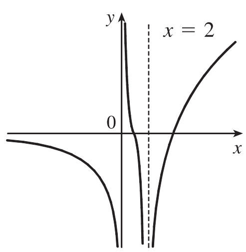
f is odd



7. $\lim\limits_{x\to 2} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) = \infty$,

$\lim\limits_{x\to -\infty} f(x) = 0$, $\lim\limits_{x\to 0^+} f(x) = \infty$,

$\lim\limits_{x\to 0^-} f(x) = -\infty$

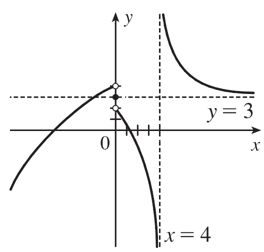


9. $f(0) = 3$, $\lim\limits_{x\to 0^-} f(x) = 4$,

$\lim\limits_{x\to 0^+} f(x) = 2$,

$\lim\limits_{x\to -\infty} f(x) = -\infty$, $\lim\limits_{x\to 4^-} f(x) = -\infty$,

$\lim\limits_{x\to 4^+} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = 3$



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

13. $\displaystyle\lim_{x\to\infty}\frac{3x^2-x+4}{2x^2+5x-8}=\lim_{x\to\infty}\frac{(3x^2-x+4)/x^2}{(2x^2+5x-8)/x^2}$ [divide both the numerator and denominator by x^2 (the highest power of x thatappears in the denominator)]

$\displaystyle=\frac{\lim_{x\to\infty}(3-1/x+4/x^2)}{\lim_{x\to\infty}(2+5/x-8/x^2)}$ [Limit Law 5]

$\displaystyle=\frac{\lim_{x\to\infty}3-\lim_{x\to\infty}(1/x)+\lim_{x\to\infty}(4/x^2)}{\lim_{x\to\infty}2+\lim_{x\to\infty}(5/x)-\lim_{x\to\infty}(8/x^2)}$ [Limit Laws 1 and 2]

$\displaystyle=\frac{3-\lim_{x\to\infty}(1/x)+4\lim_{x\to\infty}(1/x^2)}{2+5\lim_{x\to\infty}(1/x)-8\lim_{x\to\infty}(1/x^2)}$ [Limit Laws 7 and 3]

$\displaystyle=\frac{3-0+4(0)}{2+5(0)-8(0)}$ [Theorem 5 of Section 2.5]

$\displaystyle=\frac{3}{2}$

15. $\displaystyle\lim_{x\to\infty}\frac{1}{2x+3}=\lim_{x\to\infty}\frac{1/x}{(2x+3)/x}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}(2+3/x)}=\frac{\lim_{x\to\infty}(1/x)}{\lim_{x\to\infty}2+3\lim_{x\to\infty}(1/x)}=\frac{0}{2+3(0)}=\frac{0}{2}=0$

17. $\displaystyle\lim_{x\to-\infty}\frac{1-x-x^2}{2x^2-7}=\lim_{x\to-\infty}\frac{(1-x-x^2)/x^2}{(2x^2-7)/x^2}=\frac{\lim_{x\to-\infty}(1/x^2-1/x-1)}{\lim_{x\to-\infty}(2-7/x^2)}$

$$=\frac{\lim_{x\to-\infty}(1/x^2)-\lim_{x\to-\infty}(1/x)-\lim_{x\to-\infty}1}{\lim_{x\to-\infty}2-7\lim_{x\to-\infty}(1/x^2)}=\frac{0-0-1}{2-7(0)}=-\frac{1}{2}$$

19. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x\to\infty}\frac{x^3+5x}{2x^3-x^2+4}=\lim_{x\to\infty}\frac{\dfrac{x^3+5x}{x^3}}{\dfrac{2x^3-x^2+4}{x^3}}=\lim_{x\to\infty}\frac{1+\dfrac{5}{x^2}}{2-\dfrac{1}{x}+\dfrac{4}{x^3}}=\frac{\lim_{x\to\infty}\left(1+\dfrac{5}{x^2}\right)}{\lim_{x\to\infty}\left(2-\dfrac{1}{x}+\dfrac{4}{x^3}\right)}$$

$$=\frac{\lim_{x\to\infty}1+5\lim_{x\to\infty}\dfrac{1}{x^2}}{\lim_{x\to\infty}2-\lim_{x\to\infty}\dfrac{1}{x}+4\lim_{x\to\infty}\dfrac{1}{x^3}}=\frac{1+5(0)}{2-0+4(0)}=\frac{1}{2}$$

21. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$$\lim_{u\to\infty}\frac{4u^4+5}{(u^2-2)(2u^2-1)}=\lim_{u\to\infty}\frac{4u^4+5}{2u^4-5u^2+2}=\lim_{u\to\infty}\frac{\dfrac{4u^4+5}{u^4}}{\dfrac{2u^4-5u^2+2}{u^4}}=\lim_{u\to\infty}\frac{4+\dfrac{5}{u^4}}{2-\dfrac{5}{u^2}+\dfrac{2}{u^4}}$$

$$=\frac{\lim_{u\to\infty}\left(4+\dfrac{5}{u^4}\right)}{\lim_{u\to\infty}\left(2-\dfrac{5}{u^2}+\dfrac{2}{u^4}\right)}=\frac{\lim_{u\to\infty}4+5\lim_{u\to\infty}\dfrac{1}{u^4}}{\lim_{u\to\infty}2-5\lim_{u\to\infty}\dfrac{1}{u^2}+2\lim_{u\to\infty}\dfrac{1}{u^4}}=\frac{4+5(0)}{2-5(0)+2(0)}=\frac{4}{2}=2$$

23. $\lim\limits_{x\to\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1} = \lim\limits_{x\to\infty} \dfrac{\sqrt{9x^6 - x}\,/x^3}{(x^3 + 1)/x^3} = \dfrac{\lim\limits_{x\to\infty} \sqrt{(9x^6 - x)/x^6}}{\lim\limits_{x\to\infty} (1 + 1/x^3)}$ $\quad$ [since $x^3 = \sqrt{x^6}$ for $x > 0$]

$$= \frac{\lim\limits_{x\to\infty} \sqrt{9 - 1/x^5}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} (1/x^3)} = \frac{\sqrt{\lim\limits_{x\to\infty} 9 - \lim\limits_{x\to\infty} (1/x^5)}}{1 + 0}$$

$$= \sqrt{9 - 0} = 3$$

25. $\lim\limits_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{9x^2 + x} - 3x\right)\left(\sqrt{9x^2 + x} + 3x\right)}{\sqrt{9x^2 + x} + 3x} = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{9x^2 + x}\right)^2 - (3x)^2}{\sqrt{9x^2 + x} + 3x}$

$$= \lim_{x\to\infty} \frac{(9x^2 + x) - 9x^2}{\sqrt{9x^2 + x} + 3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2 + x} + 3x} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{x/x}{\sqrt{9x^2/x^2 + x/x^2} + 3x/x} = \lim_{x\to\infty} \frac{1}{\sqrt{9 + 1/x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

27. $\lim\limits_{x\to\infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)\left(\sqrt{x^2 + ax} + \sqrt{x^2 + bx}\right)}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}}$

$$= \lim_{x\to\infty} \frac{(x^2 + ax) - (x^2 + bx)}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x\to\infty} \frac{[(a - b)x]/x}{\left(\sqrt{x^2 + ax} + \sqrt{x^2 + bx}\right)/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{a - b}{\sqrt{1 + a/x} + \sqrt{1 + b/x}} = \frac{a - b}{\sqrt{1 + 0} + \sqrt{1 + 0}} = \frac{a - b}{2}$$

29. $\lim\limits_{x\to\infty} \dfrac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim\limits_{x\to\infty} \dfrac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4}$ $\quad$ [divide by the highest power of x in the denominator]

$$= \lim_{x\to\infty} \frac{1/x^3 + 1/x + x}{1/x^4 - 1/x^2 + 1} = \infty$$

because $(1/x^3 + 1/x + x) \to \infty$ and $(1/x^4 - 1/x^2 + 1) \to 1$ as $x \to \infty$.

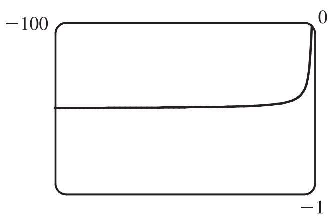
31. $\lim\limits_{x\to-\infty} (x^4 + x^5) = \lim\limits_{x\to-\infty} x^5(\frac{1}{x} + 1)$ $\quad$ [factor out the largest power of x] $\quad = -\infty$ $\quad$ because $x^5 \to -\infty$ and $1/x + 1 \to 1$ as $x \to -\infty$.

Or: $\lim\limits_{x\to-\infty} \left(x^4 + x^5\right) = \lim\limits_{x\to-\infty} x^4\,(1 + x) = -\infty$.

33. $\lim\limits_{x\to\infty} \dfrac{1 - e^x}{1 + 2e^x} = \lim\limits_{x\to\infty} \dfrac{(1 - e^x)/e^x}{(1 + 2e^x)/e^x} = \lim\limits_{x\to\infty} \dfrac{1/e^x - 1}{1/e^x + 2} = \dfrac{0 - 1}{0 + 2} = -\dfrac{1}{2}$

35. Since $-1 \le \cos x \le 1$ and $e^{-2x} > 0$, we have $-e^{-2x} \le e^{-2x} \cos x \le e^{-2x}$. We know that $\lim\limits_{x\to\infty} (-e^{-2x}) = 0$ and $\lim\limits_{x\to\infty} \left(e^{-2x}\right) = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty} (e^{-2x} \cos x) = 0$.

37. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim\limits_{x\to-\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	-0.4999625
$-100{,}000$	-0.4999962
$-1{,}000{,}000$	-0.4999996

From the table, we estimate the limit to be -0.5.

(c) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\dfrac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim\limits_{x\to-\infty}\dfrac{\left(x^2+x+1\right)-x^2}{\sqrt{x^2+x+1}-x}$

$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}$$

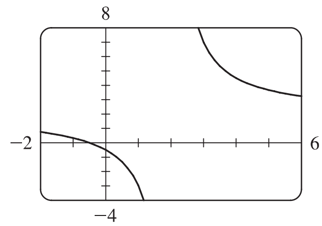
$$= \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x<0$, we have $\sqrt{x^2}=|x|=-x$, so when we divide the radical by x, with $x<0$, we get $\dfrac{1}{x}\sqrt{x^2+x+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+(1/x^2)}$.

39. $\lim\limits_{x\to\infty}\dfrac{2x+1}{x-2} = \lim\limits_{x\to\infty}\dfrac{\frac{2x+1}{x}}{\frac{x-2}{x}} = \lim\limits_{x\to\infty}\dfrac{2+\frac{1}{x}}{1-\frac{2}{x}} = \dfrac{\lim\limits_{x\to\infty}\left(2+\frac{1}{x}\right)}{\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)} = \dfrac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{2}{x}}$

$$= \frac{2+0}{1-0} = 2, \quad \text{so } y=2 \text{ is a horizontal asymptote.}$$

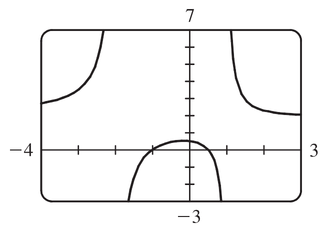
The denominator $x-2$ is zero when $x=2$ and the numerator is not zero, so we investigate $y=f(x)=\dfrac{2x+1}{x-2}$ as x approaches 2. $\lim\limits_{x\to2^-}f(x)=-\infty$ because as $x\to2^-$ the numerator is positive and the denominator approaches 0 through negative values. Similarly, $\lim\limits_{x\to2^+}f(x)=\infty$. Thus, $x=2$ is a vertical asymptote. The graph confirms our work.



41. $\lim\limits_{x\to\infty}\dfrac{2x^2+x-1}{x^2+x-2} = \lim\limits_{x\to\infty}\dfrac{\frac{2x^2+x-1}{x^2}}{\frac{x^2+x-2}{x^2}} = \lim\limits_{x\to\infty}\dfrac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}} = \dfrac{\lim\limits_{x\to\infty}\left(2+\frac{1}{x}-\frac{1}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}-\frac{2}{x^2}\right)}$

$$= \frac{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x}-2\lim\limits_{x\to\infty}\frac{1}{x^2}} = \frac{2+0-0}{1+0-2(0)} = 2, \quad \text{so } y=2 \text{ is a horizontal asymptote.}$$

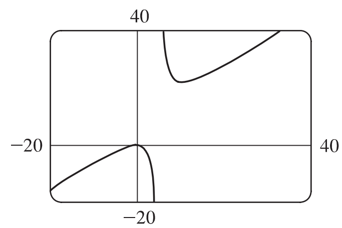
$y=f(x)=\dfrac{2x^2+x-1}{x^2+x-2}=\dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\lim\limits_{x\to-2^-}f(x)=\infty$, $\lim\limits_{x\to-2^+}f(x)=-\infty$, $\lim\limits_{x\to1^-}f(x)=-\infty$, and $\lim\limits_{x\to1^+}f(x)=\infty$. Thus, $x=-2$ and $x=1$ are vertical asymptotes. The graph confirms our work.



43. $y=f(x)=\dfrac{x^3-x}{x^2-6x+5}=\dfrac{x(x^2-1)}{(x-1)(x-5)}=\dfrac{x(x+1)(x-1)}{(x-1)(x-5)}=\dfrac{x(x+1)}{x-5}=g(x)$ for $x\neq1$.

The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x=1$. By long division, $g(x)=\dfrac{x^2+x}{x-5}=x+6+\dfrac{30}{x-5}$.

As $x\to\pm\infty$, $g(x)\to\pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x=5$. $\lim\limits_{x\to5^-}g(x)=-\infty$ and $\lim\limits_{x\to5^+}g(x)=\infty$, so $x=5$ is a vertical asymptote. The graph confirms our work.

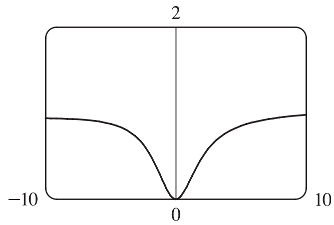


45. From the graph, it appears $y = 1$ is a horizontal asymptote.

$$\lim_{x\to\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x\to\infty} \frac{\dfrac{3x^3 + 500x^2}{x^3}}{\dfrac{x^3 + 500x^2 + 100x + 2000}{x^3}} = \lim_{x\to\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)}$$

$$= \frac{3 + 0}{1 + 0 + 0 + 0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000,\ 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.



47. Let's look for a rational function.

(1) $\lim_{x\to\pm\infty} f(x) = 0 \quad \Rightarrow \quad$ degree of numerator $<$ degree of denominator

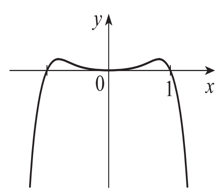
(2) $\lim_{x\to 0} f(x) = -\infty \quad \Rightarrow \quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim_{x\to 3^-} f(x) = \infty$ and $\lim_{x\to 3^+} f(x) = -\infty \quad \Rightarrow \quad$ vertical asymptote at $x = 3$; there is a factor of $(x - 3)$ in the denominator.

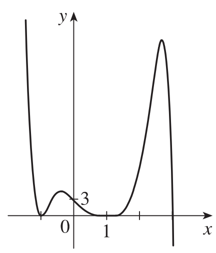
(4) $f(2) = 0 \quad \Rightarrow \quad$ 2 is an x-intercept; there is at least one factor of $(x - 2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2 - x}{x^2(x - 3)}$ as one possibility.

49. $y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1 + x)(1 - x)$. The y-intercept is $f(0) = 0$. The x-intercepts are $0, -1$, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1 - x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1 - x^2) \to -\infty$.

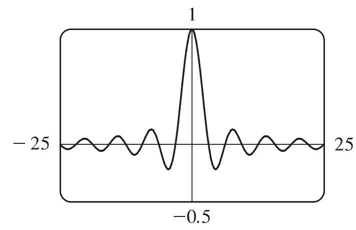


51. $y = f(x) = (3 - x)(1 + x)^2(1 - x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are $3, -1$, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3 - x$ is negative and the other factors are positive, so $\lim_{x\to\infty} f(x) = -\infty$. When x is large negative, $3 - x$ is positive, so $\lim_{x\to-\infty} f(x) = \infty$.



53. (a) Since $-1 \le \sin x \le 1$ for all x, $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times*.



55. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

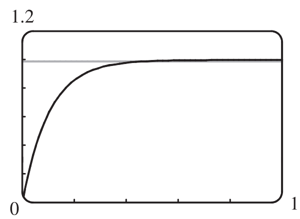
57. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10 - (21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x - 21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

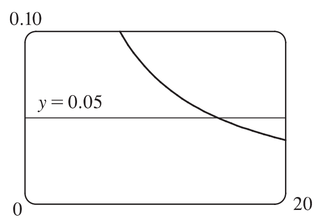
we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

59. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

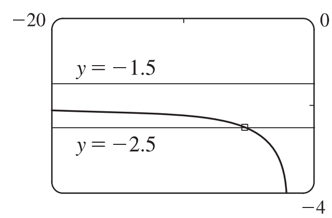


61. Let $g(x) = \dfrac{3x^2+1}{2x^2+x+1}$ and $f(x) = |g(x) - 1.5|$. Note that $\lim\limits_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim\limits_{x\to\infty} f(x) = 0$. We are interested in finding the x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$, so we choose $N = 15$ (or any larger number).

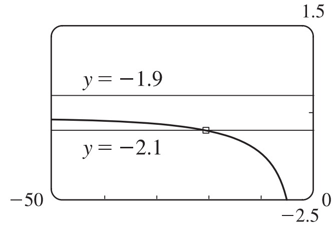


63. For $\varepsilon = 0.5$, we need to find N such that $\left|\dfrac{\sqrt{4x^2+1}}{x+1} - (-2)\right| < 0.5 \quad\Leftrightarrow$

$-2.5 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \le N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \le -6$. So we choose $N = -6$ (or any smaller number).



For $\varepsilon = 0.1$, we need $-2.1 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \le N$. From the graph, it seems that this inequality holds for $x \le -22$. So we choose $N = -22$ (or any smaller number).



65. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10\,000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\displaystyle\lim_{x\to\infty} \frac{1}{x^2} = 0$.

67. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\displaystyle\lim_{x\to-\infty} (1/x) = 0$.

69. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad e^x > M$. Now $e^x > M \quad \Leftrightarrow \quad x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \quad \Rightarrow \quad e^x > \max(e, M) \geq M$, so $\displaystyle\lim_{x\to\infty} e^x = \infty$.

71. Suppose that $\displaystyle\lim_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \quad \Leftrightarrow \quad 0 < 1/x < 1/N \quad \Leftrightarrow \quad 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\displaystyle\lim_{t\to0^+} f(1/t) = L = \lim_{x\to\infty} f(x)$.

Now suppose that $\displaystyle\lim_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \quad \Leftrightarrow \quad 1/N < 1/x < 0 \quad \Leftrightarrow \quad 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\displaystyle\lim_{t\to0^-} f(1/t) = L = \lim_{x\to-\infty} f(x)$.

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \displaystyle\lim_{x\to3} \frac{f(x) - f(3)}{x - 3}$.

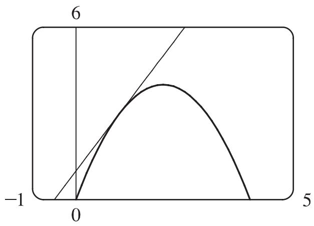
3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$\begin{aligned} m &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x\to1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x\to1} \frac{-(x - 1)(x - 3)}{x - 1} \\ &= \lim_{x\to1} (3 - x) = 3 - 1 = 2 \end{aligned}$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$\begin{aligned} m &= \lim_{h\to0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to0} \frac{f(1 + h) - f(1)}{h} = \lim_{h\to0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h} \\ &= \lim_{h\to0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h\to0} \frac{-h^2 + 2h}{h} = \lim_{h\to0} \frac{h(-h + 2)}{h} = \lim_{h\to0} (-h + 2) = 2 \end{aligned}$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \quad \Rightarrow \quad y - f(1) = f'(1)(x - 1) \quad \Rightarrow \quad y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c) 

The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

5. Using (1) with $f(x) = \dfrac{x-1}{x-2}$ and $P(3, 2)$,

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 3} \frac{\dfrac{x-1}{x-2} - 2}{x - 3} = \lim_{x\to 3} \frac{\dfrac{x - 1 - 2(x-2)}{x-2}}{x-3}$$
$$= \lim_{x\to 3} \frac{3 - x}{(x-2)(x-3)} = \lim_{x\to 3} \frac{-1}{x-2} = \frac{-1}{1} = -1$$

Tangent line: $y - 2 = -1(x - 3) \quad\Leftrightarrow\quad y - 2 = -x + 3 \quad\Leftrightarrow\quad y = -x + 5$

7. Using (1), $m = \lim\limits_{x\to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim\limits_{x\to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim\limits_{x\to 1} \dfrac{x - 1}{(x-1)(\sqrt{x} + 1)} = \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}$.

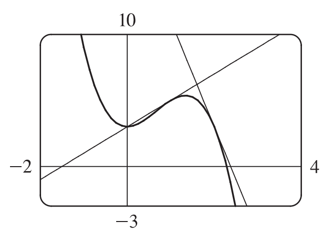
Tangent line: $y - 1 = \frac{1}{2}(x - 1) \quad\Leftrightarrow\quad y = \frac{1}{2}x + \frac{1}{2}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - (3 + 4a^2 - 2a^3)}{h}$$
$$= \lim_{h\to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h\to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$
$$= \lim_{h\to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

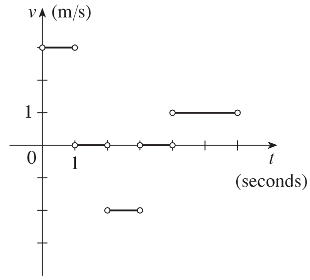
(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \quad\Leftrightarrow\quad y = 2x + 3$.

At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \quad\Leftrightarrow\quad y = -8x + 19$.

(c) 

11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0, 1)$ and $(4, 6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2, 3)$. The particle is standing still when s is constant; that is, on the intervals $(1, 2)$ and $(3, 4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\dfrac{3-0}{1-0} = 3$. On the interval $(2, 3)$, the slope is $\dfrac{1-3}{3-2} = -2$. On the interval $(4, 6)$, the slope is $\dfrac{3-1}{6-4} = 1$.



13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t)-s(2)}{t-2} = \lim_{t\to 2} \frac{\left(40t-16t^2\right)-16}{t-2} = \lim_{t\to 2} \frac{-16t^2+40t-16}{t-2} = \lim_{t\to 2} \frac{-8\left(2t^2-5t+2\right)}{t-2}$$

$$= \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8 \lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

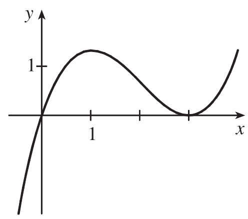
15. $$v(a) = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2-\left(a^2+2ah+h^2\right)}{ha^2(a+h)^2}$$

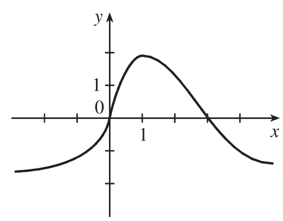
$$= \lim_{h\to 0} \frac{-\left(2ah+h^2\right)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s}$$

So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

19. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.





21. Using Definition 2 with $f(x) = 3x^2 - 5x$ and the point $(2, 2)$, we have

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left[3(2+h)^2-5(2+h)\right]-2}{h} = \lim_{h\to 0} \frac{(12+12h+3h^2-10-5h)-2}{h}$$

$$= \lim_{h\to 0} \frac{3h^2+7h}{h} = \lim_{h\to 0}(3h+7) = 7$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

23. (a) Using Definition 2 with $F(x) = 5x/(1+x^2)$ and the point $(2,2)$, we have

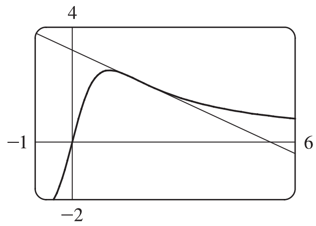
$$F'(2) = \lim_{h\to 0} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2+h)}{1+(2+h)^2} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{5h+10}{h^2+4h+5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h+10-2(h^2+4h+5)}{h^2+4h+5}}{h}$$

$$= \lim_{h\to 0} \frac{-2h^2-3h}{h(h^2+4h+5)} = \lim_{h\to 0} \frac{h(-2h-3)}{h(h^2+4h+5)} = \lim_{h\to 0} \frac{-2h-3}{h^2+4h+5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2,2)$ is $y - 2 = -\frac{3}{5}(x-2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)



25. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[3-2(a+h)+4(a+h)^2]-(3-2a+4a^2)}{h}$$

$$= \lim_{h\to 0} \frac{(3-2a-2h+4a^2+8ah+4h^2)-(3-2a+4a^2)}{h}$$

$$= \lim_{h\to 0} \frac{-2h+8ah+4h^2}{h} = \lim_{h\to 0} \frac{h(-2+8a+4h)}{h} = \lim_{h\to 0}(-2+8a+4h) = -2+8a$$

27. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} = \lim_{h\to 0} \frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{(2a^2+6a+2ah+6h+a+3)-(2a^2+2ah+6a+a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

29. Use Definition 2 with $f(x) = 1/\sqrt{x+2}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(a+h)+2}} - \dfrac{1}{\sqrt{a+2}}}{h} = \lim_{h\to 0} \frac{\dfrac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\,\sqrt{a+2}}}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sqrt{a+2}-\sqrt{a+h+2}}{h\,\sqrt{a+h+2}\,\sqrt{a+2}} \cdot \frac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right] = \lim_{h\to 0} \frac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)} = \lim_{h\to 0} \frac{-1}{\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \frac{-1}{\left(\sqrt{a+2}\right)^2\left(2\sqrt{a+2}\right)} = -\frac{1}{2(a+2)^{3/2}}$$

Note that the answers to Exercises 31–36 are not unique.

31. By Definition 2, $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10}-1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By Definition 2, $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10}-1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

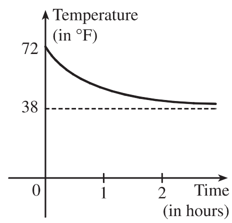
33. By Equation 3, $\lim\limits_{x\to 5} \dfrac{2^x - 32}{x-5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

35. By Definition 2, $\lim\limits_{h\to 0} \dfrac{\cos(\pi + h) + 1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim\limits_{h\to 0} \dfrac{\cos(\pi + h) + 1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

37. $$v(5) = f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{[100 + 50(5+h) - 4.9(5+h)^2] - [100 + 50(5) - 4.9(5)^2]}{h}$$
$$= \lim_{h\to 0} \frac{(100 + 250 + 50h - 4.9h^2 - 49h - 122.5) - (100 + 250 - 122.5)}{h} = \lim_{h\to 0} \frac{-4.9h^2 + h}{h}$$
$$= \lim_{h\to 0} \frac{h(-4.9h + 1)}{h} = \lim_{h\to 0} (-4.9h + 1) = 1 \text{ m/s}$$
The speed when $t = 5$ is $|1| = 1$ m/s.

39. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



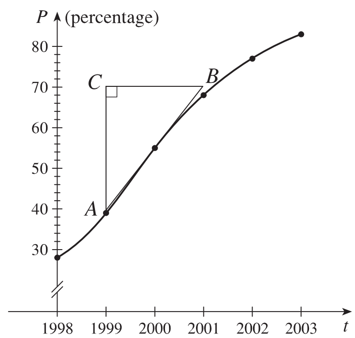
41. (a) (i) $[2000, 2002]$: $\dfrac{P(2002) - P(2000)}{2002 - 2000} = \dfrac{77 - 55}{2} = \dfrac{22}{2} = 11$ percent/year

(ii) $[2000, 2001]$: $\dfrac{P(2001) - P(2000)}{2001 - 2000} = \dfrac{68 - 55}{1} = 13$ percent/year

(iii) $[1999, 2000]$: $\dfrac{P(2000) - P(1999)}{2000 - 1999} = \dfrac{55 - 39}{1} = 16$ percent/year

(b) Using the values from (ii) and (iii), we have $\dfrac{13 + 16}{2} = 14.5$ percent/year.

(c) Estimating A as $(1999, 40)$ and B as $(2001, 70)$, the slope at 2000 is $\dfrac{70 - 40}{2001 - 1999} = \dfrac{30}{2} = 15$ percent/year.



43. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}$.

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}$.

(b) $$\frac{C(100+h) - C(100)}{h} = \frac{\left[5000 + 10(100+h) + 0.05(100+h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$
$$= 20 + 0.05h, \; h \neq 0$$

So the instantaneous rate of change is $\lim\limits_{h\to 0} \dfrac{C(100+h) - C(100)}{h} = \lim\limits_{h\to 0} (20 + 0.05h) = \$20/\text{unit}$.

45. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

47. $T'(10)$ is the rate at which the temperature is changing at 10:00 AM. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let $A = \dfrac{T(8) - T(10)}{8 - 10} = \dfrac{72 - 81}{-2} = 4.5$ and $B = \dfrac{T(12) - T(10)}{12 - 10} = \dfrac{88 - 81}{2} = 3.5$. Then $T'(10) = \lim\limits_{t \to 10} \dfrac{T(t) - T(10)}{t - 10} \approx \dfrac{A + B}{2} = \dfrac{4.5 + 3.5}{2} = 4^\circ\text{F/h}$.

49. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of $0.25\ (\text{mg/L})/^\circ\text{C}$.

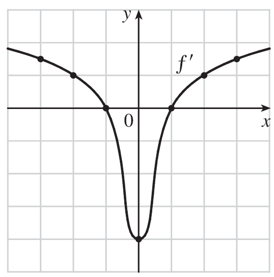
51. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$
(c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$
(e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$
(g) $f'(3) \approx 1.5$



3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

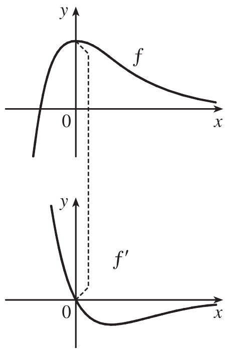
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

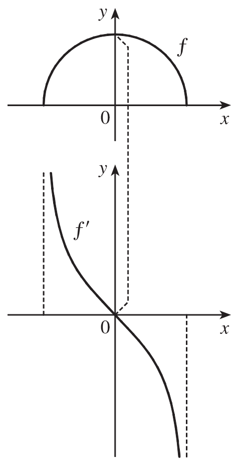
(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

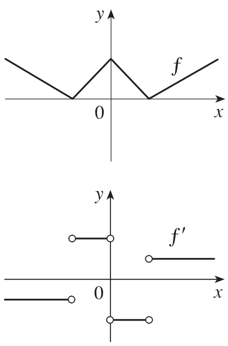
5.



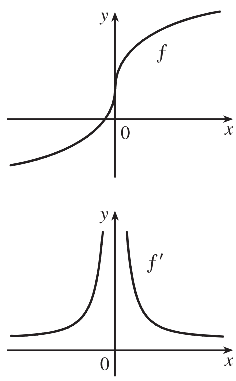
7.



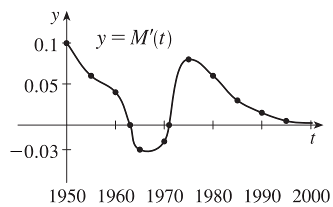
9.



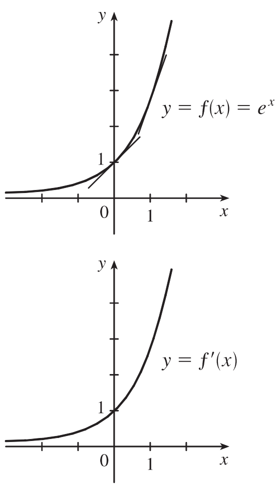
11.



13. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.

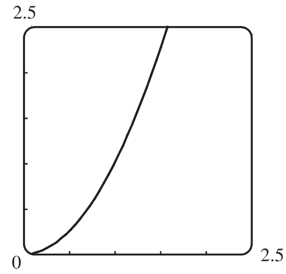


15.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

(d) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0}\frac{\left(x^2+2hx+h^2\right)-x^2}{h} = \lim_{h\to 0}\frac{2hx+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

19. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[\frac{1}{2}(x+h)-\frac{1}{3}\right]-\left(\frac{1}{2}x-\frac{1}{3}\right)}{h} = \lim_{h\to 0}\frac{\frac{1}{2}x+\frac{1}{2}h-\frac{1}{3}-\frac{1}{2}x+\frac{1}{3}}{h}$$
$$= \lim_{h\to 0}\frac{\frac{1}{2}h}{h} = \lim_{h\to 0}\frac{1}{2} = \frac{1}{2}$$

Domain of f = domain of $f' = \mathbb{R}$.

21. $$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{\left[5(t+h)-9(t+h)^2\right]-(5t-9t^2)}{h}$$
$$= \lim_{h\to 0}\frac{5t+5h-9(t^2+2th+h^2)-5t+9t^2}{h} = \lim_{h\to 0}\frac{5t+5h-9t^2-18th-9h^2-5t+9t^2}{h}$$
$$= \lim_{h\to 0}\frac{5h-18th-9h^2}{h} = \lim_{h\to 0}\frac{h(5-18t-9h)}{h} = \lim_{h\to 0}(5-18t-9h) = 5-18t$$

Domain of f = domain of $f' = \mathbb{R}$.

23. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[(x+h)^3-3(x+h)+5\right]-(x^3-3x+5)}{h}$$
$$= \lim_{h\to 0}\frac{\left(x^3+3x^2h+3xh^2+h^3-3x-3h+5\right)-\left(x^3-3x+5\right)}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$$
$$= \lim_{h\to 0}\frac{h\left(3x^2+3xh+h^2-3\right)}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2-3\right) = 3x^2-3$$

Domain of f = domain of $f' = \mathbb{R}$.

25. $$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\left[\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right]$$
$$= \lim_{h\to 0}\frac{(1+2x+2h)-(1+2x)}{h\left[\sqrt{1+2(x+h)}+\sqrt{1+2x}\right]} = \lim_{h\to 0}\frac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}} = \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$$

Domain of $g = \left[-\frac{1}{2},\infty\right)$, domain of $g' = \left(-\frac{1}{2},\infty\right)$.

27. $G'(t) = \lim\limits_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4(t+h)}{(t+h)+1} - \dfrac{4t}{t+1}}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4(t+h)(t+1) - 4t(t+h+1)}{(t+h+1)(t+1)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(4t^2+4ht+4t+4h) - (4t^2+4ht+4t)}{h(t+h+1)(t+1)} = \lim\limits_{h\to 0} \dfrac{4h}{h(t+h+1)(t+1)}$

$= \lim\limits_{h\to 0} \dfrac{4}{(t+h+1)(t+1)} = \dfrac{4}{(t+1)^2}$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

29. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^4 - x^4}{h} = \lim\limits_{h\to 0} \dfrac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4) - x^4}{h}$

$= \lim\limits_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim\limits_{h\to 0} (4x^3+6x^2h+4xh^2+h^3) = 4x^3$

Domain of f = domain of $f' = \mathbb{R}$.

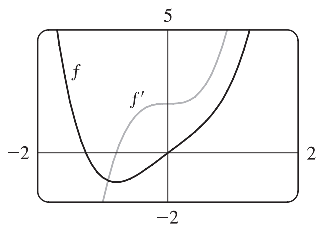
31. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^4 + 2(x+h)] - (x^4+2x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^4+4x^3h+6x^2h^2+4xh^3+h^4+2x+2h-x^4-2x}{h}$

$= \lim\limits_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4+2h}{h} = \lim\limits_{h\to 0} \dfrac{h(4x^3+6x^2h+4xh^2+h^3+2)}{h}$

$= \lim\limits_{h\to 0} (4x^3+6x^2h+4xh^2+h^3+2) = 4x^3+2$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 1993: $U'(1993) \approx \dfrac{U(1994)-U(1993)}{1994-1993} = \dfrac{6.1-6.9}{1} = -0.80$

For 1994: We estimate $U'(1994)$ by using $h = -1$ and $h = 1$, and then average the two results to obtain a final estimate.

$h = -1 \;\Rightarrow\; U'(1994) \approx \dfrac{U(1993)-U(1994)}{1993-1994} = \dfrac{6.9-6.1}{-1} = -0.80;$

$h = 1 \;\Rightarrow\; U'(1994) \approx \dfrac{U(1995)-U(1994)}{1995-1994} = \dfrac{5.6-6.1}{1} = -0.50.$

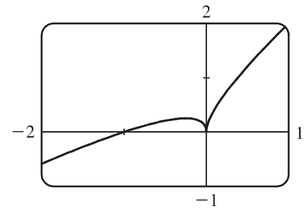
So we estimate that $U'(1994) \approx \frac{1}{2}[(-0.80)+(-0.50)] = -0.65$.

t	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
$U'(t)$	−0.80	−0.65	−0.35	−0.35	−0.45	−0.35	−0.25	0.25	0.90	1.10

35. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

37. f is not differentiable at $x = -1$, because the graph has a vertical tangent there, and at $x = 4$, because the graph has a corner there.

39. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.

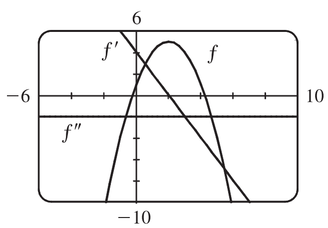


41. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

43. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

45.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[1 + 4(x+h) - (x+h)^2\right] - (1 + 4x - x^2)}{h}$$
$$= \lim_{h\to 0} \frac{(1 + 4x + 4h - x^2 - 2xh - h^2) - (1 + 4x - x^2)}{h} = \lim_{h\to 0} \frac{4h - 2xh - h^2}{h} = \lim_{h\to 0} (4 - 2x - h) = 4 - 2x$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[4 - 2(x+h)] - (4 - 2x)}{h} = \lim_{h\to 0} \frac{-2h}{h} = \lim_{h\to 0} (-2) = -2$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

47. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[2(x+h)^2-(x+h)^3\right]-(2x^2-x^3)}{h}$

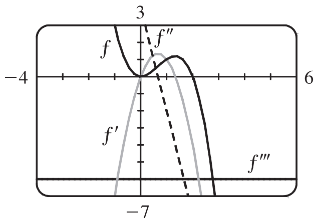
$= \lim\limits_{h\to 0} \dfrac{h(4x+2h-3x^2-3xh-h^2)}{h} = \lim\limits_{h\to 0} (4x+2h-3x^2-3xh-h^2) = 4x-3x^2$

$f''(x) = \lim\limits_{h\to 0} \dfrac{f'(x+h)-f'(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[4(x+h)-3(x+h)^2\right]-(4x-3x^2)}{h} = \lim\limits_{h\to 0} \dfrac{h(4-6x-3h)}{h}$

$= \lim\limits_{h\to 0} (4-6x-3h) = 4-6x$

$f'''(x) = \lim\limits_{h\to 0} \dfrac{f''(x+h)-f''(x)}{h} = \lim\limits_{h\to 0} \dfrac{[4-6(x+h)]-(4-6x)}{h} = \lim\limits_{h\to 0} \dfrac{-6h}{h} = \lim\limits_{h\to 0} (-6) = -6$

$f^{(4)}(x) = \lim\limits_{h\to 0} \dfrac{f'''(x+h)-f'''(x)}{h} = \lim\limits_{h\to 0} \dfrac{-6-(-6)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = \lim\limits_{h\to 0} (0) = 0$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

49. (a) Note that we have factored $x-a$ as the difference of two cubes in the third step.

$f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{x^{1/3}-a^{1/3}}{x-a} = \lim\limits_{x\to a} \dfrac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$

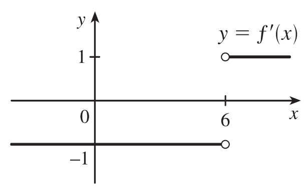
$= \lim\limits_{x\to a} \dfrac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \dfrac{1}{3a^{2/3}}$ or $\frac{1}{3}a^{-2/3}$

(b) $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{h}-0}{h} = \lim\limits_{h\to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x\to 0} |f'(x)| = \lim\limits_{x\to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x=0$ (root function), so f has a vertical tangent at $x=0$.

51. $f(x) = |x-6| = \begin{cases} x-6 & \text{if } x-6 \geq 6 \\ -(x-6) & \text{if } x-6 < 0 \end{cases} = \begin{cases} x-6 & \text{if } x \geq 6 \\ 6-x & \text{if } x < 6 \end{cases}$

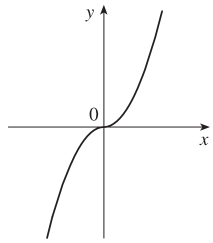
So the right-hand limit is $\lim\limits_{x\to 6^+} \dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^+} \dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^+} \dfrac{x-6}{x-6} = \lim\limits_{x\to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x\to 6^-} \dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^-} \dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^-} \dfrac{6-x}{x-6} = \lim\limits_{x\to 6^-} (-1) = -1$. Since these limits are not equal, $f'(6) = \lim\limits_{x\to 6} \dfrac{f(x)-f(6)}{x-6}$ does not exist and f is not differentiable at 6.



However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

Another way of writing the formula is $f'(x) = \dfrac{x-6}{|x-6|}$.

53. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 2.8.17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x\,|x|}{x} = \lim\limits_{x \to 0} |x| = 0$.

So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{rl} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2\,|x|$.

55. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

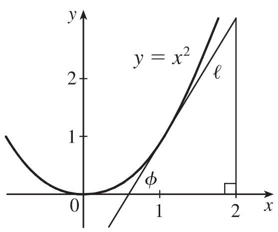
Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

57.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y / \Delta x = \tan \phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.

2 Review

CONCEPT CHECK

1. (a) $\lim_{x \to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

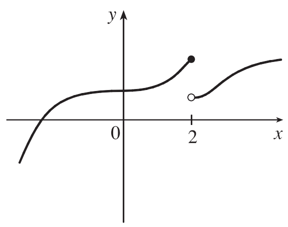
(b) $\lim_{x \to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim_{x \to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

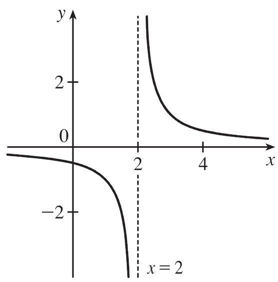
(d) $\lim_{x \to a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.

(e) $\lim_{x \to \infty} f(x) = L$: See Definition 2.6.1 and Figure 2 in Section 2.6.

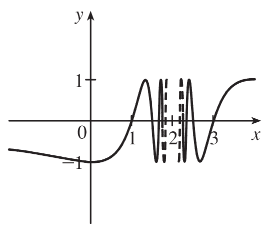
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

3. (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

4. See Theorem 3 in Section 2.3.

5. (a) See Definition 2.2.6 and Figures 12–14 in Section 2.2.

(b) See Definition 2.6.3 and Figures 3 and 4 in Section 2.6.

6. (a) $y = x^4$: No asymptote

(b) $y = \sin x$: No asymptote

(c) $y = \tan x$: Vertical asymptotes $x = \frac{\pi}{2} + \pi n$, n an integer

(d) $y = \tan^{-1} x$: Horizontal asymptotes $y = \pm\frac{\pi}{2}$

(e) $y = e^x$: Horizontal asymptote $y = 0$

$\left(\lim_{x \to -\infty} e^x = 0\right)$

(f) $y = \ln x$: Vertical asymptote $x = 0$

$\left(\lim_{x \to 0^+} \ln x = -\infty\right)$

(g) $y = 1/x$: Vertical asymptote $x = 0$, horizontal asymptote $y = 0$

(h) $y = \sqrt{x}$: No asymptote

7. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

8. See Theorem 2.5.10.

9. See Definition 2.7.1.

10. See the paragraph containing Formula 3 in Section 2.7.

11. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\displaystyle\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

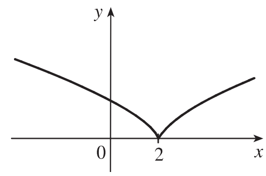
12. See Definition 2.7.2. The pages following the definition discuss interpretations of $f'(a)$ as the slope of a tangent line to the graph of f at $x = a$ and as an instantaneous rate of change of $f(x)$ with respect to x when $x = a$.

13. See the paragraphs before and after Example 6 in Section 2.8.

14. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.

(b) See Theorem 2.8.4. This theorem also tells us that if f is *not* continuous at a, then f is *not* differentiable at a.

(c)



15. See the discussion and Figure 7 on page 159.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

3. True. Limit Law 5 applies.

5. False. Consider $\displaystyle\lim_{x \to 5} \frac{x(x-5)}{x-5}$ or $\displaystyle\lim_{x \to 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

7. True. A polynomial is continuous everywhere, so $\displaystyle\lim_{x \to b} p(x)$ exists and is equal to $p(b)$.

9. True. See Figure 8 in Section 2.6.

11. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

13. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

15. True, by the definition of a limit with $\varepsilon = 1$.

17. False. See the note after Theorem 4 in Section 2.8.

19. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x) = 3$ (ii) $\lim_{x\to -3^+} f(x) = 0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x\to 4} f(x) = 2$

(v) $\lim_{x\to 0} f(x) = \infty$ (vi) $\lim_{x\to 2^-} f(x) = -\infty$

(vii) $\lim_{x\to\infty} f(x) = 4$ (viii) $\lim_{x\to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3, 0, 2,$ and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

5. $$\lim_{x\to -3}\frac{x^2-9}{x^2+2x-3} = \lim_{x\to -3}\frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3}\frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$$

7. $$\lim_{h\to 0}\frac{(h-1)^3+1}{h} = \lim_{h\to 0}\frac{\left(h^3-3h^2+3h-1\right)+1}{h} = \lim_{h\to 0}\frac{h^3-3h^2+3h}{h} = \lim_{h\to 0}\left(h^2-3h+3\right) = 3$$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to 0}\frac{(h-1)^3+1}{h} &= \lim_{h\to 0}\frac{(h-1)^3+1^3}{h} = \lim_{h\to 0}\frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}\\ &= \lim_{h\to 0}\left[(h-1)^2-h+2\right] = 1-0+2 = 3\end{aligned}$$

9. $\lim_{r\to 9}\dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0$ as $r\to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r\neq 9$.

11. $$\lim_{u\to 1}\frac{u^4-1}{u^3+5u^2-6u} = \lim_{u\to 1}\frac{(u^2+1)(u^2-1)}{u(u^2+5u-6)} = \lim_{u\to 1}\frac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)} = \lim_{u\to 1}\frac{(u^2+1)(u+1)}{u(u+6)} = \frac{2(2)}{1(7)} = \frac{4}{7}$$

13. Since x is positive, $\sqrt{x^2} = |x| = x$. Thus,

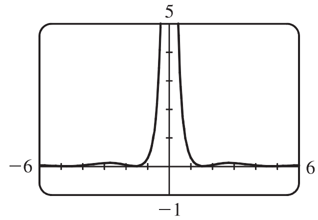
$$\lim_{x\to\infty}\frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x} = \lim_{x\to\infty}\frac{\sqrt{1-9/x^2}}{2-6/x} = \frac{\sqrt{1-0}}{2-0} = \frac{1}{2}$$

15. Let $t = \sin x$. Then as $x\to\pi^-$, $\sin x\to 0^+$, so $t\to 0^+$. Thus, $\lim_{x\to\pi^-}\ln(\sin x) = \lim_{t\to 0^+}\ln t = -\infty$.

17. $$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right) &= \lim_{x\to\infty}\left[\frac{\sqrt{x^2+4x+1}-x}{1}\cdot\frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right] = \lim_{x\to\infty}\frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}\\ &= \lim_{x\to\infty}\frac{(4x+1)/x}{\left(\sqrt{x^2+4x+1}+x\right)/x} \qquad \left[\text{divide by } x=\sqrt{x^2} \text{ for } x>0\right]\\ &= \lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1} = \frac{4+0}{\sqrt{1+0+0}+1} = \frac{4}{2} = 2\end{aligned}$$

19. Let $t = 1/x$. Then as $x \to 0^+$, $t \to \infty$, and $\lim\limits_{x \to 0^+} \tan^{-1}(1/x) = \lim\limits_{t \to \infty} \tan^{-1} t = \dfrac{\pi}{2}$.

21. From the graph of $y = (\cos^2 x)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1 \;\Rightarrow\; \dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2} \;\Rightarrow\; 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x \to \pm\infty} 0 = 0$ and $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{x \to \pm\infty} \dfrac{\cos^2 x}{x^2} = 0$.



Thus, $y = 0$ is the horizontal asymptote. $\lim\limits_{x \to 0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2 \to 0$ as $x \to 0$, so $x = 0$ is the vertical asymptote.

23. Since $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim\limits_{x \to 1} (2x - 1) = 1 = \lim\limits_{x \to 1} x^2$, we have $\lim\limits_{x \to 1} f(x) = 1$ by the Squeeze Theorem.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(14 - 5x) - 4| < \varepsilon$. But $|(14 - 5x) - 4| < \varepsilon \;\Leftrightarrow\; |-5x + 10| < \varepsilon \;\Leftrightarrow\; |-5|\,|x - 2| < \varepsilon \;\Leftrightarrow\; |x - 2| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $0 < |x - 2| < \delta \;\Rightarrow\; |(14 - 5x) - 4| < \varepsilon$. Thus, $\lim\limits_{x \to 2} (14 - 5x) = 4$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $\left|x^2 - 3x - (-2)\right| < \varepsilon$. First, note that if $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \;\Rightarrow\; |x - 1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x - 2| < \delta \;\Rightarrow\; \left|x^2 - 3x - (-2)\right| = |(x - 2)(x - 1)| = |x - 2|\,|x - 1| < (\varepsilon/2)(2) = \varepsilon$.

Thus, $\lim\limits_{x \to 2} \left(x^2 - 3x\right) = -2$ by the definition of a limit.

29. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \le x < 3$, $f(x) = (x - 3)^2$ if $x > 3$.

(i) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (3 - x) = 3$

(ii) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim\limits_{x \to 0} f(x)$ does not exist.

(iv) $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (3 - x) = 0$

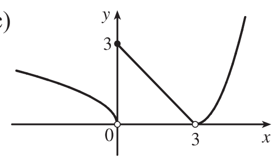
(v) $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (x - 3)^2 = 0$

(vi) Because of (iv) and (v), $\lim\limits_{x \to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim\limits_{x \to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

(c)



31. $\sin x$ is continuous on $\mathbb{R}$ by Theorem 7 in Section 2.5. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 9 in Section 2.5. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 4 in Section 2.5.

33. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

35. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{9-2x^2-1}{x-2}=\lim_{x\to 2}\frac{8-2x^2}{x-2}=\lim_{x\to 2}\frac{-2(x^2-4)}{x-2}=\lim_{x\to 2}\frac{-2(x-2)(x+2)}{x-2}$$
$$=\lim_{x\to 2}\left[-2(x+2)\right]=-2\cdot 4=-8$$

(b) An equation of this tangent line is $y-1=-8(x-2)$ or $y=-8x+17$.

37. (a) $s=s(t)=1+2t+t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}}=\frac{s(1+h)-s(1)}{(1+h)-1}=\frac{1+2(1+h)+(1+h)^2/4-13/4}{h}=\frac{10h+h^2}{4h}=\frac{10+h}{4}$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h=2$, $v_{\text{ave}}=(10+2)/4=3$ m/s

(ii) $[1, 2]$: $h=1$, $v_{\text{ave}}=(10+1)/4=2.75$ m/s

(iii) $[1, 1.5]$: $h=0.5$, $v_{\text{ave}}=(10+0.5)/4=2.625$ m/s

(iv) $[1, 1.1]$: $h=0.1$, $v_{\text{ave}}=(10+0.1)/4=2.525$ m/s

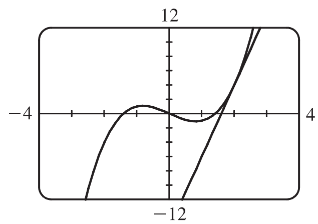
(b) When $t=1$, the instantaneous velocity is $\displaystyle\lim_{h\to 0}\frac{s(1+h)-s(1)}{h}=\lim_{h\to 0}\frac{10+h}{4}=\frac{10}{4}=2.5$ m/s.

39. (a) $\displaystyle f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{x^3-2x-4}{x-2}$

$$=\lim_{x\to 2}\frac{(x-2)\left(x^2+2x+2\right)}{x-2}$$
$$=\lim_{x\to 2}\left(x^2+2x+2\right)=10$$

(b) $y-4=10(x-2)$ or $y=10x-16$

(c)

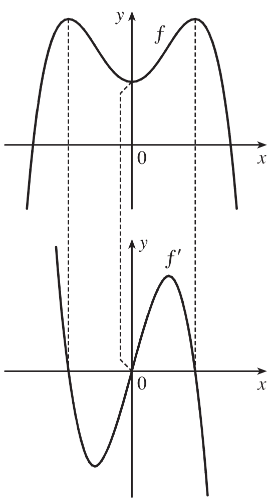


41. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

43.



45. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$

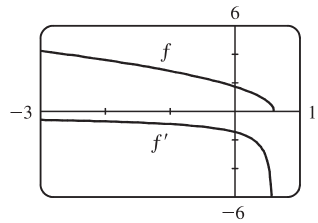
$$= \lim_{h \to 0} \frac{[3-5(x+h)] - (3-5x)}{h\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)} = \lim_{h \to 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3 - 5x \geq 0 \quad \Rightarrow$

$5x \leq 3 \quad \Rightarrow \quad x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$



(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

47. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

49. $C'(1990)$ is the rate at which the total value of US currency in circulation is changing in billions of dollars per year. To estimate the value of $C'(1990)$, we will average the difference quotients obtained using the times $t = 1985$ and $t = 1995$.

Let $A = \dfrac{C(1985) - C(1990)}{1985 - 1990} = \dfrac{187.3 - 271.9}{-5} = \dfrac{-84.6}{-5} = 16.92$ and

$B = \dfrac{C(1995) - C(1990)}{1995 - 1990} = \dfrac{409.3 - 271.9}{5} = \dfrac{137.4}{5} = 27.48$. Then

$C'(1990) = \lim\limits_{t \to 1990} \dfrac{C(t) - C(1990)}{t - 1990} \approx \dfrac{A + B}{2} = \dfrac{16.92 + 27.48}{2} = \dfrac{44.4}{2} = 22.2$ billion dollars/year.

51. $|f(x)| \leq g(x) \quad \Leftrightarrow \quad -g(x) \leq f(x) \leq g(x)$ and $\lim\limits_{x \to a} g(x) = 0 = \lim\limits_{x \to a} -g(x)$.

Thus, by the Squeeze Theorem, $\lim\limits_{x \to a} f(x) = 0$.

☐ PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)\,(t^2+t+1)} = \lim_{t\to 1} \frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0} \frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0} \frac{-4x}{x} = \lim_{x\to 0}(-4) = -4.$

5. Since $[\![x]\!] \le x < [\![x]\!]+1$, we have $\dfrac{[\![x]\!]}{[\![x]\!]} \le \dfrac{x}{[\![x]\!]} < \dfrac{[\![x]\!]+1}{[\![x]\!]} \quad\Rightarrow\quad 1 \le \dfrac{x}{[\![x]\!]} < 1 + \dfrac{1}{[\![x]\!]}$ for $x \ge 1$. As $x\to\infty$, $[\![x]\!]\to\infty$,

so $\dfrac{1}{[\![x]\!]} \to 0$ and $1+\dfrac{1}{[\![x]\!]} \to 1$. Thus, $\displaystyle\lim_{x\to\infty} \frac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

7. f is continuous on $(-\infty, a)$ and (a,∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \quad\Rightarrow\quad \lim_{x\to a^+} x^2 = \lim_{x\to a^-}(x+1) \quad\Rightarrow\quad a^2 = a+1 \quad\Rightarrow\quad a^2-a-1=0 \quad\Rightarrow$$

[by the quadratic formula] $\quad a = \left(1\pm\sqrt{5}\right)/2 \approx 1.618$ or -0.618.

9.
$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(\tfrac{1}{2}[f(x)+g(x)] + \tfrac{1}{2}[f(x)-g(x)]\right) = \tfrac{1}{2}\lim_{x\to a}[f(x)+g(x)] + \tfrac{1}{2}\lim_{x\to a}[f(x)-g(x)]\\ &= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 1 = \tfrac{3}{2},\end{aligned}$$

and $\displaystyle\lim_{x\to a} g(x) = \lim_{x\to a}\Big([f(x)+g(x)] - f(x)\Big) = \lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a} f(x) = 2 - \tfrac{3}{2} = \tfrac{1}{2}.$

So $\displaystyle\lim_{x\to a}[f(x)g(x)] = \Big[\lim_{x\to a} f(x)\Big]\Big[\lim_{x\to a} g(x)\Big] = \tfrac{3}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4}.$

Another solution: Since $\displaystyle\lim_{x\to a}[f(x)+g(x)]$ and $\displaystyle\lim_{x\to a}[f(x)-g(x)]$ exist, we must have

$\displaystyle\lim_{x\to a}[f(x)+g(x)]^2 = \Big(\lim_{x\to a}[f(x)+g(x)]\Big)^2$ and $\displaystyle\lim_{x\to a}[f(x)-g(x)]^2 = \Big(\lim_{x\to a}[f(x)-g(x)]\Big)^2$, so

$$\begin{aligned}\lim_{x\to a}[f(x)\,g(x)] &= \lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2 - [f(x)-g(x)]^2\right) \qquad \text{[because all of the } f^2 \text{ and } g^2 \text{ cancel]}\\ &= \tfrac{1}{4}\Big(\lim_{x\to a}[f(x)+g(x)]^2 - \lim_{x\to a}[f(x)-g(x)]^2\Big) = \tfrac{1}{4}\left(2^2-1^2\right) = \tfrac{3}{4}.\end{aligned}$$

11. (a) Consider $G(x) = T(x+180^\circ) - T(x)$. Fix any number a. If $G(a)=0$, we are done: Temperature at $a =$ Temperature at $a+180^\circ$. If $G(a)>0$, then $G(a+180^\circ) = T(a+360^\circ) - T(a+180^\circ) = T(a) - T(a+180^\circ) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a+180^\circ]$. If $G(a)<0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad \Rightarrow \quad f(0) = 2f(0)$.

Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h)}{h} = \lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

(c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{f(h) + x^2 h + xh^2}{h}$

$= \lim\limits_{h \to 0} \left[\dfrac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$

3 □ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

(b)

x	$\dfrac{2.7^x - 1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$\dfrac{2.8^x - 1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim\limits_{h\to 0} \dfrac{2.7^h - 1}{h} = 0.99$ and $\lim\limits_{h\to 0} \dfrac{2.8^h - 1}{h} = 1.03$.

Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

3. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

5. $f(t) = 2 - \frac{2}{3}t \quad \Rightarrow \quad f'(t) = 0 - \frac{2}{3} = -\frac{2}{3}$

7. $f(x) = x^3 - 4x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 4(1) + 0 = 3x^2 - 4$

9. $f(t) = \frac{1}{4}(t^4 + 8) \quad \Rightarrow \quad f'(t) = \frac{1}{4}(t^4 + 8)' = \frac{1}{4}(4t^{4-1} + 0) = t^3$

11. $y = x^{-2/5} \quad \Rightarrow \quad y' = -\frac{2}{5}x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\dfrac{2}{5x^{7/5}}$

13. $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$

15. $A(s) = -\dfrac{12}{s^5} = -12s^{-5} \quad \Rightarrow \quad A'(s) = -12(-5s^{-6}) = 60s^{-6}$ or $60/s^6$

17. $G(x) = \sqrt{x} - 2e^x = x^{1/2} - 2e^x \quad \Rightarrow \quad G'(x) = \frac{1}{2}x^{-1/2} - 2e^x = \dfrac{1}{2\sqrt{x}} - 2e^x$

19. $F(x) = (\frac{1}{2}x)^5 = \left(\frac{1}{2}\right)^5 x^5 = \frac{1}{32}x^5 \quad \Rightarrow \quad F'(x) = \frac{1}{32}(5x^4) = \frac{5}{32}x^4$

21. $y = ax^2 + bx + c \quad \Rightarrow \quad y' = 2ax + b$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

25. $y = 4\pi^2 \quad \Rightarrow \quad y' = 0$ since $4\pi^2$ is a constant.

27. We first expand using the Binomial Theorem (see Reference Page 1).

$H(x) = (x + x^{-1})^3 = x^3 + 3x^2x^{-1} + 3x(x^{-1})^2 + (x^{-1})^3 = x^3 + 3x + 3x^{-1} + x^{-3} \quad \Rightarrow$

$H'(x) = 3x^2 + 3 + 3(-1x^{-2}) + (-3x^{-4}) = 3x^2 + 3 - 3x^{-2} - 3x^{-4}$

29. $u = \sqrt[5]{t} + 4\sqrt{t^5} = t^{1/5} + 4t^{5/2} \quad \Rightarrow \quad u' = \frac{1}{5}t^{-4/5} + 4\left(\frac{5}{2}t^{3/2}\right) = \frac{1}{5}t^{-4/5} + 10t^{3/2}$ or $1/\left(5\sqrt[5]{t^4}\right) + 10\sqrt{t^3}$

31. $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad \Rightarrow \quad z' = -10Ay^{-11} + Be^y = -\dfrac{10A}{y^{11}} + Be^y$

33. $y = \sqrt[4]{x} = x^{1/4} \quad \Rightarrow \quad y' = \frac{1}{4}x^{-3/4} = \dfrac{1}{4\sqrt[4]{x^3}}$. At $(1, 1)$, $y' = \frac{1}{4}$ and an equation of the tangent line is

$y - 1 = \frac{1}{4}(x - 1)$ or $y = \frac{1}{4}x + \frac{3}{4}$.

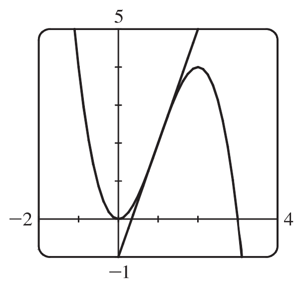
35. $y = x^4 + 2e^x \quad \Rightarrow \quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is

$y - 2 = -\frac{1}{2}(x - 0)$ or $y = -\frac{1}{2}x + 2$.

37. $y = 3x^2 - x^3 \quad \Rightarrow \quad y' = 6x - 3x^2$.

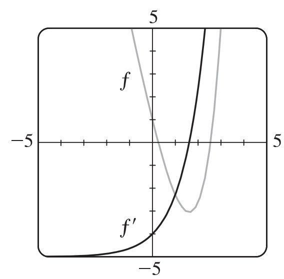
At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is

$y - 2 = 3(x - 1)$ or $y = 3x - 1$.



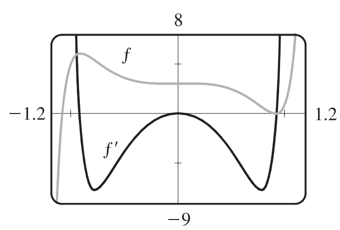
39. $f(x) = e^x - 5x \quad \Rightarrow \quad f'(x) = e^x - 5$.

Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

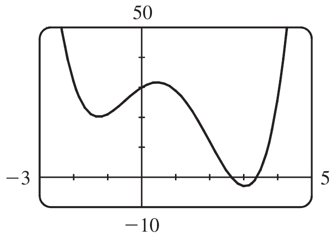


41. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow \quad f'(x) = 45x^{14} - 15x^2$.

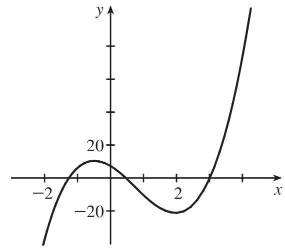
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



43. (a)

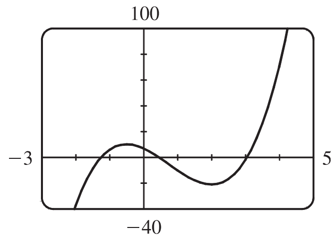


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).



(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

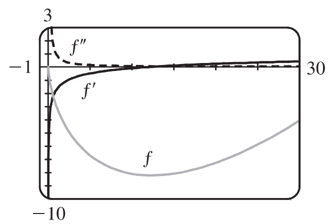
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



45. $f(x) = x^4 - 3x^3 + 16x \quad \Rightarrow \quad f'(x) = 4x^3 - 9x^2 + 16 \quad \Rightarrow \quad f''(x) = 12x^2 - 18x$

47. $f(x) = 2x - 5x^{3/4} \quad \Rightarrow \quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad \Rightarrow \quad f''(x) = \frac{15}{16}x^{-5/4}$

Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.



49. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ and $a(1) = 6 \text{ m/s}^2$.

51. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \quad \Leftrightarrow \quad 6(x^2 + x - 2) = 0 \quad \Leftrightarrow$ $6(x+2)(x-1) = 0 \quad \Leftrightarrow \quad x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

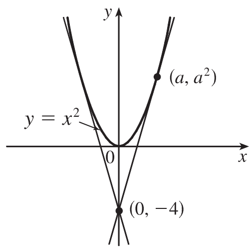
53. $y = 6x^3 + 5x - 3 \quad \Rightarrow \quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

55. The slope of the line $12x - y = 1$ (or $y = 12x - 1$) is 12, so the slope of both lines tangent to the curve is 12.

$y = 1 + x^3 \quad \Rightarrow \quad y' = 3x^2$. Thus, $3x^2 = 12 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$, which are the x-coordinates at which the tangent lines have slope 12. The points on the curve are $(2, 9)$ and $(-2, -7)$, so the tangent line equations are $y - 9 = 12(x - 2)$ or $y = 12x - 15$ and $y + 7 = 12(x + 2)$ or $y = 12x + 17$.

57. The slope of $y = x^2 - 5x + 4$ is given by $m = y' = 2x - 5$. The slope of $x - 3y = 5 \quad\Leftrightarrow\quad y = \frac{1}{3}x - \frac{5}{3}$ is $\frac{1}{3}$, so the desired normal line must have slope $\frac{1}{3}$, and hence, the tangent line to the parabola must have slope -3. This occurs if $2x - 5 = -3 \quad\Rightarrow\quad 2x = 2 \quad\Rightarrow\quad x = 1$. When $x = 1$, $y = 1^2 - 5(1) + 4 = 0$, and an equation of the normal line is $y - 0 = \frac{1}{3}(x - 1)$ or $y = \frac{1}{3}x - \frac{1}{3}$.

59.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad\Leftrightarrow\quad y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

61. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim\limits_{h\to 0} \dfrac{x - (x+h)}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-h}{hx(x+h)} = \lim\limits_{h\to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$

63. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \quad\Rightarrow\quad 2a = 2 \quad\Rightarrow\quad a = 1$.

$P'(2) = 3 \quad\Rightarrow\quad 2(1)(2) + b = 3 \quad\Rightarrow\quad 4 + b = 3 \quad\Rightarrow\quad b = -1$.

$P(2) = 5 \quad\Rightarrow\quad 1(2)^2 + (-1)(2) + c = 5 \quad\Rightarrow\quad 2 + c = 5 \quad\Rightarrow\quad c = 3$. So $P(x) = x^2 - x + 3$.

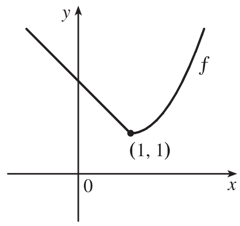
65. $y = f(x) = ax^3 + bx^2 + cx + d \quad\Rightarrow\quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad\Rightarrow\quad -8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad\Rightarrow\quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad\Rightarrow\quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad\Rightarrow\quad 12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad\Rightarrow\quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad\Rightarrow\quad 3 = 16a \quad\Rightarrow\quad a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

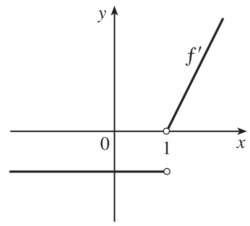
67. $f(x) = 2 - x$ if $x \le 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 2.8.54:

$f'_-(1) = \lim\limits_{h\to 0^-} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^-} \dfrac{2 - (1+h) - 1}{h} = \lim\limits_{h\to 0^-} \dfrac{-h}{h} = \lim\limits_{h\to 0^-} -1 = -1$ and

$f'_+(1) = \lim\limits_{h\to 0^+} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{(1+h)^2 - 2(1+h) + 2 - 1}{h} = \lim\limits_{h\to 0^+} \dfrac{h^2}{h} = \lim\limits_{h\to 0^+} h = 0$.

Thus, $f'(1)$ does not exist since $f'_-(1) \ne f'_+(1)$, so f is not differentiable at 1. But $f'(x) = -1$ for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.





69. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \;\Leftrightarrow\; |x| < 3 \;\Leftrightarrow\; -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \quad\Rightarrow\quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

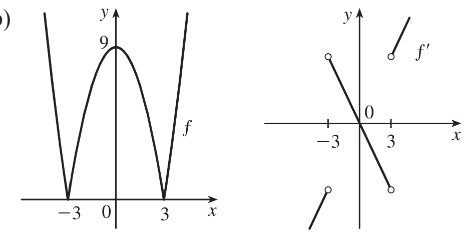
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.54.

$$f'_-(3) = \lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^-} \frac{[-(3+h)^2 + 9] - 0}{h} = \lim_{h \to 0^-} (-6 - h) = -6 \quad \text{and}$$

$$f'_+(3) = \lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0^+} \frac{[(3+h)^2 - 9] - 0}{h} = \lim_{h \to 0^+} \frac{6h + h^2}{h} = \lim_{h \to 0^+} (6 + h) = 6.$$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)



71. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \;\Rightarrow\; 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

73. $y = f(x) = ax^2 \;\Rightarrow\; f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \;\Leftrightarrow\; y = -2x + b$, is seen to be -2, so we must have $4a = -2 \;\Leftrightarrow\; a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \;\Leftrightarrow\; b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

75. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

77. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$.

But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}$$

79. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\frac{a^2 - c}{a - 0}$, so $\frac{a^2 - c}{a} = -\frac{1}{2a} \Rightarrow$ $a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

3.2 The Product and Quotient Rules

1. Product Rule: $y = (x^2 + 1)(x^3 + 1) \Rightarrow$

$$y' = (x^2 + 1)(3x^2) + (x^3 + 1)(2x) = 3x^4 + 3x^2 + 2x^4 + 2x = 5x^4 + 3x^2 + 2x.$$

Multiplying first: $y = (x^2 + 1)(x^3 + 1) = x^5 + x^3 + x^2 + 1 \Rightarrow y' = 5x^4 + 3x^2 + 2x$ (equivalent).

3. By the Product Rule, $f(x) = (x^3 + 2x)e^x \Rightarrow$

$$\begin{aligned} f'(x) &= (x^3 + 2x)(e^x)' + e^x(x^3 + 2x)' = (x^3 + 2x)e^x + e^x(3x^2 + 2) \\ &= e^x[(x^3 + 2x) + (3x^2 + 2)] = e^x(x^3 + 3x^2 + 2x + 2) \end{aligned}$$

5. By the Quotient Rule, $y = \dfrac{e^x}{x^2} \Rightarrow y' = \dfrac{x^2 \dfrac{d}{dx}(e^x) - e^x \dfrac{d}{dx}(x^2)}{(x^2)^2} = \dfrac{x^2(e^x) - e^x(2x)}{x^4} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3}$.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{3x - 1}{2x + 1} \overset{\text{QR}}{\Rightarrow} g'(x) = \dfrac{(2x + 1)(3) - (3x - 1)(2)}{(2x + 1)^2} = \dfrac{6x + 3 - 6x + 2}{(2x + 1)^2} = \dfrac{5}{(2x + 1)^2}$

9. $V(x) = (2x^3 + 3)(x^4 - 2x) \overset{\text{PR}}{\Rightarrow}$

$V'(x) = (2x^3 + 3)(4x^3 - 2) + (x^4 - 2x)(6x^2) = (8x^6 + 8x^3 - 6) + (6x^6 - 12x^3) = 14x^6 - 4x^3 - 6$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} F'(y) &= (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5}) \\ &= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2}) \\ &= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4 \end{aligned}$$

13. $y = \dfrac{x^3}{1 - x^2} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(1 - x^2)(3x^2) - x^3(-2x)}{(1 - x^2)^2} = \dfrac{x^2(3 - 3x^2 + 2x^2)}{(1 - x^2)^2} = \dfrac{x^2(3 - x^2)}{(1 - x^2)^2}$

15. $y = \dfrac{t^2+2}{t^4-3t^2+1} \stackrel{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^4-3t^2+1)(2t)-(t^2+2)(4t^3-6t)}{(t^4-3t^2+1)^2} = \frac{2t[(t^4-3t^2+1)-(t^2+2)(2t^2-3)]}{(t^4-3t^2+1)^2}$$
$$= \frac{2t(t^4-3t^2+1-2t^4-4t^2+3t^2+6)}{(t^4-3t^2+1)^2} = \frac{2t(-t^4-4t^2+7)}{(t^4-3t^2+1)^2}$$

17. $y = (r^2-2r)e^r \stackrel{\text{PR}}{\Rightarrow} y' = (r^2-2r)(e^r) + e^r(2r-2) = e^r(r^2-2r+2r-2) = e^r(r^2-2)$

19. $y = \dfrac{v^3-2v\sqrt{v}}{v} = v^2 - 2\sqrt{v} = v^2 - 2v^{1/2} \Rightarrow y' = 2v - 2\left(\frac{1}{2}\right)v^{-1/2} = 2v - v^{-1/2}$.

We can change the form of the answer as follows: $2v - v^{-1/2} = 2v - \dfrac{1}{\sqrt{v}} = \dfrac{2v\sqrt{v}-1}{\sqrt{v}} = \dfrac{2v^{3/2}-1}{\sqrt{v}}$

21. $f(t) = \dfrac{2t}{2+\sqrt{t}} \stackrel{\text{QR}}{\Rightarrow} f'(t) = \dfrac{(2+t^{1/2})(2)-2t\left(\frac{1}{2}t^{-1/2}\right)}{(2+\sqrt{t})^2} = \dfrac{4+2t^{1/2}-t^{1/2}}{(2+\sqrt{t})^2} = \dfrac{4+t^{1/2}}{(2+\sqrt{t})^2}$ or $\dfrac{4+\sqrt{t}}{(2+\sqrt{t})^2}$

23. $f(x) = \dfrac{A}{B+Ce^x} \stackrel{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(B+Ce^x)\cdot 0 - A(Ce^x)}{(B+Ce^x)^2} = -\dfrac{ACe^x}{(B+Ce^x)^2}$

25. $f(x) = \dfrac{x}{x+c/x} \Rightarrow f'(x) = \dfrac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\dfrac{c}{x}\right)^2} = \dfrac{x+c/x-x+c/x}{\left(\dfrac{x^2+c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2+c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2+c)^2}$

27. $f(x) = x^4e^x \Rightarrow f'(x) = x^4e^x + e^x \cdot 4x^3 = (x^4+4x^3)e^x$ $\left[\text{or } x^3e^x(x+4)\right] \Rightarrow$

$$f''(x) = (x^4+4x^3)e^x + e^x(4x^3+12x^2) = (x^4+4x^3+4x^3+12x^2)e^x$$
$$= (x^4+8x^3+12x^2)e^x \quad \left[\text{or } x^2e^x(x+2)(x+6)\right]$$

29. $f(x) = \dfrac{x^2}{1+2x} \Rightarrow f'(x) = \dfrac{(1+2x)(2x)-x^2(2)}{(1+2x)^2} = \dfrac{2x+4x^2-2x^2}{(1+2x)^2} = \dfrac{2x^2+2x}{(1+2x)^2} \Rightarrow$

$$f''(x) = \frac{(1+2x)^2(4x+2)-(2x^2+2x)(1+4x+4x^2)'}{[(1+2x)^2]^2} = \frac{2(1+2x)^2(2x+1)-2x(x+1)(4+8x)}{(1+2x)^4}$$
$$= \frac{2(1+2x)[(1+2x)^2-4x(x+1)]}{(1+2x)^4} = \frac{2(1+4x+4x^2-4x^2-4x)}{(1+2x)^3} = \frac{2}{(1+2x)^3}$$

31. $y = \dfrac{2x}{x+1} \Rightarrow y' = \dfrac{(x+1)(2)-(2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$.

At $(1,1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x-1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

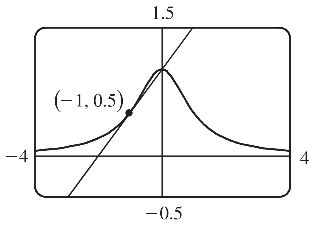
33. $y = 2xe^x \Rightarrow y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x+1)$.

At $(0,0)$, $y' = 2e^0(0+1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x-0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x$.

35. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad\Rightarrow$

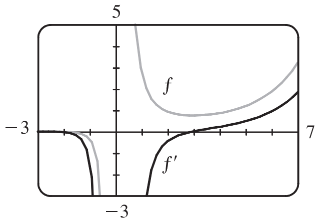
$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

(b)



37. (a) $f(x) = \dfrac{e^x}{x^3} \quad\Rightarrow\quad f'(x) = \dfrac{x^3(e^x) - e^x(3x^2)}{(x^3)^2} = \dfrac{x^2e^x(x-3)}{x^6} = \dfrac{e^x(x-3)}{x^4}$

(b)

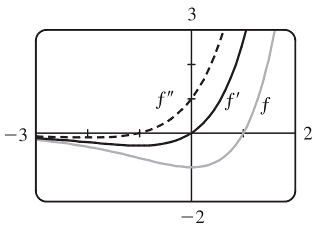


$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

39. (a) $f(x) = (x-1)e^x \;\Rightarrow\; f'(x) = (x-1)e^x + e^x(1) = e^x(x-1+1) = xe^x$.

$f''(x) = x(e^x) + e^x(1) = e^x(x+1)$

(b)



$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

41. $f(x) = \dfrac{x^2}{1+x} \quad\Rightarrow\quad f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2+2x}{x^2+2x+1} \quad\Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$
$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

43. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

45. $f(x) = e^xg(x) \quad\Rightarrow\quad f'(x) = e^xg'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5+2) = 7$

47. (a) From the graphs of f and g, we obtain the following values: $f(1)=2$ since the point $(1,2)$ is on the graph of f; $g(1)=1$ since the point $(1,1)$ is on the graph of g; $f'(1)=2$ since the slope of the line segment between $(0,0)$ and $(2,4)$ is $\dfrac{4-0}{2-0}=2$; $g'(1)=-1$ since the slope of the line segment between $(-2,4)$ and $(2,0)$ is $\dfrac{0-4}{2-(-2)}=-1$.

Now $u(x)=f(x)g(x)$, so $u'(1)=f(1)g'(1)+g(1)\,f'(1)=2\cdot(-1)+1\cdot 2=0$.

(b) $v(x)=f(x)/g(x)$, so $v'(5)=\dfrac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\dfrac{2\left(-\frac{1}{3}\right)-3\cdot\frac{2}{3}}{2^2}=\dfrac{-\frac{8}{3}}{4}=-\dfrac{2}{3}$

49. (a) $y=xg(x) \quad\Rightarrow\quad y'=xg'(x)+g(x)\cdot 1=xg'(x)+g(x)$

(b) $y=\dfrac{x}{g(x)} \quad\Rightarrow\quad y'=\dfrac{g(x)\cdot 1-xg'(x)}{[g(x)]^2}=\dfrac{g(x)-xg'(x)}{[g(x)]^2}$

(c) $y=\dfrac{g(x)}{x} \quad\Rightarrow\quad y'=\dfrac{xg'(x)-g(x)\cdot 1}{(x)^2}=\dfrac{xg'(x)-g(x)}{x^2}$

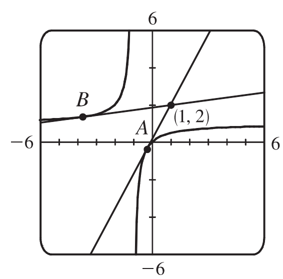
51. If $y=f(x)=\dfrac{x}{x+1}$, then $f'(x)=\dfrac{(x+1)(1)-x(1)}{(x+1)^2}=\dfrac{1}{(x+1)^2}$. When $x=a$, the equation of the tangent line is $y-\dfrac{a}{a+1}=\dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1,2)$ when $2-\dfrac{a}{a+1}=\dfrac{1}{(a+1)^2}(1-a) \quad\Leftrightarrow$

$2(a+1)^2-a(a+1)=1-a \quad\Leftrightarrow\quad 2a^2+4a+2-a^2-a-1+a=0 \quad\Leftrightarrow\quad a^2+4a+1=0.$

The quadratic formula gives the roots of this equation as $a=\dfrac{-4\pm\sqrt{4^2-4(1)(1)}}{2(1)}=\dfrac{-4\pm\sqrt{12}}{2}=-2\pm\sqrt{3}$,

so there are two such tangent lines. Since

$$\begin{aligned} f\left(-2\pm\sqrt{3}\right) &= \frac{-2\pm\sqrt{3}}{-2\pm\sqrt{3}+1}=\frac{-2\pm\sqrt{3}}{-1\pm\sqrt{3}}\cdot\frac{-1\mp\sqrt{3}}{-1\mp\sqrt{3}} \\ &= \frac{2\pm 2\sqrt{3}\mp\sqrt{3}-3}{1-3}=\frac{-1\pm\sqrt{3}}{-2}=\frac{1\mp\sqrt{3}}{2}, \end{aligned}$$



the lines touch the curve at $A\left(-2+\sqrt{3},\frac{1-\sqrt{3}}{2}\right)\approx(-0.27,-0.37)$

and $B\left(-2-\sqrt{3},\frac{1+\sqrt{3}}{2}\right)\approx(-3.73,1.37)$.

53. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t)=P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t)=P(t)A'(t)+A(t)P'(t) \quad\Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999)+A(1999)P'(1999)=(961{,}400)(\$1400/\text{yr})+(\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr}+\$281{,}455{,}600/\text{yr}=\$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t)\approx\$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t)\approx\$281$ million represents the portion of the rate of change of total income due to increasing population.

We will sometimes use the form $f'g + fg'$ rather than the form $fg' + gf'$ for the Product Rule.

55. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\frac{d}{dx}\left(e^{3x}\right) = \frac{d}{dx}\left(e^x\right)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

57. For $f(x) = x^2 e^x$, $f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$\begin{aligned} f''(x) &= e^x(x^2 + 4x + 2) \\ f'''(x) &= e^x(x^2 + 6x + 6) \\ f^{(4)}(x) &= e^x(x^2 + 8x + 12) \\ f^{(5)}(x) &= e^x(x^2 + 10x + 20) \end{aligned}$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

   This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

3.3 Derivatives of Trigonometric Functions

1. $f(x) = 3x^2 - 2\cos x \quad\Rightarrow\quad f'(x) = 6x - 2(-\sin x) = 6x + 2\sin x$

3. $f(x) = \sin x + \frac{1}{2}\cot x \quad\Rightarrow\quad f'(x) = \cos x - \frac{1}{2}\csc^2 x$

5. $g(t) = t^3\cos t \quad\Rightarrow\quad g'(t) = t^3(-\sin t) + (\cos t)\cdot 3t^2 = 3t^2\cos t - t^3\sin t$ or $t^2(3\cos t - t\sin t)$

7. $h(\theta) = \csc\theta + e^\theta\cot\theta \quad\Rightarrow\quad h'(\theta) = -\csc\theta\,\cot\theta + e^\theta(-\csc^2\theta) + (\cot\theta)e^\theta = -\csc\theta\,\cot\theta + e^\theta(\cot\theta - \csc^2\theta)$

9. $y = \dfrac{x}{2 - \tan x} \quad\Rightarrow\quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \quad\Rightarrow$

$$f'(\theta) = \frac{(1 + \sec\theta)(\sec\theta\,\tan\theta) - (\sec\theta)(\sec\theta\,\tan\theta)}{(1 + \sec\theta)^2} = \frac{(\sec\theta\,\tan\theta)\,[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \frac{\sec\theta\,\tan\theta}{(1 + \sec\theta)^2}$$

13. $y = \dfrac{\sin x}{x^2} \quad\Rightarrow\quad y' = \dfrac{x^2 \cos x - (\sin x)(2x)}{(x^2)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

15. Using Exercise 3.2.55(a), $f(x) = xe^x \csc x \quad\Rightarrow$

$$f'(x) = (x)'e^x \csc x + x(e^x)' \csc x + xe^x(\csc x)' = 1e^x \csc x + xe^x \csc x + xe^x(-\cot x\,\csc x)$$
$$= e^x \csc x\,(1 + x - x\cot x)$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\,\cot x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

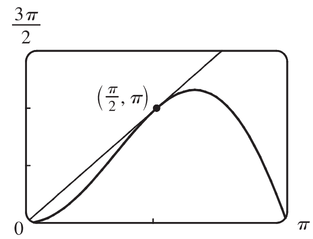
21. $y = \sec x \quad\Rightarrow\quad y' = \sec x\,\tan x$, so $y'(\frac{\pi}{3}) = \sec\frac{\pi}{3}\tan\frac{\pi}{3} = 2\sqrt{3}$. An equation of the tangent line to the curve $y = \sec x$ at the point $\left(\frac{\pi}{3}, 2\right)$ is $y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 2\sqrt{3}\,x + 2 - \frac{2}{3}\sqrt{3}\,\pi$.

23. $y = x + \cos x \quad\Rightarrow\quad y' = 1 - \sin x$. At $(0,1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

25. (a) $y = 2x\sin x \quad\Rightarrow\quad y' = 2(x\cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$,

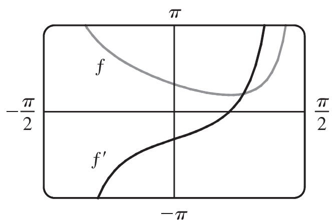
$y' = 2\left(\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



27. (a) $f(x) = \sec x - x \quad\Rightarrow\quad f'(x) = \sec x\,\tan x - 1$

(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

29. $H(\theta) = \theta\sin\theta \quad\Rightarrow\quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad\Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

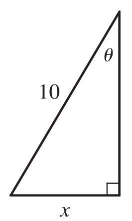
(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad \Leftrightarrow \quad 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow$ $x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

35. (a) $x(t) = 8\sin t \quad \Rightarrow \quad v(t) = x'(t) = 8\cos t \quad \Rightarrow \quad a(t) = x''(t) = -8\sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

37.



From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

39.
$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 3x}{x} &= \lim_{x\to 0}\frac{3\sin 3x}{3x} && \text{[multiply numerator and denominator by 3]}\\
&= 3\lim_{3x\to 0}\frac{\sin 3x}{3x} && [\text{as } x \to 0,\ 3x \to 0]\\
&= 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} && [\text{let } \theta = 3x]\\
&= 3(1) && \text{[Equation 2]}\\
&= 3
\end{aligned}$$

41.
$$\begin{aligned}
\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}\\
&= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3
\end{aligned}$$

43.
$$\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta} = \frac{\sin\left(\lim\limits_{\theta\to 0}\cos\theta\right)}{\lim\limits_{\theta\to 0}\sec\theta} = \frac{\sin 1}{1} = \sin 1$$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}} = \frac{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1+\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\ \lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}} = \frac{1}{1+1\cdot 1} = \frac{1}{2}$$

47.
$$\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\pi/4}\frac{\left(1-\dfrac{\sin x}{\cos x}\right)\cdot\cos x}{(\sin x-\cos x)\cdot\cos x} = \lim_{x\to\pi/4}\frac{\cos x-\sin x}{(\sin x-\cos x)\cos x} = \lim_{x\to\pi/4}\frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$$

49. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \quad \Rightarrow \quad \sec^2 x = \dfrac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad \Rightarrow \quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1+\cot x}{\csc x} \quad \Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1+\cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1+\cot x)\,\cot x]}{\csc^2 x}$$

$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1+\cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

51. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\dfrac{\theta}{2} = \dfrac{d/2}{r} \quad\Rightarrow\quad d = 2r\sin\dfrac{\theta}{2}$. So $\displaystyle\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1.$

[This is just the reciprocal of the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

3.4 The Chain Rule

1. Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(4) = 4\cos 4x$.

3. Let $u = g(x) = 1 - x^2$ and $y = f(u) = u^{10}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (10u^9)(-2x) = -20x(1-x^2)^9$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

7. $F(x) = (x^4+3x^2-2)^5 \quad\Rightarrow\quad F'(x) = 5(x^4+3x^2-2)^4\cdot\dfrac{d}{dx}(x^4+3x^2-2) = 5(x^4+3x^2-2)^4(4x^3+6x)$

$\left[\text{or } 10x(x^4+3x^2-2)^4(2x^2+3)\right]$

9. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \quad\Rightarrow$

$$F'(x) = \tfrac{1}{4}(1+2x+x^3)^{-3/4}\cdot\frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}}\cdot(2+3x^2) = \frac{2+3x^2}{4(1+2x+x^3)^{3/4}}$$

$$= \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

11. $g(t) = \dfrac{1}{(t^4+1)^3} = (t^4+1)^{-3} \quad\Rightarrow\quad g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \dfrac{-12t^3}{(t^4+1)^4}$

13. $y = \cos(a^3+x^3) \quad\Rightarrow\quad y' = -\sin(a^3+x^3)\cdot 3x^2 \quad [a^3 \text{ is just a constant}] \quad = -3x^2\sin(a^3+x^3)$

15. $y = xe^{-kx} \quad\Rightarrow\quad y' = x\left[e^{-kx}(-k)\right] + e^{-kx}\cdot 1 = e^{-kx}(-kx+1) \quad \left[\text{or } (1-kx)e^{-kx}\right]$

17. $g(x) = (1+4x)^5(3+x-x^2)^8 \quad\Rightarrow$

$$g'(x) = (1+4x)^5\cdot 8(3+x-x^2)^7(1-2x) + (3+x-x^2)^8\cdot 5(1+4x)^4\cdot 4$$

$$= 4(1+4x)^4(3+x-x^2)^7\left[2(1+4x)(1-2x) + 5(3+x-x^2)\right]$$

$$= 4(1+4x)^4(3+x-x^2)^7\left[(2+4x-16x^2) + (15+5x-5x^2)\right] = 4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)$$

19. $y = (2x-5)^4(8x^2-5)^{-3} \quad \Rightarrow$

$$\begin{aligned} y' &= 4(2x-5)^3(2)(8x^2-5)^{-3} + (2x-5)^4(-3)(8x^2-5)^{-4}(16x) \\ &= 8(2x-5)^3(8x^2-5)^{-3} - 48x(2x-5)^4(8x^2-5)^{-4} \end{aligned}$$

[This simplifies to $8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$.]

21. $y = \left(\dfrac{x^2+1}{x^2-1}\right)^3 \quad \Rightarrow$

$$\begin{aligned} y' &= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2} \\ &= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x[x^2-1-(x^2+1)]}{(x^2-1)^2} = 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{2x(-2)}{(x^2-1)^2} = \frac{-12x(x^2+1)^2}{(x^2-1)^4} \end{aligned}$$

23. $y = e^{x\cos x} \quad \Rightarrow \quad y' = e^{x\cos x} \cdot \dfrac{d}{dx}(x\cos x) = e^{x\cos x}\left[x(-\sin x) + (\cos x)\cdot 1\right] = e^{x\cos x}(\cos x - x\sin x)$

25. $F(z) = \sqrt{\dfrac{z-1}{z+1}} = \left(\dfrac{z-1}{z+1}\right)^{1/2} \quad \Rightarrow$

$$\begin{aligned} F'(z) &= \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2} \cdot \frac{d}{dz}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2} \cdot \frac{(z+1)(1)-(z-1)(1)}{(z+1)^2} \\ &= \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{z+1-z+1}{(z+1)^2} = \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}} \cdot \frac{2}{(z+1)^2} = \frac{1}{(z-1)^{1/2}(z+1)^{3/2}} \end{aligned}$$

27. $y = \dfrac{r}{\sqrt{r^2+1}} \quad \Rightarrow$

$$\begin{aligned} y' &= \frac{\sqrt{r^2+1}\,(1) - r\cdot\frac{1}{2}(r^2+1)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2} = \frac{\sqrt{r^2+1} - \dfrac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} = \frac{\dfrac{\sqrt{r^2+1}\sqrt{r^2+1}-r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} \\ &= \frac{\left(r^2+1\right)-r^2}{\left(\sqrt{r^2+1}\right)^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } (r^2+1)^{-3/2} \end{aligned}$$

Another solution: Write y as a product and make use of the Product Rule. $y = r(r^2+1)^{-1/2} \quad \Rightarrow$

$y' = r\cdot -\frac{1}{2}(r^2+1)^{-3/2}(2r) + (r^2+1)^{-1/2}\cdot 1 = (r^2+1)^{-3/2}[-r^2+(r^2+1)^1] = (r^2+1)^{-3/2}(1) = (r^2+1)^{-3/2}$.

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

29. $y = \sin(\tan 2x) \quad \Rightarrow \quad y' = \cos(\tan 2x)\cdot \dfrac{d}{dx}(\tan 2x) = \cos(\tan 2x)\cdot \sec^2(2x)\cdot \dfrac{d}{dx}(2x) = 2\cos(\tan 2x)\sec^2(2x)$

31. Using Formula 5 and the Chain Rule, $y = 2^{\sin \pi x} \quad \Rightarrow$

$y' = 2^{\sin\pi x}(\ln 2)\cdot \dfrac{d}{dx}(\sin \pi x) = 2^{\sin\pi x}(\ln 2)\cdot \cos\pi x\cdot \pi = 2^{\sin\pi x}(\pi\ln 2)\cos\pi x$

33. $y = \sec^2 x + \tan^2 x = (\sec x)^2 + (\tan x)^2 \quad \Rightarrow$

$$y' = 2(\sec x)(\sec x \, \tan x) + 2(\tan x)(\sec^2 x) = 2\sec^2 x \, \tan x + 2\sec^2 x \, \tan x = 4\sec^2 x \, \tan x$$

35. $y = \cos\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right) \quad \Rightarrow$

$$y' = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{d}{dx}\left(\frac{1-e^{2x}}{1+e^{2x}}\right) = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{(1+e^{2x})(-2e^{2x}) - (1-e^{2x})(2e^{2x})}{(1+e^{2x})^2}$$

$$= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{-2e^{2x}\left[(1+e^{2x}) + (1-e^{2x})\right]}{(1+e^{2x})^2} = -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \cdot \frac{-2e^{2x}(2)}{(1+e^{2x})^2} = \frac{4e^{2x}}{(1+e^{2x})^2} \cdot \sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)$$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad \Rightarrow$

$$y' = 2[\cot(\sin\theta)] \cdot \frac{d}{d\theta}\,[\cot(\sin\theta)] = 2\cot(\sin\theta) \cdot [-\csc^2(\sin\theta) \cdot \cos\theta] = -2\cos\theta \, \cot(\sin\theta) \, \csc^2(\sin\theta)$$

39. $f(t) = \tan(e^t) + e^{\tan t} \quad \Rightarrow \quad f'(t) = \sec^2(e^t) \cdot \dfrac{d}{dt}\,(e^t) + e^{\tan t} \cdot \dfrac{d}{dt}\,(\tan t) = \sec^2(e^t) \cdot e^t + e^{\tan t} \cdot \sec^2 t$

41. $f(t) = \sin^2\left(e^{\sin^2 t}\right) = \left[\sin\left(e^{\sin^2 t}\right)\right]^2 \quad \Rightarrow$

$$f'(t) = 2\left[\sin\left(e^{\sin^2 t}\right)\right] \cdot \frac{d}{dt}\sin\left(e^{\sin^2 t}\right) = 2\sin\left(e^{\sin^2 t}\right) \cdot \cos\left(e^{\sin^2 t}\right) \cdot \frac{d}{dt}\,e^{\sin^2 t}$$

$$= 2\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) \cdot e^{\sin^2 t} \cdot \frac{d}{dt}\,\sin^2 t = 2\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \cdot 2\sin t \, \cos t$$

$$= 4\sin\left(e^{\sin^2 t}\right)\cos\left(e^{\sin^2 t}\right) e^{\sin^2 t} \sin t \, \cos t$$

43. $g(x) = (2ra^{rx} + n)^p \quad \Rightarrow$

$$g'(x) = p(2ra^{rx} + n)^{p-1} \cdot \frac{d}{dx}(2ra^{rx} + n) = p(2ra^{rx} + n)^{p-1} \cdot 2ra^{rx}(\ln a) \cdot r = 2r^2 p(\ln a)(2ra^{rx} + n)^{p-1} a^{rx}$$

45. $y = \cos\sqrt{\sin(\tan\pi x)} = \cos(\sin(\tan\pi x))^{1/2} \quad \Rightarrow$

$$y' = -\sin(\sin(\tan\pi x))^{1/2} \cdot \frac{d}{dx}\,(\sin(\tan\pi x))^{1/2} = -\sin(\sin(\tan\pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan\pi x))^{-1/2} \cdot \frac{d}{dx}\,(\sin(\tan\pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}} \cdot \cos(\tan\pi x) \cdot \frac{d}{dx}\tan\pi x = \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}} \cdot \cos(\tan\pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi\cos(\tan\pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}$$

47. $h(x) = \sqrt{x^2+1} \quad \Rightarrow \quad h'(x) = \dfrac{1}{2}(x^2+1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}} \quad \Rightarrow$

$$h''(x) = \frac{\sqrt{x^2+1} \cdot 1 - x\left[\tfrac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2+1}\right)^2} = \frac{(x^2+1)^{-1/2}\left[(x^2+1) - x^2\right]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$$

49. $y = e^{\alpha x} \sin \beta x \quad \Rightarrow \quad y' = e^{\alpha x} \cdot \beta \cos \beta x + \sin \beta x \cdot \alpha e^{\alpha x} = e^{\alpha x}(\beta \cos \beta x + \alpha \sin \beta x) \quad \Rightarrow$

$$\begin{aligned} y'' &= e^{\alpha x}(-\beta^2 \sin \beta x + \alpha\beta \cos \beta x) + (\beta \cos \beta x + \alpha \sin \beta x) \cdot \alpha e^{\alpha x} \\ &= e^{\alpha x}(-\beta^2 \sin \beta x + \alpha\beta \cos \beta x + \alpha\beta \cos \beta x + \alpha^2 \sin \beta x) = e^{\alpha x}(\alpha^2 \sin \beta x - \beta^2 \sin \beta x + 2\alpha\beta \cos \beta x) \\ &= e^{\alpha x}\left[(\alpha^2 - \beta^2) \sin \beta x + 2\alpha\beta \cos \beta x\right] \end{aligned}$$

51. $y = (1 + 2x)^{10} \quad \Rightarrow \quad y' = 10(1 + 2x)^9 \cdot 2 = 20(1 + 2x)^9$.

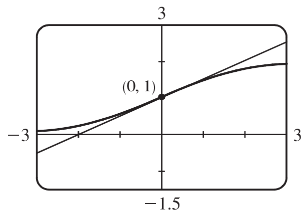
At $(0, 1)$, $y' = 20(1 + 0)^9 = 20$, and an equation of the tangent line is $y - 1 = 20(x - 0)$, or $y = 20x + 1$.

53. $y = \sin(\sin x) \quad \Rightarrow \quad y' = \cos(\sin x) \cdot \cos x$. At $(\pi, 0)$, $y' = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot (-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

55. (a) $y = \dfrac{2}{1 + e^{-x}} \quad \Rightarrow \quad y' = \dfrac{(1 + e^{-x})(0) - 2(-e^{-x})}{(1 + e^{-x})^2} = \dfrac{2e^{-x}}{(1 + e^{-x})^2}$.

At $(0, 1)$, $y' = \dfrac{2e^0}{(1 + e^0)^2} = \dfrac{2(1)}{(1 + 1)^2} = \dfrac{2}{2^2} = \dfrac{1}{2}$. So an equation of the tangent line is $y - 1 = \frac{1}{2}(x - 0)$ or $y = \frac{1}{2}x + 1$.

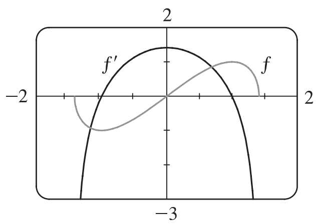
(b)



57. (a) $f(x) = x\sqrt{2 - x^2} = x(2 - x^2)^{1/2} \quad \Rightarrow$

$$f'(x) = x \cdot \tfrac{1}{2}(2 - x^2)^{-1/2}(-2x) + (2 - x^2)^{1/2} \cdot 1 = (2 - x^2)^{-1/2}\left[-x^2 + (2 - x^2)\right] = \frac{2 - 2x^2}{\sqrt{2 - x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

59. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\, \cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

61. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$

63. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

65. (a) $u(x) = f(g(x)) \quad \Rightarrow \quad u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\dfrac{3 - 4}{6 - 2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$ to $(2, 0)$, so its slope is $\dfrac{0 - 6}{2 - 0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \quad \Rightarrow \quad v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \quad \Rightarrow \quad w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

67. (a) $F(x) = f(e^x) \quad \Rightarrow \quad F'(x) = f'(e^x)\dfrac{d}{dx}(e^x) = f'(e^x)e^x$

(b) $G(x) = e^{f(x)} \quad \Rightarrow \quad G'(x) = e^{f(x)}\dfrac{d}{dx}f(x) = e^{f(x)}f'(x)$

69. $r(x) = f(g(h(x))) \quad \Rightarrow \quad r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so
$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$

71. $F(x) = f(3f(4f(x))) \quad \Rightarrow$

$$\begin{aligned} F'(x) &= f'(3f(4f(x))) \cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot \frac{d}{dx}(4f(x)) \\ &= f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot 4f'(x), \quad \text{so} \end{aligned}$$

$$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0) = f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$$

73. $y = Ae^{-x} + Bxe^{-x} \quad \Rightarrow$
$y' = A(-e^{-x}) + B[x(-e^{-x}) + e^{-x} \cdot 1] = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \quad \Rightarrow$
$y'' = (B - A)(-e^{-x}) - B[x(-e^{-x}) + e^{-x} \cdot 1] = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$,

$$\begin{aligned} \text{so} \quad y'' + 2y' + y &= (A - 2B)e^{-x} + Bxe^{-x} + 2[(B - A)e^{-x} - Bxe^{-x}] + Ae^{-x} + Bxe^{-x} \\ &= [(A - 2B) + 2(B - A) + A]e^{-x} + [B - 2B + B]xe^{-x} = 0. \end{aligned}$$

75. The use of $D, D^2, \ldots, D^n$ is just a derivative notation (see text page 157). In general, $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 4f''(2x), \ldots, D^n f(2x) = 2^n f^{(n)}(2x)$. Since $f(x) = \cos x$ and $50 = 4(12) + 2$, we have $f^{(50)}(x) = f^{(2)}(x) = -\cos x$, so $D^{50} \cos 2x = -2^{50} \cos 2x$.

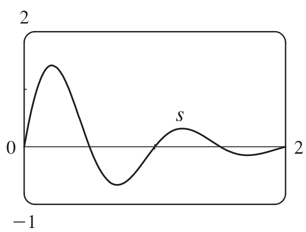
77. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \quad \Rightarrow \quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

79. (a) $B(t) = 4.0 + 0.35 \sin \dfrac{2\pi t}{5.4} \quad \Rightarrow \quad \dfrac{dB}{dt} = \left(0.35 \cos \dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4} \cos \dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$

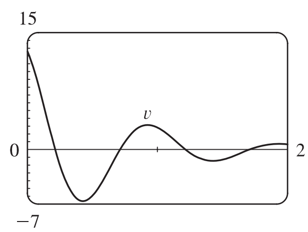
(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi}{5.4} \approx 0.16$.

81. $s(t) = 2e^{-1.5t} \sin 2\pi t \quad \Rightarrow$
$v(t) = s'(t) = 2[e^{-1.5t}(\cos 2\pi t)(2\pi) + (\sin 2\pi t)e^{-1.5t}(-1.5)] = 2e^{-1.5t}(2\pi \cos 2\pi t - 1.5 \sin 2\pi t)$

Graph of position



Graph of velocity



83. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}\,v(t) = v(t)\,\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

85. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a \approx 100.0124369$ and $b \approx 0.000045145933$.

(b) Use $Q'(t) = ab^t \ln b$ (from Formula 5) with the values of a and b from part (a) to get $Q'(0.04) \approx -670.63\ \mu\text{A}$.

The result of Example 2 in Section 2.1 was $-670\ \mu\text{A}$.

87. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get $g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the expression given by Derive.

(b) Derive gives $y' = 2(x^3-x+1)^3(2x+1)^4(17x^3+6x^2-9x+3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x+1)^4(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1)$. If we use Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

89. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\,\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

91. (a)
$$\begin{aligned}
\frac{d}{dx}(\sin^n x\,\cos nx) &= n\sin^{n-1} x\,\cos x\,\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]}\\
&= n\sin^{n-1} x\,(\cos nx\,\cos x - \sin nx\,\sin x) && [\text{factor out } n\sin^{n-1} x]\\
&= n\sin^{n-1} x\cos(nx+x) && \text{[Addition Formula for cosine]}\\
&= n\sin^{n-1} x\cos[(n+1)x] && [\text{factor out } x]
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dx}(\cos^n x\,\cos nx) &= n\cos^{n-1} x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]}\\
&= -n\cos^{n-1} x\,(\cos nx\,\sin x + \sin nx\,\cos x) && [\text{factor out } -n\cos^{n-1} x]\\
&= -n\cos^{n-1} x\sin(nx+x) && \text{[Addition Formula for sine]}\\
&= -n\cos^{n-1} x\sin[(n+1)x] && [\text{factor out } x]
\end{aligned}$$

93. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

95. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \qquad \text{[Product Rule]}$$
$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

3.5 Implicit Differentiation

1. (a) $\frac{d}{dx}\left(xy+2x+3x^2\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad (x\cdot y'+y\cdot 1)+2+6x=0 \quad\Rightarrow\quad xy'=-y-2-6x \quad\Rightarrow$

$y'=\frac{-y-2-6x}{x}$ or $y'=-6-\frac{y+2}{x}$.

(b) $xy+2x+3x^2=4 \quad\Rightarrow\quad xy=4-2x-3x^2 \quad\Rightarrow\quad y=\frac{4-2x-3x^2}{x}=\frac{4}{x}-2-3x$, so $y'=-\frac{4}{x^2}-3$.

(c) From part (a), $y'=\frac{-y-2-6x}{x}=\frac{-(4/x-2-3x)-2-6x}{x}=\frac{-4/x-3x}{x}=-\frac{4}{x^2}-3$.

3. (a) $\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad -\frac{1}{x^2}-\frac{1}{y^2}\,y'=0 \quad\Rightarrow\quad -\frac{1}{y^2}\,y'=\frac{1}{x^2} \quad\Rightarrow\quad y'=-\frac{y^2}{x^2}$

(b) $\frac{1}{x}+\frac{1}{y}=1 \quad\Rightarrow\quad \frac{1}{y}=1-\frac{1}{x}=\frac{x-1}{x} \quad\Rightarrow\quad y=\frac{x}{x-1}$, so $y'=\frac{(x-1)(1)-(x)(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}$.

(c) $y'=-\frac{y^2}{x^2}=-\frac{[x/(x-1)]^2}{x^2}=-\frac{x^2}{x^2(x-1)^2}=-\frac{1}{(x-1)^2}$

5. $\frac{d}{dx}\left(x^3+y^3\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad 3x^2+3y^2\cdot y'=0 \quad\Rightarrow\quad 3y^2\,y'=-3x^2 \quad\Rightarrow\quad y'=-\frac{x^2}{y^2}$

7. $\frac{d}{dx}\left(x^2+xy-y^2\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad 2x+x\cdot y'+y\cdot 1-2y\,y'=0 \quad\Rightarrow$

$xy'-2y\,y'=-2x-y \quad\Rightarrow\quad (x-2y)\,y'=-2x-y \quad\Rightarrow\quad y'=\frac{-2x-y}{x-2y}=\frac{2x+y}{2y-x}$

9. $\frac{d}{dx}\left[x^4(x+y)\right]=\frac{d}{dx}\left[y^2(3x-y)\right] \quad\Rightarrow\quad x^4(1+y')+(x+y)\cdot 4x^3=y^2(3-y')+(3x-y)\cdot 2y\,y' \quad\Rightarrow$

$x^4+x^4\,y'+4x^4+4x^3y=3y^2-y^2\,y'+6xy\,y'-2y^2\,y' \quad\Rightarrow\quad x^4\,y'+3y^2\,y'-6xy\,y'=3y^2-5x^4-4x^3y \quad\Rightarrow$

$(x^4+3y^2-6xy)\,y'=3y^2-5x^4-4x^3y \quad\Rightarrow\quad y'=\frac{3y^2-5x^4-4x^3y}{x^4+3y^2-6xy}$

11. $\frac{d}{dx}\left(x^2y^2+x\sin y\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad x^2\cdot 2y\,y'+y^2\cdot 2x+x\cos y\cdot y'+\sin y\cdot 1=0 \quad\Rightarrow$

$2x^2y\,y'+x\cos y\cdot y'=-2xy^2-\sin y \quad\Rightarrow\quad (2x^2y+x\cos y)y'=-2xy^2-\sin y \quad\Rightarrow\quad y'=\frac{-2xy^2-\sin y}{2x^2y+x\cos y}$

13. $\frac{d}{dx}\left(4\cos x\,\sin y\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad 4\left[\cos x\cdot\cos y\cdot y'+\sin y\cdot(-\sin x)\right]=0 \quad\Rightarrow$

$y'(4\cos x\,\cos y)=4\sin x\,\sin y \quad\Rightarrow\quad y'=\frac{4\sin x\,\sin y}{4\cos x\,\cos y}=\tan x\,\tan y$

15. $\frac{d}{dx}\left(e^{x/y}\right)=\frac{d}{dx}(x-y) \quad\Rightarrow\quad e^{x/y}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)=1-y' \quad\Rightarrow$

$e^{x/y}\cdot\frac{y\cdot 1-x\cdot y'}{y^2}=1-y' \quad\Rightarrow\quad e^{x/y}\cdot\frac{1}{y}-\frac{xe^{x/y}}{y^2}\cdot y'=1-y' \quad\Rightarrow\quad y'-\frac{xe^{x/y}}{y^2}\cdot y'=1-\frac{e^{x/y}}{y} \quad\Rightarrow$

$y'\left(1-\frac{xe^{x/y}}{y^2}\right)=\frac{y-e^{x/y}}{y} \quad\Rightarrow\quad y'=\frac{\dfrac{y-e^{x/y}}{y}}{\dfrac{y^2-xe^{x/y}}{y^2}}=\frac{y(y-e^{x/y})}{y^2-xe^{x/y}}$

17. $\sqrt{xy} = 1 + x^2 y \;\Rightarrow\; \frac{1}{2}(xy)^{-1/2}(xy' + y \cdot 1) = 0 + x^2 y' + y \cdot 2x \;\Rightarrow\; \dfrac{x}{2\sqrt{xy}}\, y' + \dfrac{y}{2\sqrt{xy}} = x^2 y' + 2xy \;\Rightarrow$

$y'\left(\dfrac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \dfrac{y}{2\sqrt{xy}} \;\Rightarrow\; y'\left(\dfrac{x - 2x^2\sqrt{xy}}{2\sqrt{xy}}\right) = \dfrac{4xy\sqrt{xy} - y}{2\sqrt{xy}} \;\Rightarrow\; y' = \dfrac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

19. $\dfrac{d}{dx}\left(e^y \cos x\right) = \dfrac{d}{dx}\left[1 + \sin(xy)\right] \;\Rightarrow\; e^y(-\sin x) + \cos x \cdot e^y \cdot y' = \cos(xy) \cdot (xy' + y \cdot 1) \;\Rightarrow$

$-e^y \sin x + e^y \cos x \cdot y' = x\cos(xy) \cdot y' + y\cos(xy) \;\Rightarrow\; e^y \cos x \cdot y' - x\cos(xy) \cdot y' = e^y \sin x + y\cos(xy) \;\Rightarrow$

$\left[e^y \cos x - x\cos(xy)\right] y' = e^y \sin x + y\cos(xy) \;\Rightarrow\; y' = \dfrac{e^y \sin x + y\cos(xy)}{e^y \cos x - x\cos(xy)}$

21. $\dfrac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \dfrac{d}{dx}(10) \;\Rightarrow\; f'(x) + x^2 \cdot 3[f(x)]^2 \cdot f'(x) + [f(x)]^3 \cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2 \cdot 3[f(1)]^2 \cdot f'(1) + [f(1)]^3 \cdot 2(1) = 0 \;\Rightarrow\; f'(1) + 1 \cdot 3 \cdot 2^2 \cdot f'(1) + 2^3 \cdot 2 = 0 \;\Rightarrow$

$f'(1) + 12f'(1) = -16 \;\Rightarrow\; 13f'(1) = -16 \;\Rightarrow\; f'(1) = -\frac{16}{13}.$

23. $\dfrac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \dfrac{d}{dy}(0) \;\Rightarrow\; x^4 \cdot 2y + y^2 \cdot 4x^3\, x' - (x^3 \cdot 1 + y \cdot 3x^2\, x') + 2(x \cdot 3y^2 + y^3 \cdot x') = 0 \;\Rightarrow$

$4x^3y^2\, x' - 3x^2y\, x' + 2y^3\, x' = -2x^4y + x^3 - 6xy^2 \;\Rightarrow\; (4x^3y^2 - 3x^2y + 2y^3)\, x' = -2x^4y + x^3 - 6xy^2 \;\Rightarrow$

$x' = \dfrac{dx}{dy} = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

25. $x^2 + xy + y^2 = 3 \;\Rightarrow\; 2x + x\,y' + y \cdot 1 + 2yy' = 0 \;\Rightarrow\; x\,y' + 2y\,y' = -2x - y \;\Rightarrow\; y'(x + 2y) = -2x - y \;\Rightarrow$

$y' = \dfrac{-2x - y}{x + 2y}.$ When $x = 1$ and $y = 1$, we have $y' = \dfrac{-2 - 1}{1 + 2 \cdot 1} = \dfrac{-3}{3} = -1$, so an equation of the tangent line is

$y - 1 = -1(x - 1)$ or $y = -x + 2.$

27. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \;\Rightarrow\; 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1).$ When $x = 0$ and $y = \frac{1}{2}$, we have

$0 + y' = 2(\frac{1}{2})(2y' - 1) \;\Rightarrow\; y' = 2y' - 1 \;\Rightarrow\; y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$

or $y = x + \frac{1}{2}.$

29. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \;\Rightarrow\; 4(x^2 + y^2)(2x + 2y\,y') = 25(2x - 2y\,y') \;\Rightarrow$

$4(x + y\,y')(x^2 + y^2) = 25(x - y\,y') \;\Rightarrow\; 4y\,y'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \;\Rightarrow$

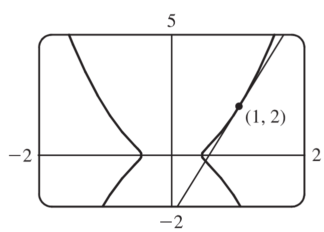
$y' = \dfrac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}.$ When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$,

so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}.$

31. (a) $y^2 = 5x^4 - x^2 \;\Rightarrow\; 2y\,y' = 5(4x^3) - 2x \;\Rightarrow\; y' = \dfrac{10x^3 - x}{y}.$

So at the point $(1, 2)$ we have $y' = \dfrac{10(1)^3 - 1}{2} = \dfrac{9}{2}$, and an equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}.$

(b)



33. $9x^2 + y^2 = 9 \quad \Rightarrow \quad 18x + 2y\,y' = 0 \quad \Rightarrow \quad 2y\,y' = -18x \quad \Rightarrow \quad y' = -9x/y \quad \Rightarrow$

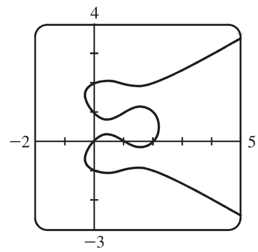
$y'' = -9\left(\dfrac{y \cdot 1 - x \cdot y'}{y^2}\right) = -9\left(\dfrac{y - x(-9x/y)}{y^2}\right) = -9 \cdot \dfrac{y^2 + 9x^2}{y^3} = -9 \cdot \dfrac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2 + y^2 = 9$]. Thus, $y'' = -81/y^3$.

35. $x^3 + y^3 = 1 \quad \Rightarrow \quad 3x^2 + 3y^2\,y' = 0 \quad \Rightarrow \quad y' = -\dfrac{x^2}{y^2} \quad \Rightarrow$

$$y'' = -\frac{y^2(2x) - x^2 \cdot 2y\,y'}{(y^2)^2} = -\frac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\frac{2xy^4 + 2x^4y}{y^6} = -\frac{2xy(y^3 + x^3)}{y^6} = -\frac{2x}{y^5},$$

since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

37. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

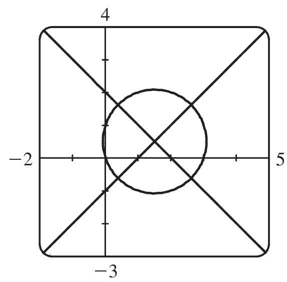


(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

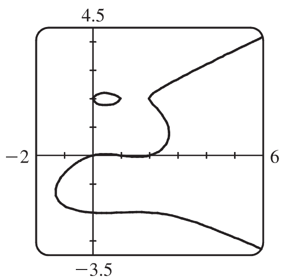
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

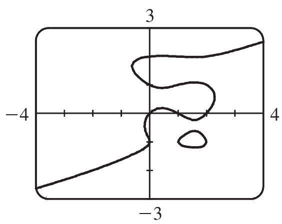
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



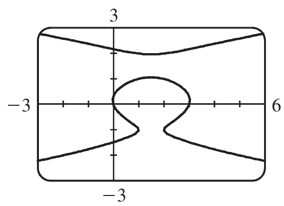
$y(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



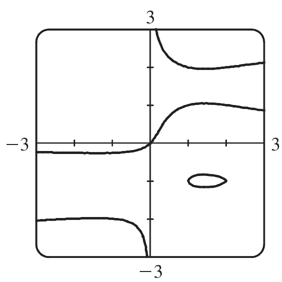
$y(y^2 - 4)(y - 2)$
$= x(x - 1)(x - 2)$



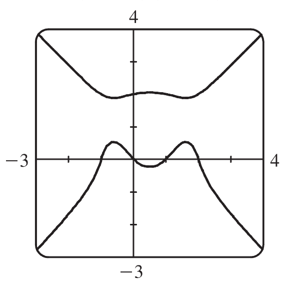
$y(y + 1)(y^2 - 1)(y - 2)$
$= x(x - 1)(x - 2)$



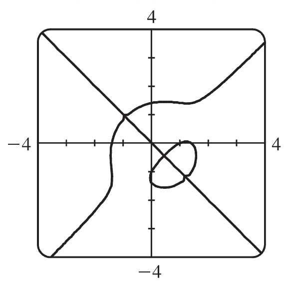
$(y + 1)(y^2 - 1)(y - 2)$
$= (x - 1)(x - 2)$



$x(y + 1)(y^2 - 1)(y - 2)$
$= y(x - 1)(x - 2)$



$y(y^2 + 1)(y - 2)$
$= x(x^2 - 1)(x - 2)$



$y(y + 1)(y^2 - 2)$
$= x(x - 1)(x^2 - 2)$

39. From Exercise 29, a tangent to the lemniscate will be horizontal if $y' = 0 \Rightarrow 25x - 4x(x^2 + y^2) = 0 \Rightarrow$ $x[25 - 4(x^2 + y^2)] = 0 \Rightarrow x^2 + y^2 = \frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

41. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

43. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

45. $y = \tan^{-1}\sqrt{x} \Rightarrow y' = \dfrac{1}{1 + \left(\sqrt{x}\right)^2} \cdot \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x}\,(1+x)}$

47. $y = \sin^{-1}(2x + 1) \Rightarrow$

$$y' = \frac{1}{\sqrt{1 - (2x+1)^2}} \cdot \frac{d}{dx}(2x + 1) = \frac{1}{\sqrt{1 - (4x^2 + 4x + 1)}} \cdot 2 = \frac{2}{\sqrt{-4x^2 - 4x}} = \frac{1}{\sqrt{-x^2 - x}}$$

49. $G(x) = \sqrt{1 - x^2}\arccos x \Rightarrow G'(x) = \sqrt{1 - x^2} \cdot \dfrac{-1}{\sqrt{1 - x^2}} + \arccos x \cdot \dfrac{1}{2}(1 - x^2)^{-1/2}(-2x) = -1 - \dfrac{x\arccos x}{\sqrt{1 - x^2}}$

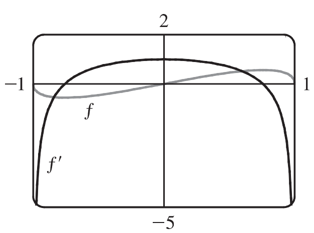
51. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \Rightarrow$

$$h'(t) = -\frac{1}{1 + t^2} - \frac{1}{1 + (1/t)^2} \cdot \frac{d}{dt}\frac{1}{t} = -\frac{1}{1 + t^2} - \frac{t^2}{t^2 + 1} \cdot \left(-\frac{1}{t^2}\right) = -\frac{1}{1 + t^2} + \frac{1}{t^2 + 1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = \dfrac{3\pi}{2}$ for $t < 0$.

53. $y = \cos^{-1}(e^{2x}) \Rightarrow y' = -\dfrac{1}{\sqrt{1 - (e^{2x})^2}} \cdot \dfrac{d}{dx}(e^{2x}) = -\dfrac{2e^{2x}}{\sqrt{1 - e^{4x}}}$

55. $f(x) = \sqrt{1 - x^2}\arcsin x \Rightarrow f'(x) = \sqrt{1 - x^2} \cdot \dfrac{1}{\sqrt{1 - x^2}} + \arcsin x \cdot \dfrac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x) = 1 - \dfrac{x\arcsin x}{\sqrt{1 - x^2}}$

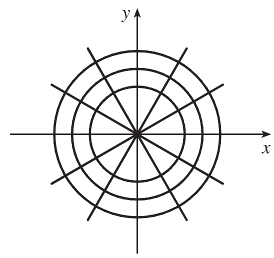


Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

57. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \leq y \leq \pi \quad \Rightarrow \quad -\sin y \, \dfrac{dy}{dx} = 1 \quad \Rightarrow$

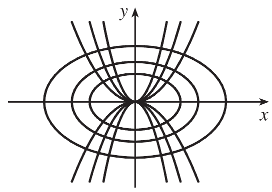
$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}. \quad \text{[Note that } \sin y \geq 0 \text{ for } 0 \leq y \leq \pi.]$$

59. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \quad \Rightarrow \quad 2x + 2yy' = 0 \quad \Rightarrow \quad y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



61. $y = cx^2 \quad \Rightarrow \quad y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow$

$2yy' = -x \quad \Rightarrow \quad y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if $c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.



63. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

$y = 0 \quad \Rightarrow \quad x^2 - x(0) + 0^2 = 3 \quad \Leftrightarrow \quad x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $\left(\pm\sqrt{3}, 0\right)$.

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \quad \Rightarrow \quad y'(2y - x) = y - 2x \quad \Leftrightarrow \quad y' = \dfrac{y - 2x}{2y - x}$.

So y' at $\left(\sqrt{3}, 0\right)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $\left(-\sqrt{3}, 0\right)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

65. $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \quad \Leftrightarrow \quad y'(2x^2y + x) = -2xy^2 - y \quad \Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \quad \Leftrightarrow \quad 2xy^2 + y = 2x^2y + x \quad \Leftrightarrow \quad y(2xy + 1) = x(2xy + 1) \quad \Leftrightarrow$

$y(2xy + 1) - x(2xy + 1) = 0 \quad \Leftrightarrow \quad (2xy + 1)(y - x) = 0 \quad \Leftrightarrow \quad xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \quad \Rightarrow$

$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \quad \Rightarrow \quad x^4 + x^2 = 2 \quad \Leftrightarrow \quad x^4 + x^2 - 2 = 0 \quad \Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

67. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x,

we get $f'(y)\,\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad \Rightarrow \quad \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \quad \Rightarrow \quad f^{-1}(5) = 4$. By part (a), $\left(f^{-1}\right)'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = 1\big/\left(\frac{2}{3}\right) = \frac{3}{2}$.

69. $x^2 + 4y^2 = 5 \;\Rightarrow\; 2x + 4(2yy') = 0 \;\Rightarrow\; y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a+5} \;\Leftrightarrow\; 4b^2 = -a^2 - 5a \;\Leftrightarrow\; a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \;\Leftrightarrow\; a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \;\Rightarrow\; b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a+5} = \dfrac{1}{-1+5} = \dfrac{1}{4} \;\Rightarrow\; h = 2$. So the lamp is located 2 units above the x-axis.

3.6 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

3. $f(x) = \sin(\ln x) \;\Rightarrow\; f'(x) = \cos(\ln x) \cdot \dfrac{d}{dx} \ln x = \cos(\ln x) \cdot \dfrac{1}{x} = \dfrac{\cos(\ln x)}{x}$

5. $f(x) = \log_2(1 - 3x) \;\Rightarrow\; f'(x) = \dfrac{1}{(1-3x)\ln 2} \dfrac{d}{dx}(1 - 3x) = \dfrac{-3}{(1-3x)\ \ln 2}$ or $\dfrac{3}{(3x-1)\ \ln 2}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \;\Rightarrow\; f'(x) = \frac{1}{5}(\ln x)^{-4/5} \dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}} \cdot \dfrac{1}{x} = \dfrac{1}{5x\ \sqrt[5]{(\ln x)^4}}$

9. $f(x) = \sin x \ln(5x) \;\Rightarrow\; f'(x) = \sin x \cdot \dfrac{1}{5x} \cdot \dfrac{d}{dx}(5x) + \ln(5x) \cdot \cos x = \dfrac{\sin x \cdot 5}{5x} + \cos x \ln(5x) = \dfrac{\sin x}{x} + \cos x \ln(5x)$

11. $F(t) = \ln \dfrac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3\ln(2t+1) - 4\ln(3t-1) \;\Rightarrow$

$F'(t) = 3 \cdot \dfrac{1}{2t+1} \cdot 2 - 4 \cdot \dfrac{1}{3t-1} \cdot 3 = \dfrac{6}{2t+1} - \dfrac{12}{3t-1}$, or combined, $\dfrac{-6(t+3)}{(2t+1)(3t-1)}$.

13. $g(x) = \ln\left(x\sqrt{x^2-1}\right) = \ln x + \ln(x^2-1)^{1/2} = \ln x + \frac{1}{2}\ln(x^2-1) \;\Rightarrow$

$g'(x) = \dfrac{1}{x} + \dfrac{1}{2} \cdot \dfrac{1}{x^2-1} \cdot 2x = \dfrac{1}{x} + \dfrac{x}{x^2-1} = \dfrac{x^2 - 1 + x \cdot x}{x(x^2-1)} = \dfrac{2x^2-1}{x(x^2-1)}$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \;\Rightarrow$

$f'(u) = \dfrac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1+\ln(2u)]^2} = \dfrac{\frac{1}{u}[1 + \ln(2u) - \ln u]}{[1+\ln(2u)]^2} = \dfrac{1 + (\ln 2 + \ln u) - \ln u}{u[1+\ln(2u)]^2} = \dfrac{1 + \ln 2}{u[1+\ln(2u)]^2}$

17. $y = \ln\left|2 - x - 5x^2\right| \;\Rightarrow\; y' = \dfrac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \dfrac{-10x - 1}{2 - x - 5x^2}$ or $\dfrac{10x+1}{5x^2 + x - 2}$

19. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \;\Rightarrow$

$y' = -1 + \dfrac{1}{1+x} = \dfrac{-1 - x + 1}{1+x} = -\dfrac{x}{1+x}$

21. $y = 2x\log_{10}\sqrt{x} = 2x\log_{10}x^{1/2} = 2x\cdot\frac{1}{2}\log_{10}x = x\log_{10}x \quad\Rightarrow\quad y' = x\cdot\dfrac{1}{x\ln 10} + \log_{10}x\cdot 1 = \dfrac{1}{\ln 10} + \log_{10}x$

Note: $\dfrac{1}{\ln 10} = \dfrac{\ln e}{\ln 10} = \log_{10}e$, so the answer could be written as $\dfrac{1}{\ln 10} + \log_{10}x = \log_{10}e + \log_{10}x = \log_{10}ex$.

23. $y = x^2\ln(2x) \quad\Rightarrow\quad y' = x^2\cdot\dfrac{1}{2x}\cdot 2 + \ln(2x)\cdot(2x) = x + 2x\ln(2x) \quad\Rightarrow$

$y'' = 1 + 2x\cdot\dfrac{1}{2x}\cdot 2 + \ln(2x)\cdot 2 = 1 + 2 + 2\ln(2x) = 3 + 2\ln(2x)$

25. $y = \ln\left(x + \sqrt{1+x^2}\right) \quad\Rightarrow$

$$y' = \frac{1}{x+\sqrt{1+x^2}}\frac{d}{dx}\left(x+\sqrt{1+x^2}\right) = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \tfrac{1}{2}(1+x^2)^{-1/2}(2x)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} \quad\Rightarrow$$

$$y'' = -\tfrac{1}{2}(1+x^2)^{-3/2}(2x) = \frac{-x}{(1+x^2)^{3/2}}$$

27. $f(x) = \dfrac{x}{1 - \ln(x-1)} \quad\Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\mathrm{Dom}(f) = \{x \mid x-1>0 \quad\text{and}\quad 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \quad\text{and}\quad \ln(x-1)\neq 1\}$$

$$= \left\{x \mid x>1 \quad\text{and}\quad x-1\neq e^1\right\} = \{x \mid x>1 \quad\text{and}\quad x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$$

29. $f(x) = \ln(x^2-2x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{x^2-2x}(2x-2) = \dfrac{2(x-1)}{x(x-2)}$.

$\mathrm{Dom}(f) = \{x \mid x(x-2)>0\} = (-\infty,0)\cup(2,\infty)$.

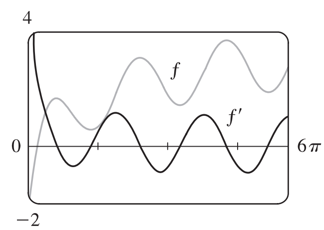
31. $f(x) = \dfrac{\ln x}{x^2} \quad\Rightarrow\quad f'(x) = \dfrac{x^2(1/x)-(\ln x)(2x)}{(x^2)^2} = \dfrac{x-2x\ln x}{x^4} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$,

so $f'(1) = \dfrac{1-2\ln 1}{1^3} = \dfrac{1-2\cdot 0}{1} = 1$.

33. $y = \ln\left(xe^{x^2}\right) = \ln x + \ln e^{x^2} = \ln x + x^2 \quad\Rightarrow\quad y' = \dfrac{1}{x} + 2x$. At $(1,1)$, the slope of the tangent line is $y'(1) = 1+2 = 3$, and an equation of the tangent line is $y-1 = 3(x-1)$, or $y = 3x-2$.

35. $f(x) = \sin x + \ln x \quad\Rightarrow\quad f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



37. $y = (2x+1)^5(x^4-3)^6 \quad\Rightarrow\quad \ln y = \ln\big((2x+1)^5(x^4-3)^6\big) \quad\Rightarrow\quad \ln y = 5\ln(2x+1) + 6\ln(x^4-3) \quad\Rightarrow$

$\dfrac{1}{y}\,y' = 5\cdot\dfrac{1}{2x+1}\cdot 2 + 6\cdot\dfrac{1}{x^4-3}\cdot 4x^3 \quad\Rightarrow$

$y' = y\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right).$

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

39. $y = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2} \quad\Rightarrow\quad \ln y = \ln(\sin^2 x\,\tan^4 x) - \ln(x^2+1)^2 \quad\Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \quad\Rightarrow\quad \ln y = 2\,\ln|\sin x| + 4\,\ln|\tan x| - 2\,\ln(x^2+1) \quad\Rightarrow$

$\dfrac{1}{y}\,y' = 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 4\cdot\dfrac{1}{\tan x}\cdot\sec^2 x - 2\cdot\dfrac{1}{x^2+1}\cdot 2x \quad\Rightarrow\quad y' = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

41. $y = x^x \quad\Rightarrow\quad \ln y = \ln x^x \quad\Rightarrow\quad \ln y = x\ln x \quad\Rightarrow\quad y'/y = x(1/x) + (\ln x)\cdot 1 \quad\Rightarrow\quad y' = y(1+\ln x) \quad\Rightarrow$

$y' = x^x(1+\ln x)$

43. $y = x^{\sin x} \quad\Rightarrow\quad \ln y = \ln x^{\sin x} \quad\Rightarrow\quad \ln y = \sin x \ln x \quad\Rightarrow\quad \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \quad\Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x\cos x\right) \quad\Rightarrow\quad y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\,\cos x\right)$

45. $y = (\cos x)^x \quad\Rightarrow\quad \ln y = \ln(\cos x)^x \quad\Rightarrow\quad \ln y = x\ln\cos x \quad\Rightarrow\quad \dfrac{1}{y}\,y' = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \quad\Rightarrow$

$y' = y\left(\ln\cos x - \dfrac{x\sin x}{\cos x}\right) \quad\Rightarrow\quad y' = (\cos x)^x(\ln\cos x - x\tan x)$

47. $y = (\tan x)^{1/x} \quad\Rightarrow\quad \ln y = \ln(\tan x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln\tan x \quad\Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{x}\cdot\dfrac{1}{\tan x}\cdot\sec^2 x + \ln\tan x\cdot\left(-\dfrac{1}{x^2}\right) \quad\Rightarrow\quad y' = y\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right) \quad\Rightarrow$

$y' = (\tan x)^{1/x}\left(\dfrac{\sec^2 x}{x\tan x} - \dfrac{\ln\tan x}{x^2}\right) \quad$ or $\quad y' = (\tan x)^{1/x}\cdot\dfrac{1}{x}\left(\csc x\sec x - \dfrac{\ln\tan x}{x}\right)$

49. $y = \ln(x^2+y^2) \quad\Rightarrow\quad y' = \dfrac{1}{x^2+y^2}\,\dfrac{d}{dx}\,(x^2+y^2) \quad\Rightarrow\quad y' = \dfrac{2x+2yy'}{x^2+y^2} \quad\Rightarrow\quad x^2y' + y^2y' = 2x + 2yy' \quad\Rightarrow$

$x^2y' + y^2y' - 2yy' = 2x \quad\Rightarrow\quad (x^2+y^2-2y)y' = 2x \quad\Rightarrow\quad y' = \dfrac{2x}{x^2+y^2-2y}$

51. $f(x) = \ln(x-1) \quad\Rightarrow\quad f'(x) = \dfrac{1}{(x-1)} = (x-1)^{-1} \quad\Rightarrow\quad f''(x) = -(x-1)^{-2} \quad\Rightarrow\quad f'''(x) = 2(x-1)^{-3} \quad\Rightarrow$

$f^{(4)}(x) = -2\cdot 3(x-1)^{-4} \quad\Rightarrow\quad \cdots \quad\Rightarrow\quad f^{(n)}(x) = (-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot(n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

53. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1.$

3.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 12t^2 + 36t \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 24t + 36$

(b) $v(3) = 27 - 72 + 36 = -9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\quad 3t^2 - 24t + 36 = 0 \quad \Leftrightarrow \quad 3(t-2)(t-6) = 0 \quad \Leftrightarrow \quad t = 2$ s or 6 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \quad \Leftrightarrow \quad 0 \leq t < 2$ or $t > 6$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$, and $[6, 8]$ separately.

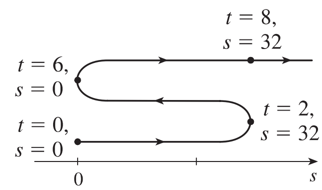
$$|f(2) - f(0)| = |32 - 0| = 32.$$

$$|f(6) - f(2)| = |0 - 32| = 32.$$

$$|f(8) - f(6)| = |32 - 0| = 32.$$

The total distance is $32 + 32 + 32 = 96$ ft.

(f)

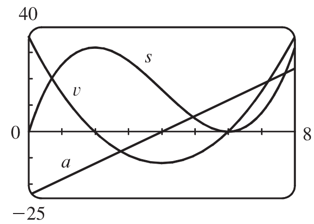


(g) $v(t) = 3t^2 - 24t + 36 \quad \Rightarrow$

$a(t) = v'(t) = 6t - 24.$

$a(3) = 6(3) - 24 = -6\,(\text{ft/s})/\text{s}$ or ft/s^2.

(h)



(i) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ [v and a are both negative] and when $t > 6$ [v and a are both positive]. It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $4 < t < 6$.

3. (a) $s = f(t) = \cos(\pi t/4) \quad \Rightarrow \quad v(t) = f'(t) = -\sin(\pi t/4) \cdot (\pi/4)$

(b) $v(3) = -\frac{\pi}{4} \sin \frac{3\pi}{4} = -\frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} = -\frac{\pi\sqrt{2}}{8}$ ft/s $[\approx -0.56]$

(c) The particle is at rest when $v(t) = 0$. $\quad -\frac{\pi}{4} \sin \frac{\pi t}{4} = 0 \quad \Rightarrow \quad \sin \frac{\pi t}{4} = 0 \quad \Rightarrow \quad \frac{\pi t}{4} = \pi n \quad \Rightarrow \quad t = 0, 4, 8$ s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $\quad -\frac{\pi}{4} \sin \frac{\pi t}{4} > 0 \quad \Rightarrow \quad \sin \frac{\pi t}{4} < 0 \quad \Rightarrow \quad 4 < t < 8$.

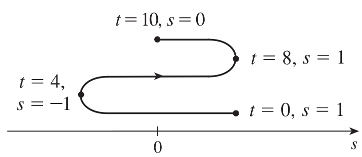
(e) From part (c), $v(t) = 0$ for $t = 0, 4, 8$. As in Exercise 1, we'll find the distance traveled in the intervals $[0, 4]$ and $[4, 8]$.

$$|f(4) - f(0)| = |-1 - 1| = 2$$

$$|f(8) - f(4)| = |1 - (-1)| = 2.$$

The total distance is $2 + 2 = 4$ ft.

(f)

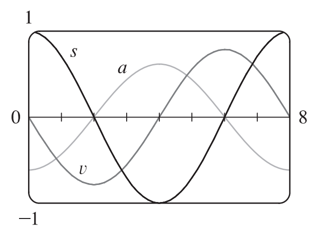


(g) $v(t) = -\dfrac{\pi}{4} \sin \dfrac{\pi t}{4} \quad \Rightarrow$

$$a(t) = v'(t) = -\frac{\pi}{4} \cos \frac{\pi t}{4} \cdot \frac{\pi}{4} = -\frac{\pi^2}{16} \cos \frac{\pi t}{4}.$$

$$a(3) = -\frac{\pi^2}{16} \cos \frac{3\pi}{4} = -\frac{\pi^2}{16}\left(-\frac{\sqrt{2}}{2}\right) = \frac{\pi^2\sqrt{2}}{32}\ (\text{ft/s})/\text{s} \text{ or } \text{ft/s}^2.$$

(h)



(i) The particle is speeding up when v and a have the same sign. This occurs when $0 < t < 2$ or $8 < t < 10$ [v and a are both negative] and when $4 < t < 6$ [v and a are both positive]. It is slowing down when v and a have opposite signs; that is, when $2 < t < 4$ and when $6 < t < 8$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ [$v > 0, a > 0$] and on $(3, 4)$ [$v < 0, a < 0$]. The particle is slowing down on $(0, 1)$ and $(2, 3)$ [$v > 0, a < 0$].

7. (a) $s(t) = t^3 - 4.5t^2 - 7t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 9t - 7 = 5 \quad \Leftrightarrow \quad 3t^2 - 9t - 12 = 0 \quad \Leftrightarrow$

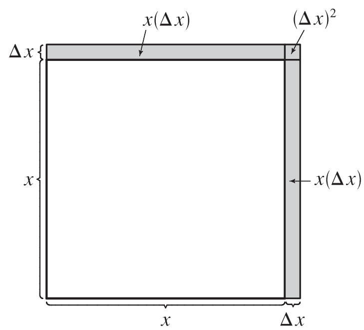
$3(t-4)(t+1) = 0 \quad \Leftrightarrow \quad t = 4$ or -1. Since $t \geq 0$, the particle reaches a velocity of 5 m/s at $t = 4$ s.

(b) $a(t) = v'(t) = 6t - 9 = 0 \quad \Leftrightarrow \quad t = 1.5$. The acceleration changes from negative to positive, so the velocity changes from decreasing to increasing. Thus, at $t = 1.5$ s, the velocity has its minimum value.

9. (a) $h = 10t - 0.83t^2 \quad \Rightarrow \quad v(t) = \dfrac{dh}{dt} = 10 - 1.66t$, so $v(3) = 10 - 1.66(3) = 5.02$ m/s.

(b) $h = 25 \quad \Rightarrow \quad 10t - 0.83t^2 = 25 \quad \Rightarrow \quad 0.83t^2 - 10t + 25 = 0 \quad \Rightarrow \quad t = \frac{10 \pm \sqrt{17}}{1.66} \approx 3.54$ or 8.51.

The value $t_1 = \frac{10 - \sqrt{17}}{1.66}$ corresponds to the time it takes for the stone to rise 25 m and $t_2 = \frac{10 + \sqrt{17}}{1.66}$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1) = 10 - 1.66\left(\frac{10 - \sqrt{17}}{1.66}\right) = \sqrt{17} \approx 4.12$ m/s.



11. (a) $A(x) = x^2 \quad \Rightarrow \quad A'(x) = 2x$. $A'(15) = 30$ mm^2/mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

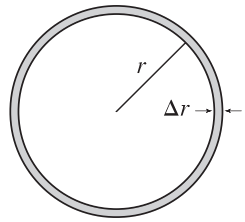
(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$

(ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \quad \Rightarrow \quad A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$.
So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



15. $S(r) = 4\pi r^2 \quad \Rightarrow \quad S'(r) = 8\pi r \quad \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

21. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad \Rightarrow \quad V = \frac{C}{P} \quad \Rightarrow \quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

23. In Example 6, the population function was $n = 2^t n_0$. Since we are tripling instead of doubling and the initial population is 400, the population function is $n(t) = 400 \cdot 3^t$. The rate of growth is $n'(t) = 400 \cdot 3^t \cdot \ln 3$, so the rate of growth after 2.5 hours is $n'(2.5) = 400 \cdot 3^{2.5} \cdot \ln 3 \approx 6850$ bacteria/hour.

25. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \dfrac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \dfrac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \dfrac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \dfrac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12{,}822.97902$, and $d \approx -7{,}743{,}770.396$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad \Rightarrow \quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) $P'(1920) = 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12{,}822.97902$

≈ 14.48 million/year [smaller than the answer in part (a), but close to it]

$P'(1980) \approx 75.29$ million/year (smaller, but close)

(e) $P'(1985) \approx 81.62$ million/year, so the rate of growth in 1985 was about 81.62 million/year.

27. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}(0.01^2 - r^2)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \;\Rightarrow\; v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

29. (a) $C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3 \;\Rightarrow\; C'(x) = 12 - 0.2x + 0.0015x^2$ \$/yard, which is the marginal cost function.

(b) $C'(200) = 12 - 0.2(200) + 0.0015(200)^2 = \$32/\text{yard}$, and this is the rate at which costs are increasing with respect to the production level when $x = 200$. $C'(200)$ predicts the cost of producing the 201st yard.

(c) The cost of manufacturing the 201st yard of fabric is $C(201) - C(200) = 3632.2005 - 3600 \approx \$32.20$, which is approximately $C'(200)$.

31. (a) $A(x) = \dfrac{p(x)}{x} \;\Rightarrow\; A'(x) = \dfrac{xp'(x) - p(x)\cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.

$A'(x) > 0 \;\Rightarrow\; A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\Rightarrow\; p'(x) > A(x) \;\Rightarrow\; p'(x) > \dfrac{p(x)}{x} \;\Rightarrow\; xp'(x) > p(x) \;\Rightarrow$

$xp'(x) - p(x) > 0 \;\Rightarrow\; \dfrac{xp'(x) - p(x)}{x^2} > 0 \;\Rightarrow\; A'(x) > 0$.

33. $PV = nRT \;\Rightarrow\; T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$\dfrac{dT}{dt} = \dfrac{1}{0.821}[P(t)V'(t) + V(t)P'(t)] = \dfrac{1}{0.821}[(8)(-0.15) + (10)(0.10)] \approx -0.2436$ K/min.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \;\Rightarrow\; C = 10W$. Substituting $C = 10W$ into **(1)**

results in $0.05(10W) - 0.001(10W)W = 0 \quad \Leftrightarrow \quad 0.5W - 0.01W^2 = 0 \quad \Leftrightarrow \quad 50W - W^2 = 0 \quad \Leftrightarrow$ $W(50 - W) = 0 \quad \Leftrightarrow \quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.8 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.7944$, so $\dfrac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$.

Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

3. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 100e^{kt}$. Now $P(1) = 100e^{k(1)} = 420 \quad \Rightarrow \quad e^k = \frac{420}{100} \quad \Rightarrow \quad k = \ln 4.2$.

So $P(t) = 100e^{(\ln 4.2)t} = 100(4.2)^t$.

(b) $P(3) = 100(4.2)^3 = 7408.8 \approx 7409$ bacteria

(c) $dP/dt = kP \quad \Rightarrow \quad P'(3) = k \cdot P(3) = (\ln 4.2)\big(100(4.2)^3\big)$ [from part (a)] $\approx 10{,}632$ bacteria/hour

(d) $P(t) = 100(4.2)^t = 10{,}000 \quad \Rightarrow \quad (4.2)^t = 100 \quad \Rightarrow \quad t = (\ln 100)/(\ln 4.2) \approx 3.2$ hours

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \quad \Rightarrow$ $\frac{980}{790} = e^{k(50)} \quad \Rightarrow \quad \ln \frac{980}{790} = 50k \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \quad \Rightarrow \quad P(1900) = 1650 = 1260e^{k(1900-1850)} \quad \Rightarrow$ $\ln \frac{1650}{1260} = k(50) \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2560 = 1650e^{k(1950-1900)} \quad \Rightarrow$ $\ln \frac{2560}{1650} = k(50) \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \quad \Rightarrow \quad y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

(b) $y(t) = Ce^{-0.0005t} = 0.9C \quad \Rightarrow \quad e^{-0.0005t} = 0.9 \quad \Rightarrow \quad -0.0005t = \ln 0.9 \quad \Rightarrow \quad t = -2000 \ln 0.9 \approx 211$ s

9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$.

$y(30) = 100e^{30k} = \frac{1}{2}(100) \;\Rightarrow\; e^{30k} = \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/30 \;\Rightarrow\; y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \;\Rightarrow\; -(\ln 2)t/30 = \ln \frac{1}{100} \;\Rightarrow\; t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$y(5730) = \frac{1}{2}y(0) \;\Rightarrow\; y(0)e^{-k(5730)} = \frac{1}{2}y(0) \;\Rightarrow\; e^{-5730k} = \frac{1}{2} \;\Rightarrow\; -5730k = \ln\frac{1}{2} \;\Rightarrow\; k = -\dfrac{\ln\frac{1}{2}}{5730} = \dfrac{\ln 2}{5730}$.

If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0) \;\Rightarrow\; 0.74 = e^{-t(\ln 2)/5730} \;\Rightarrow\; \ln 0.74 = -\dfrac{t\ln 2}{5730} \;\Rightarrow$

$t = -\dfrac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

13. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 75)$. Now let $y = T - 75$, so

$y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.

$y(30) = 110e^{30k} = 150 - 75 \;\Rightarrow\; e^{30k} = \frac{75}{110} = \frac{15}{22} \;\Rightarrow\; k = \frac{1}{30}\ln\frac{15}{22}$, so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and

$y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62^\circ$F. Thus, $T(45) \approx 62 + 75 = 137^\circ$F.

(b) $T(t) = 100 \;\Rightarrow\; y(t) = 25$. $\;y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \;\Rightarrow\; e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \;\Rightarrow\; \frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110} \;\Rightarrow$

$t = \dfrac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

15. $\dfrac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $\;y(0) = T(0) - 20 = 5 - 20 = -15$, so

$y(25) = y(0)e^{25k} = -15e^{25k}$, and $y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \;\Rightarrow\; e^{25k} = \frac{2}{3}$. Thus,

$25k = \ln\left(\frac{2}{3}\right)$ and $k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \;\Rightarrow\; e^k = \left(\frac{2}{3}\right)^{1/25} \;\Rightarrow$

$e^{kt} = \left(\frac{2}{3}\right)^{t/25} \;\Rightarrow\; y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,^\circ$C

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \;\Rightarrow\; 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \;\Rightarrow\; \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \;\Rightarrow$

$(t/25)\,\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \;\Rightarrow\; t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

17. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \;\Rightarrow\; P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$P(1000) = 101.3e^{1000k} = 87.14 \;\Rightarrow\; 1000k = \ln\left(\frac{87.14}{101.3}\right) \;\Rightarrow\; k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \;\Rightarrow$

$P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

19. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1 + \frac{0.05}{1}\right)^{1 \cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1 + \frac{0.05}{2}\right)^{2 \cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1 + \frac{0.05}{52}\right)^{52 \cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1 + \frac{0.05}{365}\right)^{365 \cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

3.9 Related Rates

1. $V = x^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\,\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2/\text{s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \quad \Rightarrow \quad \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \quad \Rightarrow \quad 3 = 25\pi\,\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dh}{dt} = \dfrac{3}{25\pi}\text{ m/min.}$

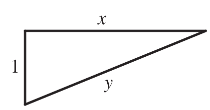
7. $y = x^3 + 2x \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\,\dfrac{dx}{dt} = (3x^2 + 2)(5) = 5(3x^2 + 2)$. When $x = 2$, $\dfrac{dy}{dt} = 5(14) = 70$.

9. $z^2 = x^2 + y^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right)$. When $x = 5$ and $y = 12$, $z^2 = 5^2 + 12^2 \quad \Rightarrow \quad z^2 = 169 \quad \Rightarrow \quad z = \pm 13$. For $\dfrac{dx}{dt} = 2$ and $\dfrac{dy}{dt} = 3$, $\dfrac{dz}{dt} = \dfrac{1}{\pm 13}\,(5 \cdot 2 + 12 \cdot 3) = \pm\dfrac{46}{13}$.

11. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



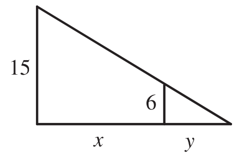
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\,\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

13. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x+y)$ when $x = 40$ ft.

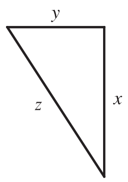
(c)



(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \quad\Rightarrow\quad 15y = 6x + 6y \quad\Rightarrow\quad 9y = 6x \quad\Rightarrow\quad y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \dfrac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

15.



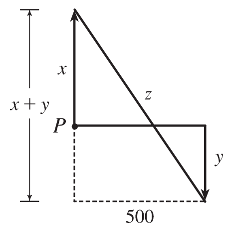
We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \quad\Rightarrow$

$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \quad\Rightarrow\quad z\,\frac{dz}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt} \quad\Rightarrow\quad \frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right).$$

After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \quad\Rightarrow\quad z = \sqrt{120^2 + 50^2} = 130$,

so $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

17.



We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

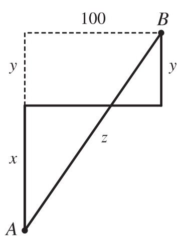
$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad\Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s}.$$

19. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \quad\Rightarrow\quad \frac{1}{2}b = 10 \quad\Rightarrow$ $b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \quad\Rightarrow\quad 4 = 20 + 10\,\dfrac{db}{dt} \quad\Rightarrow\quad \dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

21.



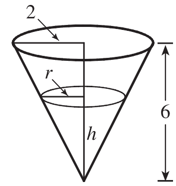
We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2 \quad\Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad\Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx 55.4 \text{ km/h}.$$

23. If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where

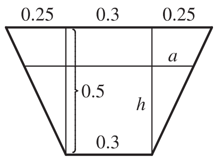


$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \quad\Rightarrow\quad r = \dfrac{1}{3}h \quad\Rightarrow$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \dfrac{\pi}{9}h^2\,\dfrac{dh}{dt}$. When $h = 200$ cm,

$\dfrac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \dfrac{\pi}{9}(200)^2(20) \quad\Rightarrow\quad C = 10{,}000 + \dfrac{800{,}000}{9}\pi \approx 289{,}253\ \text{cm}^3/\text{min}$.

25.



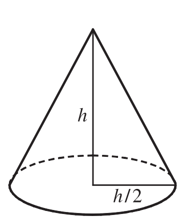
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h$.

By similar triangles, $\dfrac{a}{h} = \dfrac{0.25}{0.5} = \dfrac{1}{2}$, so $2a = h \quad\Rightarrow\quad V = 5(0.6 + h)h = 3h + 5h^2$.

Now $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \quad\Rightarrow\quad 0.2 = (3 + 10h)\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{0.2}{3 + 10h}$. When $h = 0.3$,

$\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10(0.3)} = \dfrac{0.2}{6}\ \text{m/min} = \dfrac{1}{30}\ \text{m/min or}\ \dfrac{10}{3}\ \text{cm/min}$.

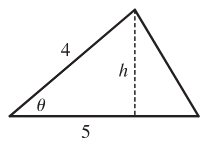
27. We are given that $\dfrac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi h^3}{12} \quad\Rightarrow$



$\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} \quad\Rightarrow\quad 30 = \dfrac{\pi h^2}{4}\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{120}{\pi h^2}$.

When $h = 10$ ft, $\dfrac{dh}{dt} = \dfrac{120}{10^2\pi} = \dfrac{6}{5\pi} \approx 0.38$ ft/min.

29. $A = \frac{1}{2}bh$, but $b = 5$ m and $\sin\theta = \dfrac{h}{4} \quad\Rightarrow\quad h = 4\sin\theta$, so $A = \frac{1}{2}(5)(4\sin\theta) = 10\sin\theta$.



We are given $\dfrac{d\theta}{dt} = 0.06$ rad/s, so $\dfrac{dA}{dt} = \dfrac{dA}{d\theta}\dfrac{d\theta}{dt} = (10\cos\theta)(0.06) = 0.6\cos\theta$.

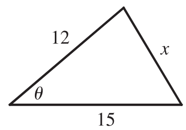
When $\theta = \dfrac{\pi}{3}$, $\dfrac{dA}{dt} = 0.6\left(\cos\dfrac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3\ \text{m}^2/\text{s}$.

31. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\dfrac{dV}{dt} + V\dfrac{dP}{dt} = 0 \quad\Rightarrow$

$\dfrac{dV}{dt} = -\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 600$, $P = 150$ and $\dfrac{dP}{dt} = 20$, so we have $\dfrac{dV}{dt} = -\dfrac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of $80\ \text{cm}^3/\text{min}$.

33. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\dfrac{dR_2}{dt} \quad\Rightarrow\quad \dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\dfrac{dR}{dt} = \dfrac{400^2}{9^2}\left[\dfrac{1}{80^2}(0.3) + \dfrac{1}{100^2}(0.2)\right] = \dfrac{107}{810} \approx 0.132\ \Omega/\text{s}$.

35. We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

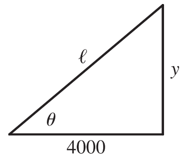


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \quad\Rightarrow$

$2x\,\dfrac{dx}{dt} = 360\sin\theta\,\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = \dfrac{180\sin\theta}{x}\,\dfrac{d\theta}{dt}$. When $\theta = 60°$,

$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21}$, so $\dfrac{dx}{dt} = \dfrac{180\sin 60°}{3\sqrt{21}}\,\dfrac{\pi}{90} = \dfrac{\pi\sqrt{3}}{3\sqrt{21}} = \dfrac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

37. (a) By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t,



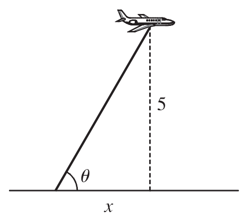
we obtain $2y\,\dfrac{dy}{dt} = 2\ell\,\dfrac{d\ell}{dt}$. We know that $\dfrac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft,

$\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft

and $\dfrac{d\ell}{dt} = \dfrac{y}{\ell}\,\dfrac{dy}{dt} = \dfrac{3000}{5000}(600) = \dfrac{1800}{5} = 360$ ft/s.

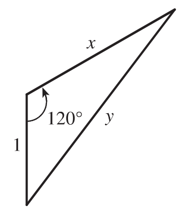
(b) Here $\tan\theta = \dfrac{y}{4000} \quad\Rightarrow\quad \dfrac{d}{dt}(\tan\theta) = \dfrac{d}{dt}\left(\dfrac{y}{4000}\right) \quad\Rightarrow\quad \sec^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{4000}\,\dfrac{dy}{dt} \quad\Rightarrow\quad \dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{4000}\,\dfrac{dy}{dt}$. When $y = 3000$ ft, $\dfrac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \dfrac{4000}{\ell} = \dfrac{4000}{5000} = \dfrac{4}{5}$, so $\dfrac{d\theta}{dt} = \dfrac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

39. $\cot\theta = \dfrac{x}{5} \quad\Rightarrow\quad -\csc^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{5}\,\dfrac{dx}{dt} \quad\Rightarrow\quad -\left(\csc\dfrac{\pi}{3}\right)^2\left(-\dfrac{\pi}{6}\right) = \dfrac{1}{5}\,\dfrac{dx}{dt} \quad\Rightarrow$



$\dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$

41. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,

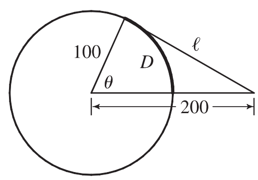


$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} + \dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\,\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad\Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.

43. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,



$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\,\dfrac{d\ell}{dt} = -40{,}000(-\sin\theta)\,\dfrac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \dfrac{1}{100}D \quad\Rightarrow\quad \dfrac{d\theta}{dt} = \dfrac{1}{100}\,\dfrac{dD}{dt} = \dfrac{7}{100}$. To substitute into the expression for $\dfrac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad\Leftrightarrow$

$\cos\theta = \frac{1}{4} \quad\Rightarrow\quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\,\dfrac{d\ell}{dt} = 40{,}000\,\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad\Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

3.10 Linear Approximations and Differentials

1. $f(x) = x^4 + 3x^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 6x$, so $f(-1) = 4$ and $f'(-1) = -10$.
Thus, $L(x) = f(-1) + f'(-1)(x - (-1)) = 4 + (-10)(x + 1) = -10x - 6$.

3. $f(x) = \cos x \quad \Rightarrow \quad f'(x) = -\sin x$, so $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$.
Thus, $L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$.

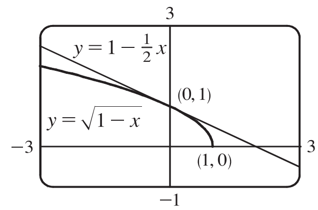
5. $f(x) = \sqrt{1 - x} \quad \Rightarrow \quad f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

Therefore,

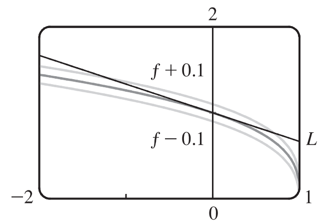
$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x$.

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$

and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.



7. $f(x) = \sqrt[3]{1 - x} = (1 - x)^{1/3} \quad \Rightarrow \quad f'(x) = -\frac{1}{3}(1 - x)^{-2/3}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{3}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{3}x$. We need $\sqrt[3]{1 - x} - 0.1 < 1 - \frac{1}{3}x < \sqrt[3]{1 - x} + 0.1$, which is true when $-1.204 < x < 0.706$.

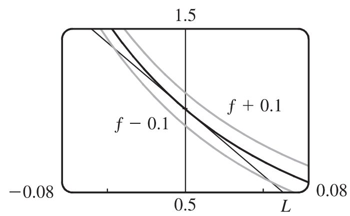


9. $f(x) = \dfrac{1}{(1 + 2x)^4} = (1 + 2x)^{-4} \quad \Rightarrow$

$f'(x) = -4(1 + 2x)^{-5}(2) = \dfrac{-8}{(1 + 2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + (-8)(x - 0) = 1 - 8x$.

We need $\dfrac{1}{(1 + 2x)^4} - 0.1 < 1 - 8x < \dfrac{1}{(1 + 2x)^4} + 0.1$, which is true when $-0.045 < x < 0.055$.



11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = x^2 \sin 2x$,

$f'(x) = x^2 \cos 2x \cdot 2 + \sin 2x \cdot 2x = 2x(x \cos 2x + \sin 2x)$, so $dy = 2x(x \cos 2x + \sin 2x)\,dx$.

(b) For $y = f(t) = \ln\sqrt{1 + t^2} = \frac{1}{2}\ln(1 + t^2)$, $f'(t) = \dfrac{1}{2} \cdot \dfrac{1}{1 + t^2} \cdot 2t = \dfrac{t}{1 + t^2}$, so $dy = \dfrac{t}{1 + t^2}\,dt$.

13. (a) For $y = f(u) = \dfrac{u + 1}{u - 1}$, $f'(u) = \dfrac{(u - 1)(1) - (u + 1)(1)}{(u - 1)^2} = \dfrac{-2}{(u - 1)^2}$, so $dy = \dfrac{-2}{(u - 1)^2}\,du$.

(b) For $y = f(r) = (1 + r^3)^{-2}$, $f'(r) = -2(1 + r^3)^{-3}(3r^2) = \dfrac{-6r^2}{(1 + r^3)^3}$, so $dy = \dfrac{-6r^2}{(1 + r^3)^3}\,dr$.

15. (a) $y = e^{x/10} \quad \Rightarrow \quad dy = e^{x/10} \cdot \frac{1}{10}\,dx = \frac{1}{10}e^{x/10}dx$

(b) $x = 0$ and $dx = 0.1 \quad \Rightarrow \quad dy = \frac{1}{10}e^{0/10}(0.1) = 0.01$.

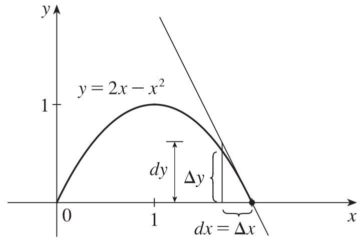
17. (a) $y = \tan x \quad \Rightarrow \quad dy = \sec^2 x\, dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = \left(\sqrt{2}\right)^2(-0.1) = -0.2$.

19. $y = f(x) = 2x - x^2$, $x = 2$, $\Delta x = -0.4 \quad \Rightarrow$

$\Delta y = f(1.6) - f(2) = 0.64 - 0 = 0.64$

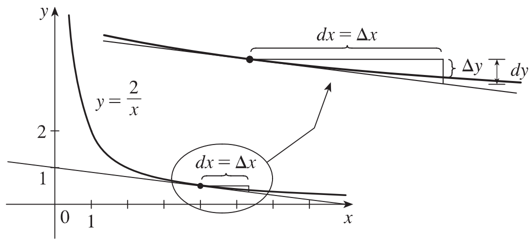
$dy = (2 - 2x)\, dx = (2 - 4)(-0.4) = 0.8$



21. $y = f(x) = 2/x$, $x = 4$, $\Delta x = 1 \quad \Rightarrow$

$\Delta y = f(5) - f(4) = \frac{2}{5} - \frac{2}{4} = -0.1$

$dy = -\dfrac{2}{x^2}\, dx = -\dfrac{2}{4^2}(1) = -0.125$



23. To estimate $(2.001)^5$, we'll find the linearization of $f(x) = x^5$ at $a = 2$. Since $f'(x) = 5x^4$, $f(2) = 32$, and $f'(2) = 80$, we have $L(x) = 32 + 80(x - 2) = 80x - 128$. Thus, $x^5 \approx 80x - 128$ when x is near 2 , so $(2.001)^5 \approx 80(2.001) - 128 = 160.08 - 128 = 32.08$.

25. To estimate $(8.06)^{2/3}$, we'll find the linearization of $f(x) = x^{2/3}$ at $a = 8$. Since $f'(x) = \frac{2}{3}x^{-1/3} = 2/\left(3\sqrt[3]{x}\right)$, $f(8) = 4$, and $f'(8) = \frac{1}{3}$, we have $L(x) = 4 + \frac{1}{3}(x - 8) = \frac{1}{3}x + \frac{4}{3}$. Thus, $x^{2/3} \approx \frac{1}{3}x + \frac{4}{3}$ when x is near 8, so $(8.06)^{2/3} \approx \frac{1}{3}(8.06) + \frac{4}{3} = \frac{12.06}{3} = 4.02$.

27. $y = f(x) = \tan x \quad \Rightarrow \quad dy = \sec^2 x\, dx$. When $x = 45^\circ$ and $dx = -1^\circ$, $dy = \sec^2 45^\circ(-\pi/180) = \left(\sqrt{2}\right)^2(-\pi/180) = -\pi/90$, so $\tan 44^\circ = f(44^\circ) \approx f(45^\circ) + dy = 1 - \pi/90 \approx 0.965$.

29. $y = f(x) = \sec x \quad \Rightarrow \quad f'(x) = \sec x\, \tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

31. $y = f(x) = \ln x \quad \Rightarrow \quad f'(x) = 1/x$, so $f(1) = 0$ and $f'(1) = 1$. The linear approximation of f at 1 is $f(1) + f'(1)(x - 1) = 0 + 1(x - 1) = x - 1$. Now $f(1.05) = \ln 1.05 \approx 1.05 - 1 = 0.05$, so the approximation is reasonable.

33. (a) If x is the edge length, then $V = x^3 \quad \Rightarrow \quad dV = 3x^2\, dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\, dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error $=$ relative error $\times\, 100\% = 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \quad \Rightarrow \quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error $\times\, 100\% = 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

35. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \quad \Rightarrow \quad S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \quad \Rightarrow \quad dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27$ cm^2. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \quad \Rightarrow \quad dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179$ cm^3.

The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

37. (a) $V = \pi r^2 h \quad \Rightarrow \quad \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$$\Delta V - dV = [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r = \pi(\Delta r)^2 h.$$

39. $V = RI \quad \Rightarrow \quad I = \dfrac{V}{R} \quad \Rightarrow \quad dI = -\dfrac{V}{R^2}\,dR$. The relative error in calculating I is $\dfrac{\Delta I}{I} \approx \dfrac{dI}{I} = \dfrac{-(V/R^2)\,dR}{V/R} = -\dfrac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

41. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u + v) = \dfrac{d}{dx}(u + v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)\,dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

43. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

3.11 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = 1$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

(b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

7. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^{-x} - e^x) = -\sinh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

11.
$$\begin{aligned}\sinh x\, \cosh y + \cosh x\, \sinh y &= \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)\end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad\Leftrightarrow\quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x + x) = \sinh x\, \cosh x + \cosh x\, \sinh x = 2 \sinh x\, \cosh x$.

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2 - 1)/x}{(x^2+1)/x} = \dfrac{x^2 - 1}{x^2 + 1}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

21. $\operatorname{sech} x = \dfrac{1}{\cosh x} \quad\Rightarrow\quad \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}$.

$\cosh^2 x - \sinh^2 x = 1 \quad\Rightarrow\quad \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \quad\Rightarrow\quad \sinh x = \frac{4}{3}$ [because $x > 0$].

$\operatorname{csch} x = \dfrac{1}{\sinh x} \quad\Rightarrow\quad \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}$.

$\tanh x = \dfrac{\sinh x}{\cosh x} \quad\Rightarrow\quad \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}$.

$\coth x = \dfrac{1}{\tanh x} \quad\Rightarrow\quad \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}$.

23. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - 0}{1 + 0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0 - 1}{0 + 1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim\limits_{x\to0^+} \coth x = \lim\limits_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\lim\limits_{x\to0^-} \coth x = \lim\limits_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$]

$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \;\Rightarrow\; y = \ln\left(x + \sqrt{1 + x^2}\right)$.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1 - x) \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\left(\frac{1+x}{1-x}\right) \;\Rightarrow\; y = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \ge 0 \;\Rightarrow\; \sinh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ [since $\sinh y \ge 0$ for $y \ge 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \;\Rightarrow\; \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \;\Rightarrow\; -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$. By Exercise 13,

$\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$,

so $\coth y = -\sqrt{x^2 + 1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \;\Rightarrow\; -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1 - x^2}}$. [Note that $y > 0$ and so $\tanh y > 0$.]

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1 - x^2}$

by Exercise 13.

31. $f(x) = x \sinh x - \cosh x \Rightarrow f'(x) = x\,(\sinh x)' + \sinh x \cdot 1 - \sinh x = x \cosh x$

33. $h(x) = \ln(\cosh x) \Rightarrow h'(x) = \dfrac{1}{\cosh x}(\cosh x)' = \dfrac{\sinh x}{\cosh x} = \tanh x$

35. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$

37. $f(t) = \operatorname{sech}^2(e^t) = [\operatorname{sech}(e^t)]^2 \Rightarrow$

$f'(t) = 2[\operatorname{sech}(e^t)]\,[\operatorname{sech}(e^t)]' = 2\operatorname{sech}(e^t)\left[-\operatorname{sech}(e^t)\,\tanh(e^t) \cdot e^t\right] = -2e^t \operatorname{sech}^2(e^t)\,\tanh(e^t)$

39. $y = \arctan(\tanh x) \Rightarrow y' = \dfrac{1}{1 + (\tanh x)^2}\,(\tanh x)' = \dfrac{\operatorname{sech}^2 x}{1 + \tanh^2 x}$

41. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x} \Rightarrow$

$$G'(x) = \frac{(1 + \cosh x)(-\sinh x) - (1 - \cosh x)(\sinh x)}{(1 + \cosh x)^2} = \frac{-\sinh x - \sinh x\,\cosh x - \sinh x + \sinh x\,\cosh x}{(1 + \cosh x)^2}$$
$$= \frac{-2\sinh x}{(1 + \cosh x)^2}$$

43. $y = \tanh^{-1}\sqrt{x} \Rightarrow y' = \dfrac{1}{1 - \left(\sqrt{x}\right)^2} \cdot \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}\,(1 - x)}$

45. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2} \Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1 + (x/3)^2}} - \frac{2x}{2\sqrt{9 + x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9 + x^2}} - \frac{x}{\sqrt{9 + x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

47. $y = \coth^{-1}\sqrt{x^2 + 1} \Rightarrow y' = \dfrac{1}{1 - (x^2 + 1)}\dfrac{2x}{2\sqrt{x^2 + 1}} = -\dfrac{1}{x\sqrt{x^2 + 1}}$

49. As the depth d of the water gets large, the fraction $\dfrac{2\pi d}{L}$ gets large, and from Figure 3 or Exercise 23(a), $\tanh\left(\dfrac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\dfrac{gL}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)} \approx \sqrt{\dfrac{gL}{2\pi}(1)} = \sqrt{\dfrac{gL}{2\pi}}$.

51. (a) $y = 20\cosh(x/20) - 15 \Rightarrow y' = 20\sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

53. (a) $y = A\sinh mx + B\cosh mx \Rightarrow y' = mA\cosh mx + mB\sinh mx \Rightarrow$

$y'' = m^2 A\sinh mx + m^2 B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2 y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \quad\Rightarrow\quad 6 = y'(0) = 3A \quad\Rightarrow\quad A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

55. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \quad\Rightarrow\quad x = \sinh^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1 + \sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1 + \sqrt{2}\right), \sqrt{2}\right)$.

57. If $ae^x + be^{-x} = \alpha\cosh(x + \beta)$ [or $\alpha\sinh(x + \beta)$], then

$ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x} e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\frac{a}{b} = \pm e^{2\beta} \quad\Rightarrow\quad (\star)\quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad\Rightarrow\quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad\Rightarrow\quad \alpha^2 = \pm 4ab \quad\Rightarrow\quad \alpha = 2\sqrt{\pm ab}$.

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

3 Review

CONCEPT CHECK

1. (a) The Power Rule: If n is any real number, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.

(b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}f(x)$.

The derivative of a constant times a function is the constant times the derivative of the function.

(c) The Sum Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.

(d) The Difference Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x) - g(x)] = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.

(e) The Product Rule: If f and g are both differentiable, then $\dfrac{d}{dx}[f(x)\,g(x)] = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

(f) The Quotient Rule: If f and g are both differentiable, then $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,\dfrac{d}{dx}\,f(x) - f(x)\,\dfrac{d}{dx}\,g(x)}{[g(x)]^2}$.

The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))\,g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

2. (a) $y = x^n \quad \Rightarrow \quad y' = nx^{n-1}$

(b) $y = e^x \quad \Rightarrow \quad y' = e^x$

(c) $y = a^x \quad \Rightarrow \quad y' = a^x \ln a$

(d) $y = \ln x \quad \Rightarrow \quad y' = 1/x$

(e) $y = \log_a x \quad \Rightarrow \quad y' = 1/(x \ln a)$

(f) $y = \sin x \quad \Rightarrow \quad y' = \cos x$

(g) $y = \cos x \quad \Rightarrow \quad y' = -\sin x$

(h) $y = \tan x \quad \Rightarrow \quad y' = \sec^2 x$

(i) $y = \csc x \quad \Rightarrow \quad y' = -\csc x\,\cot x$

(j) $y = \sec x \quad \Rightarrow \quad y' = \sec x\,\tan x$

(k) $y = \cot x \quad \Rightarrow \quad y' = -\csc^2 x$

(l) $y = \sin^{-1} x \quad \Rightarrow \quad y' = 1/\sqrt{1 - x^2}$

(m) $y = \cos^{-1} x \quad \Rightarrow \quad y' = -1/\sqrt{1 - x^2}$

(n) $y = \tan^{-1} x \quad \Rightarrow \quad y' = 1/(1 + x^2)$

(o) $y = \sinh x \quad \Rightarrow \quad y' = \cosh x$

(p) $y = \cosh x \quad \Rightarrow \quad y' = \sinh x$

(q) $y = \tanh x \quad \Rightarrow \quad y' = \operatorname{sech}^2 x$

(r) $y = \sinh^{-1} x \quad \Rightarrow \quad y' = 1/\sqrt{1 + x^2}$

(s) $y = \cosh^{-1} x \quad \Rightarrow \quad y' = 1/\sqrt{x^2 - 1}$

(t) $y = \tanh^{-1} x \quad \Rightarrow \quad y' = 1/(1 - x^2)$

3. (a) e is the number such that $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

(b) $e = \displaystyle\lim_{x \to 0} (1 + x)^{1/x}$

(c) The differentiation formula for $y = a^x$ $\;[y' = a^x \ln a]\;$ is simplest when $a = e$ because $\ln e = 1$.

(d) The differentiation formula for $y = \log_a x$ $\;[y' = 1/(x \ln a)]\;$ is simplest when $a = e$ because $\ln e = 1$.

4. (a) Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.

(b) Logarithmic differentiation consists of taking natural logarithms of both sides of an equation $y = f(x)$, simplifying, differentiating implicitly with respect to x, and then solving the resulting equation for y'.

5. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.

(b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

(c) See Figure 5 in Section 3.10.

TRUE-FALSE QUIZ

1. True. This is the Sum Rule.

3. True. This is the Chain Rule.

5. False. $\dfrac{d}{dx} f\left(\sqrt{x}\right) = \dfrac{f'\left(\sqrt{x}\right)}{2\sqrt{x}}$ by the Chain Rule.

7. False. $\dfrac{d}{dx} 10^x = 10^x \ln 10$

9. True. $\dfrac{d}{dx}(\tan^2 x) = 2\tan x\sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2\sec x\,(\sec x\,\tan x) = 2\tan x\sec^2 x$.

Or: $\dfrac{d}{dx}(\sec^2 x) = \dfrac{d}{dx}(1+\tan^2 x) = \dfrac{d}{dx}(\tan^2 x)$.

11. True. $g(x) = x^5 \;\Rightarrow\; g'(x) = 5x^4 \;\Rightarrow\; g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,

$\displaystyle\lim_{x\to 2} \frac{g(x)-g(2)}{x-2} = g'(2) = 80.$

EXERCISES

1. $y = (x^4 - 3x^2 + 5)^3 \;\Rightarrow$

$$y' = 3(x^4-3x^2+5)^2 \frac{d}{dx}(x^4-3x^2+5) = 3(x^4-3x^2+5)^2(4x^3-6x) = 6x(x^4-3x^2+5)^2(2x^2-3)$$

3. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \;\Rightarrow\; y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\sqrt[3]{x^7}}$

5. $y = 2x\sqrt{x^2+1} \;\Rightarrow$

$$y' = 2x\cdot\tfrac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1}\,(2) = \frac{2x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1} = \frac{2x^2+2(x^2+1)}{\sqrt{x^2+1}} = \frac{2(2x^2+1)}{\sqrt{x^2+1}}$$

7. $y = e^{\sin 2\theta} \;\Rightarrow\; y' = e^{\sin 2\theta}\dfrac{d}{d\theta}(\sin 2\theta) = e^{\sin 2\theta}(\cos 2\theta)(2) = 2\cos 2\theta\, e^{\sin 2\theta}$

9. $y = \dfrac{t}{1-t^2} \;\Rightarrow\; y' = \dfrac{(1-t^2)(1)-t(-2t)}{(1-t^2)^2} = \dfrac{1-t^2+2t^2}{(1-t^2)^2} = \dfrac{t^2+1}{(1-t^2)^2}$

11. $y = \sqrt{x}\cos\sqrt{x} \;\Rightarrow$

$$y' = \sqrt{x}\left(\cos\sqrt{x}\right)' + \cos\sqrt{x}\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\,\sin\sqrt{x} + \cos\sqrt{x}\right) = \frac{\cos\sqrt{x} - \sqrt{x}\,\sin\sqrt{x}}{2\sqrt{x}}$$

13. $y = \dfrac{e^{1/x}}{x^2} \;\Rightarrow\; y' = \dfrac{x^2(e^{1/x})' - e^{1/x}\left(x^2\right)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1+2x)}{x^4}$

15. $\dfrac{d}{dx}\left(xy^4 + x^2 y\right) = \dfrac{d}{dx}\left(x + 3y\right) \Rightarrow x \cdot 4y^3 y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow$

$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \Rightarrow y' = \dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

17. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \Rightarrow$

$$y' = \frac{(1+\tan 2\theta)(\sec 2\theta\,\tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,[(1+\tan 2\theta)\tan 2\theta - \sec^2 2\theta]}{(1+\tan 2\theta)^2}$$
$$= \frac{2\sec 2\theta\,(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1+\tan 2\theta)^2} \qquad [1 + \tan^2 x = \sec^2 x]$$

19. $y = e^{cx}(c\sin x - \cos x) \Rightarrow$

$$y' = e^{cx}(c\cos x + \sin x) + ce^{cx}(c\sin x - \cos x) = e^{cx}(c^2 \sin x - c\cos x + c\cos x + \sin x)$$
$$= e^{cx}(c^2 \sin x + \sin x) = e^{cx} \sin x\,(c^2 + 1)$$

21. $y = 3^{x\ln x} \Rightarrow y' = 3^{x\ln x} \cdot \ln 3 \cdot \dfrac{d}{dx}(x\ln x) = 3^{x\ln x} \cdot \ln 3\left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) = 3^{x\ln x} \cdot \ln 3(1 + \ln x)$

23. $y = (1 - x^{-1})^{-1} \Rightarrow$

$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

25. $\sin(xy) = x^2 - y \Rightarrow \cos(xy)(xy' + y \cdot 1) = 2x - y' \Rightarrow x\cos(xy)y' + y' = 2x - y\cos(xy) \Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \Rightarrow y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

27. $y = \log_5(1 + 2x) \Rightarrow y' = \dfrac{1}{(1+2x)\ln 5}\dfrac{d}{dx}(1 + 2x) = \dfrac{2}{(1+2x)\ln 5}$

29. $y = \ln\sin x - \frac{1}{2}\sin^2 x \Rightarrow y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2\sin x \cdot \cos x = \cot x - \sin x\,\cos x$

31. $y = x\tan^{-1}(4x) \Rightarrow y' = x \cdot \dfrac{1}{1 + (4x)^2} \cdot 4 + \tan^{-1}(4x) \cdot 1 = \dfrac{4x}{1 + 16x^2} + \tan^{-1}(4x)$

33. $y = \ln|\sec 5x + \tan 5x| \Rightarrow$

$$y' = \frac{1}{\sec 5x + \tan 5x}(\sec 5x\,\tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \frac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$$

35. $y = \cot(3x^2 + 5) \Rightarrow y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

37. $y = \sin\left(\tan\sqrt{1 + x^3}\right) \Rightarrow y' = \cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\left[3x^2/\left(2\sqrt{1 + x^3}\right)\right]$

39. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \Rightarrow y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

41. $y = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln y = \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3) \quad\Rightarrow\quad \dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow$

$y' = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\left[\dfrac{1}{2(x+1)} - \dfrac{5}{2-x} - \dfrac{7}{x+3}\right]$ or $y' = \dfrac{(2-x)^4(3x^2-55x-52)}{2\sqrt{x+1}\,(x+3)^8}$.

43. $y = x\sinh(x^2) \quad\Rightarrow\quad y' = x\cosh(x^2)\cdot 2x + \sinh(x^2)\cdot 1 = 2x^2\cosh(x^2) + \sinh(x^2)$

45. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

47. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = \dfrac{1}{\sqrt{(\sinh x)^2 - 1}}\cdot\cosh x = \dfrac{\cosh x}{\sqrt{\sinh^2 x - 1}}$

49. $y = \cos\left(e^{\sqrt{\tan 3x}}\right) \quad\Rightarrow$

$$y' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' = -\sin\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\cdot\tfrac{1}{2}(\tan 3x)^{-1/2}\cdot\sec^2(3x)\cdot 3$$

$$= \frac{-3\sin\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

51. $f(t) = \sqrt{4t+1} \quad\Rightarrow\quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad\Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

53. $x^6 + y^6 = 1 \quad\Rightarrow\quad 6x^5 + 6y^5y' = 0 \quad\Rightarrow\quad y' = -x^5/y^5 \quad\Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

55. We first show it is true for $n = 1$: $f(x) = xe^x \quad\Rightarrow\quad f'(x) = xe^x + e^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = (x+k)e^x + e^x = [(x+k)+1]e^x = [x+(k+1)]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

57. $y = 4\sin^2 x \quad\Rightarrow\quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

59. $y = \sqrt{1 + 4\sin x} \quad\Rightarrow\quad y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$.

At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

61. $y = (2+x)e^{-x} \quad\Rightarrow\quad y' = (2+x)(-e^{-x}) + e^{-x}\cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1)$.

At $(0,2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

The slope of the normal line is 1, so an equation of the normal line is $y - 2 = 1(x - 0)$, or $y = x + 2$.

63. (a) $f(x) = x\sqrt{5-x} \quad \Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}} = \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}}$$

$$= \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

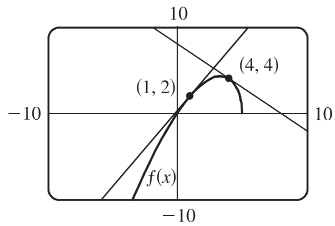
(b) At $(1,2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

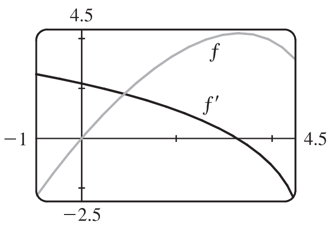
At $(4,4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

65. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

67. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

Or: $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \quad \Rightarrow$

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

69. (a) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$h'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$

(b) $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x))\,g'(x) \quad \Rightarrow \quad F'(2) = f'(g(2))\,g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

71. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

73. $f(x) = [g(x)]^2 \quad \Rightarrow \quad f'(x) = 2[g(x)] \cdot g'(x) = 2g(x)\,g'(x)$

75. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x)\,e^x$

77. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

79. $h(x) = \dfrac{f(x)\,g(x)}{f(x)+g(x)} \quad \Rightarrow$

$$h'(x) = \frac{[f(x)+g(x)]\,[f(x)\,g'(x)+g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x)+g'(x)]}{[f(x)+g(x)]^2}$$
$$= \frac{[f(x)]^2\,g'(x)+f(x)\,g(x)\,f'(x)+f(x)\,g(x)\,g'(x)+[g(x)]^2\,f'(x)-f(x)\,g(x)\,f'(x)-f(x)\,g(x)\,g'(x)}{[f(x)+g(x)]^2}$$
$$= \frac{f'(x)\,[g(x)]^2+g'(x)\,[f(x)]^2}{[f(x)+g(x)]^2}$$

81. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$$

83. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2[\ln(x+4)]^1 \cdot \dfrac{1}{x+4} \cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow$

$x+4 = e^0 \quad \Rightarrow \quad x+4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

85. $y = f(x) = ax^2+bx+c \quad \Rightarrow \quad f'(x) = 2ax+b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a+b = 6$ and $10a+b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad \Rightarrow \quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad \Rightarrow \quad 4 = a+b+c$, so $c = 4+\frac{2}{3}-\frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2+\frac{14}{3}x$.

87. $s(t) = Ae^{-ct}\cos(\omega t+\delta) \quad \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}\,[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\} = -Ae^{-ct}\,[\omega\sin(\omega t+\delta)+c\cos(\omega t+\delta)] \quad \Rightarrow$$
$$a(t) = v'(t) = -A\{e^{-ct}[\omega^2\cos(\omega t+\delta)-c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta)+c\cos(\omega t+\delta)](-ce^{-ct})\}$$
$$= -Ae^{-ct}[\omega^2\cos(\omega t+\delta)-c\omega\sin(\omega t+\delta)-c\omega\sin(\omega t+\delta)-c^2\cos(\omega t+\delta)]$$
$$= -Ae^{-ct}[(\omega^2-c^2)\cos(\omega t+\delta)-2c\omega\sin(\omega t+\delta)] = Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta)+2c\omega\sin(\omega t+\delta)]$$

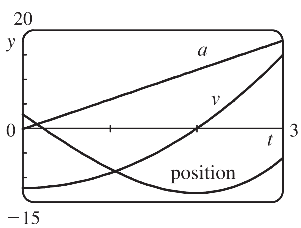
89. (a) $y = t^3-12t+3 \quad \Rightarrow \quad v(t) = y' = 3t^2-12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2-4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3)-y(2) = -6-(-13) = 7$,

Distance downward $= y(0)-y(2) = 3-(-13) = 16$. Total distance $= 7+16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

91. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1+\sqrt{x}\right) = x+x^{3/2} \quad \Rightarrow \quad \rho = dm/dx = 1+\frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1+\frac{3}{2}\sqrt{4} = 4$ kg/m.

93. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \quad \Rightarrow \quad y(0.5) = 200e^{0.5k} = 360 \quad \Rightarrow \quad e^{0.5k} = 1.8 \quad \Rightarrow \quad 0.5k = \ln 1.8 \quad \Rightarrow$

$k = 2\ln 1.8 = \ln(1.8)^2 = \ln 3.24 \quad \Rightarrow \quad y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ bacteria

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ bacteria per hour

(d) $200(3.24)^t = 10{,}000 \quad \Rightarrow \quad (3.24)^t = 50 \quad \Rightarrow \quad t\ln 3.24 = \ln 50 \quad \Rightarrow \quad t = \ln 50/\ln 3.24 \approx 3.33$ hours

95. (a) $C'(t) = -kC(t) \quad \Rightarrow \quad C(t) = C(0)e^{-kt}$ by Theorem 9.4.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \quad \Rightarrow \ln\frac{1}{2} = -30k \quad \Rightarrow$

$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t

such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \quad \Rightarrow \quad \ln 0.1 = -t(\ln 2)/30 \quad \Rightarrow \quad t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

97. If $x =$ edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\,dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$

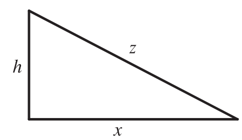
$dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}\ \text{cm}^2/\text{min}$.

99. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \quad \Rightarrow$

$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2h\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \quad \Rightarrow \quad z = \sqrt{45^2 + 60^2} = 75$,

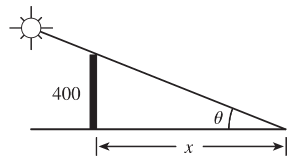
so $\dfrac{dz}{dt} = \dfrac{1}{75}[15(45) + 5(60)] = 13$ ft/s.



101. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \quad \Rightarrow$

$x = 400\cot\theta \quad \Rightarrow \quad \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.



103. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \quad \Rightarrow \quad f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is

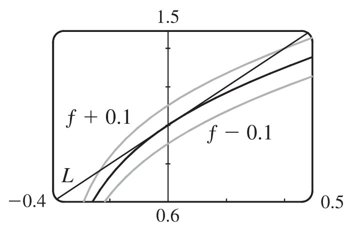
$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \quad \Rightarrow$

$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

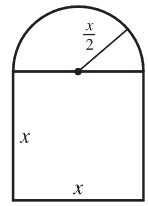
(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy

we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph,

it appears that this is true when $-0.23 < x < 0.40$.



105. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \quad\Rightarrow\quad dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\text{ cm}^2$.



107. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x\,=\,16} = \frac{1}{4}x^{-3/4}\bigg|_{x\,=\,16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

109.
$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} &= \lim_{x\to 0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} \\
&= \lim_{x\to 0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} = \lim_{x\to 0}\frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x} \\
&= \lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)} \\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3\lim_{x\to 0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)} \\
&= 1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)} = \frac{1}{4}
\end{aligned}$$

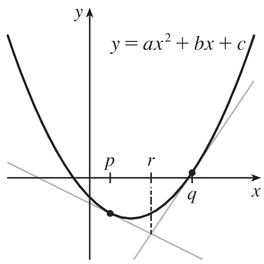
111. $\dfrac{d}{dx}\,[f(2x)] = x^2 \quad\Rightarrow\quad f'(2x)\cdot 2 = x^2 \quad\Rightarrow\quad f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad \Rightarrow \quad a = \frac{\sqrt{3}}{2}$.

Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p+q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or $$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

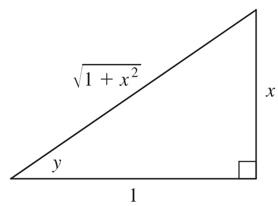
$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$\begin{aligned} [(2aq + b) - (2ap + b)]x - aq^2 + ap^2 &= 0 \\ (2aq - 2ap)x &= aq^2 - ap^2 \\ 2a(q - p)x &= a(q^2 - p^2) \\ x &= \frac{a(q+p)(q-p)}{2a(q-p)} = \frac{p+q}{2} \end{aligned}$$

Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p+q)/2$.

5. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that



$\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1 + x^2}}$. Using this fact we have that

$$\sin(\tan^{-1}(\sinh x)) = \frac{\sinh x}{\sqrt{1 + \sinh^2 x}} = \frac{\sinh x}{\cosh x} = \tanh x.$$

Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.

7. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned} \frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\,\cos x - 4\cos^3 x\,\sin x = 4\sin x\,\cos x\left(\sin^2 x - \cos^2 x\right)x \\ &= -4\sin x\,\cos x\,\cos 2x = -2\sin 2x\,\cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1 \end{aligned}$$

[continued]

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\!\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x + \cos^4 x\right) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right)\right] = \frac{d}{dx}\left[4^{k-1}\cos\!\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\!\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\!\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\!\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\!\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\!\left(4x + (k+1)\,\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\!\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\,\cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4}\cdot 4^n\cos\!\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\!\left(4x + n\dfrac{\pi}{2}\right)$.

9. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \dfrac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad\Leftrightarrow\quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \quad\Leftrightarrow\quad x^2 + x^4 - 2ax^2 + a^2 = 1 \quad\Leftrightarrow$ $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \quad\Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad\Leftrightarrow\quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

11. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360\cdot(2\pi\text{ rad})/(60\text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2\text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha\cdot\dfrac{d\alpha}{dt} = \dfrac{1}{3}\cdot 12\pi\cdot\cos 12\pi t = 4\pi\cos\theta$. When $\theta = \dfrac{\pi}{3}$, we have

$$\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6},\text{ so }\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}\text{ and }\frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56\text{ rad/s.}$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \quad\Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2\cdot 40\,|OP|\cos\theta \quad\Rightarrow\quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad\Rightarrow$

$|OP| = \frac{1}{2}\big(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\,\big) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} = 40\big(\cos\theta + \sqrt{8+\cos^2\theta}\,\big)$ cm

[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\big(\cos\theta + \sqrt{8+\cos^2\theta}\,\big)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8+\cos^2\theta}}\right)\cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8+\cos^2\theta}}\right)\ \text{cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

13. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\dfrac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \quad\Rightarrow\quad \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$. So the statement is true for $n = 1$.

Assume it is true for $n = k$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)\\ &= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]\end{aligned}$$

But

$\sin[bx+(k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta) = \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$.

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx+k\theta)+b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)] = r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)].$$

Therefore, the statement is true for all n by mathematical induction.

15. It seems from the figure that as P approaches the point $(0,2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3,0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3,\infty)$ and $y_T \in (2,\infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0,3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \quad\Rightarrow\quad \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the

tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$,

because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = 1$.

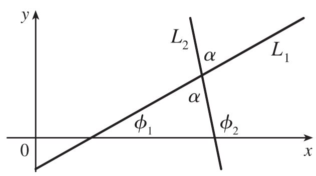
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9} = 1 \quad\Leftrightarrow\quad x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4} = 1 \quad\Leftrightarrow\quad y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \quad\Rightarrow$

$x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \quad\Rightarrow\quad y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

17. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have

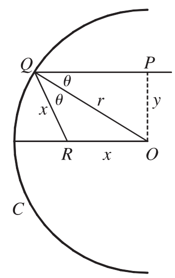


$\tan\alpha = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1}$ and so $\tan\,\alpha = \dfrac{m_2 - m_1}{1 + m_1m_2}$.

(b) (i) The parabolas intersect when $x^2 = (x - 2)^2 \quad\Rightarrow\quad x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1, 1)$ is $m_1 = 2(1) = 2$. If $y = (x - 2)^2$, then $y' = 2(x - 2)$, so the slope of the tangent to $y = (x - 2)^2$ at $(1, 1)$ is $m_2 = 2(1 - 2) = -2$. Therefore, $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_1m_2} = \dfrac{-2 - 2}{1 + 2(-2)} = \dfrac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ$ [or 127°].

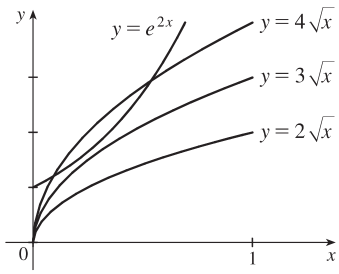
(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \quad\Leftrightarrow\quad 2x(x - 2) = 0 \quad\Rightarrow$ $x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \quad\Rightarrow\quad y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \quad\Rightarrow\quad 2x - 4 + 2yy' = 0 \quad\Rightarrow\quad y' = \dfrac{2 - x}{y}$. At $(2, 1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0 - 2}{1 + 2 \cdot 0} = -2 \quad\Rightarrow\quad \alpha \approx 117^\circ$. At $(2, -1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \dfrac{0 - (-2)}{1 + (-2)\,(0)} = 2 \quad\Rightarrow\quad \alpha \approx 63^\circ$ [or 117°].

19. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



21. $$\lim_{x\to 0}\frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}$$
$$= \lim_{x\to 0}\frac{\sin a\,\cos 2x + \cos a\,\sin 2x - 2\sin a\,\cos x - 2\cos a\,\sin x + \sin a}{x^2}$$
$$= \lim_{x\to 0}\frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2}$$
$$= \lim_{x\to 0}\frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\,\cos x - 2\sin x)}{x^2}$$
$$= \lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2}$$
$$= \lim_{x\to 0}\frac{2(\cos x - 1)[\sin a\,\cos x + \cos a\,\sin x](\cos x + 1)}{x^2(\cos x + 1)}$$
$$= \lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x + 1)} = -2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x + 1} = -2(1)^2\frac{\sin(a+0)}{\cos 0 + 1} = -\sin a$$

23.



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $[k > 0]$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

$$f(a) = g(a) \quad\Rightarrow\quad e^{2a} = k\sqrt{a} \quad (\star)$$

and $$f'(a) = g'(a) \quad\Rightarrow\quad 2e^{2a} = \frac{k}{2\sqrt{a}} \quad\Rightarrow\quad e^{2a} = \frac{k}{4\sqrt{a}}.$$

So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \Rightarrow \left(\sqrt{a}\right)^2 = \dfrac{k}{4k} \Rightarrow a = \frac{1}{4}$. From $(\star)$, $e^{2(1/4)} = k\sqrt{1/4} \Rightarrow k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

25. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}}\arctan\dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{1}{1 + \sin^2 x/(k+\cos x)^2}\cdot\frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2}$$
$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x + 1}{k^2 + 2k\cos x + 1}$$
$$= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak - 1)$.

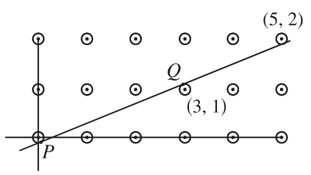
[continued]

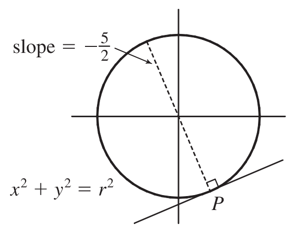
So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak+2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a+\cos x)}$. But $ak-1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$, so $y' = 1/(a+\cos x)$.

27. $y = x^4 - 2x^2 - x \quad\Rightarrow\quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad\Rightarrow\quad (a-b)(a^2+ab+b^2) = (a-b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad\Rightarrow\quad 3(a^2-b^2)(a^2+b^2) = 2(a^2-b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad\Rightarrow\quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2-b^2)(a^2+b^2) = 2(a^2-b^2)$ gives $3(a^2+b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2+ab+b^2) - (a^2+b^2) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad\Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2-1)^2$. So $3a^2 - 1 = 0 \quad\Rightarrow\quad a^2 = \dfrac{1}{3} \quad\Rightarrow\quad b^2 = \dfrac{1}{9a^2} = \dfrac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

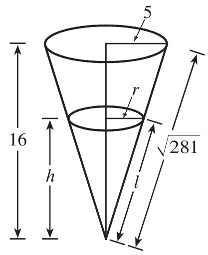
29. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.





To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \quad\Rightarrow\quad x^2 + \frac{25}{4}x^2 = r^2 \quad\Rightarrow\quad x^2 = \frac{4}{29}r^2 \quad\Rightarrow\quad x = \frac{2}{\sqrt{29}}r$, $y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r$, $y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so $m_{PQ} = \dfrac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \dfrac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \dfrac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \dfrac{2}{5} \quad\Rightarrow$ $5\sqrt{29} + 50r = 6\sqrt{29} - 8r \quad\Leftrightarrow\quad 58r = \sqrt{29} \quad\Leftrightarrow\quad r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

31.



By similar triangles, $\frac{r}{5} = \frac{h}{16} \quad \Rightarrow \quad r = \frac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5h}{16}\right)^2 h = \frac{25\pi}{768}h^3$, so $\frac{dV}{dt} = \frac{25\pi}{256}h^2\,\frac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\frac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\frac{dV}{dt}$ and substituting $h = 10$, $\frac{dh}{dt} = -0.3$, $r = \frac{5(10)}{16} = \frac{25}{8}$, and $\frac{l}{\sqrt{281}} = \frac{10}{16} \quad \Leftrightarrow$ $l = \frac{5}{8}\sqrt{281}$, we get $\frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\frac{25}{8}\cdot\frac{5}{8}\sqrt{281} \quad \Leftrightarrow \quad \frac{125k\pi\sqrt{281}}{64} = 2 + \frac{750\pi}{256}$. Solving for k gives us $k = \frac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\frac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\frac{25}{8}\cdot\frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$$

4 □ APPLICATIONS OF DIFFERENTIATION

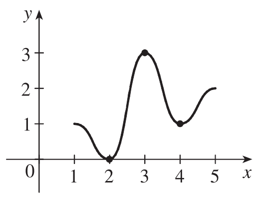
4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

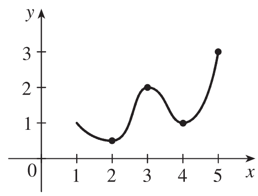
3. Absolute maximum at s, absolute minimum at r, local maximum at c, local minima at b and r, neither a maximum nor a minimum at a and d.

5. Absolute maximum value is $f(4) = 5$; there is no absolute minimum value; local maximum values are $f(4) = 5$ and $f(6) = 4$; local minimum values are $f(2) = 2$ and $f(1) = f(5) = 3$.

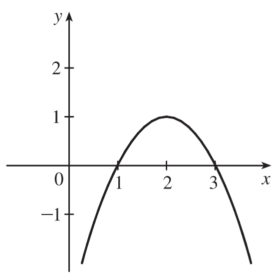
7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4



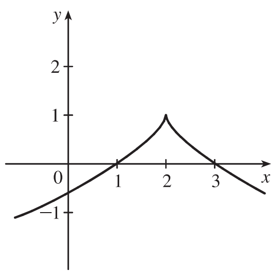
9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



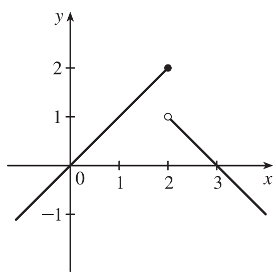
11. (a)



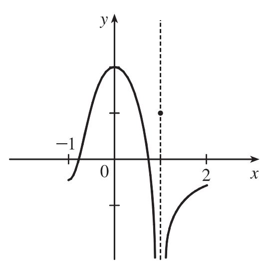
(b)



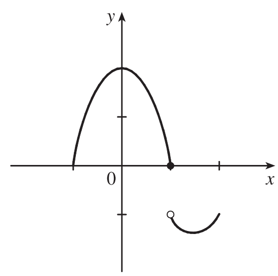
(c)



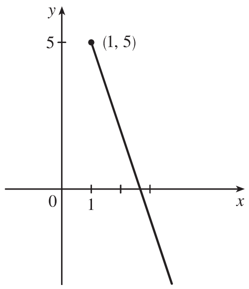
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



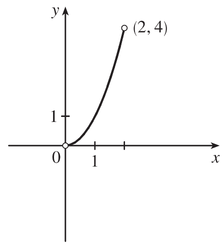
(b)



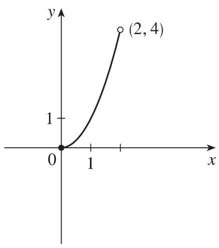
15. $f(x) = 8 - 3x$, $x \geq 1$. Absolute maximum $f(1) = 5$; no local maximum. No absolute or local minimum.



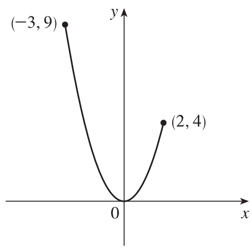
17. $f(x) = x^2$, $0 < x < 2$. No absolute or local maximum or minimum value.



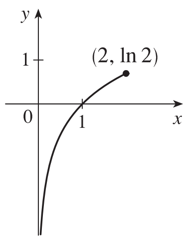
19. $f(x) = x^2$, $0 \leq x < 2$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.



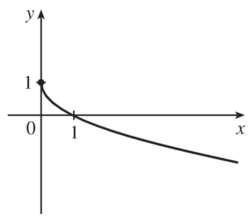
21. $f(x) = x^2$, $-3 \leq x \leq 2$. Absolute maximum $f(-3) = 9$. No local maximum. Absolute and local minimum $f(0) = 0$.



23. $f(x) = \ln x$, $0 < x \leq 2$. Absolute maximum $f(2) = \ln 2 \approx 0.69$; no local maximum. No absolute or local minimum.

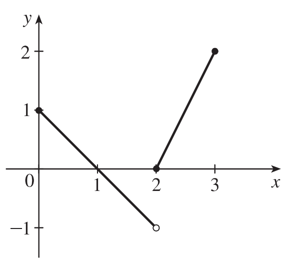


25. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.



27. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < 2 \\ 2x - 4 & \text{if } 2 \leq x \leq 3 \end{cases}$

Absolute maximum $f(3) = 2$; no local maximum. No absolute or local minimum.



29. $f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4$. $f'(x) = 0 \Rightarrow x = -\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

31. $f(x) = x^3 + 3x^2 - 24x \Rightarrow f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$.

$f'(x) = 0 \Rightarrow 3(x+4)(x-2) = 0 \Rightarrow x = -4, 2$. These are the only critical numbers.

33. $s(t) = 3t^4 + 4t^3 - 6t^2 \Rightarrow s'(t) = 12t^3 + 12t^2 - 12t$. $s'(t) = 0 \Rightarrow 12t(t^2 + t - 1) \Rightarrow$

$t = 0$ or $t^2 + t - 1 = 0$. Using the quadratic formula to solve the latter equation gives us

$t = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2} \approx 0.618, -1.618$. The three critical numbers are 0, $\dfrac{-1 \pm \sqrt{5}}{2}$.

35. $g(y) = \dfrac{y-1}{y^2 - y + 1} \Rightarrow$

$$g'(y) = \frac{(y^2 - y + 1)(1) - (y-1)(2y-1)}{(y^2 - y + 1)^2} = \frac{y^2 - y + 1 - (2y^2 - 3y + 1)}{(y^2 - y + 1)^2} = \frac{-y^2 + 2y}{(y^2 - y + 1)^2} = \frac{y(2-y)}{(y^2 - y + 1)^2}.$$

$g'(y) = 0 \Rightarrow y = 0, 2$. The expression $y^2 - y + 1$ is never equal to 0, so $g'(y)$ exists for all real numbers.

The critical numbers are 0 and 2.

37. $h(t) = t^{3/4} - 2t^{1/4} \Rightarrow h'(t) = \frac{3}{4}t^{-1/4} - \frac{2}{4}t^{-3/4} = \frac{1}{4}t^{-3/4}(3t^{1/2} - 2) = \dfrac{3\sqrt{t} - 2}{4\sqrt[4]{t^3}}$.

$h'(t) = 0 \Rightarrow 3\sqrt{t} = 2 \Rightarrow \sqrt{t} = \frac{2}{3} \Rightarrow t = \frac{4}{9}$. $h'(t)$ does not exist at $t = 0$, so the critical numbers are 0 and $\frac{4}{9}$.

39. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$

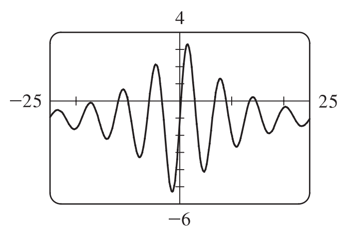
$$\begin{aligned} F'(x) &= x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x-4)[5 \cdot x \cdot 2 + (x-4) \cdot 4] \\ &= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}} \end{aligned}$$

$F'(x) = 0 \Rightarrow x = 4, \frac{8}{7}$. $F'(0)$ does not exist. Thus, the three critical numbers are 0, $\frac{8}{7}$, and 4.

41. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta(\cos\theta - 1) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ [n an integer] or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

43. $f(x) = x^2e^{-3x} \Rightarrow f'(x) = x^2(-3e^{-3x}) + e^{-3x}(2x) = xe^{-3x}(-3x + 2)$. $f'(x) = 0 \Rightarrow x = 0, \frac{2}{3}$ [e^{-3x} is never equal to 0]. $f'(x)$ always exists, so the critical numbers are 0 and $\frac{2}{3}$.

45. The graph of $f'(x) = 5e^{-0.1|x|}\sin x - 1$ has 10 zeros and exists everywhere, so f has 10 critical numbers.



47. $f(x) = 3x^2 - 12x + 5,\ [0, 3].\quad f'(x) = 6x - 12 = 0 \quad\Leftrightarrow\quad x = 2$. Applying the Closed Interval Method, we find that $f(0) = 5$, $f(2) = -7$, and $f(3) = -4$. So $f(0) = 5$ is the absolute maximum value and $f(2) = -7$ is the absolute minimum value.

49. $f(x) = 2x^3 - 3x^2 - 12x + 1,\ [-2, 3].\quad f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \quad\Leftrightarrow$ $x = 2, -1.\quad f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

51. $f(x) = x^4 - 2x^2 + 3,\ [-2, 3].\quad f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1) = 0 \quad\Leftrightarrow\quad x = -1, 0, 1.$ $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, $f(3) = 66$. So $f(3) = 66$ is the absolute maximum value and $f(\pm 1) = 2$ is the absolute minimum value.

53. $f(x) = \dfrac{x}{x^2 + 1},\ [0, 2].\quad f'(x) = \dfrac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2} = 0 \quad\Leftrightarrow\quad x = \pm 1$, but -1 is not in $[0, 2]$. $\quad f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

55. $f(t) = t\sqrt{4 - t^2},\ [-1, 2].$

$$f'(t) = t \cdot \tfrac{1}{2}(4 - t^2)^{-1/2}(-2t) + (4 - t^2)^{1/2} \cdot 1 = \frac{-t^2}{\sqrt{4 - t^2}} + \sqrt{4 - t^2} = \frac{-t^2 + (4 - t^2)}{\sqrt{4 - t^2}} = \frac{4 - 2t^2}{\sqrt{4 - t^2}}.$$

$f'(t) = 0 \quad\Rightarrow\quad 4 - 2t^2 = 0 \quad\Rightarrow\quad t^2 = 2 \quad\Rightarrow\quad t = \pm\sqrt{2}$, but $t = -\sqrt{2}$ is not in the given interval, $[-1, 2]$.

$f'(t)$ does not exist if $4 - t^2 = 0 \quad\Rightarrow\quad t = \pm 2$, but -2 is not in the given interval. $f(-1) = -\sqrt{3}$, $f\big(\sqrt{2}\big) = 2$, and $f(2) = 0$. So $f\big(\sqrt{2}\big) = 2$ is the absolute maximum value and $f(-1) = -\sqrt{3}$ is the absolute minimum value.

57. $f(t) = 2\cos t + \sin 2t,\ [0, \pi/2].$

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1).$

$f'(t) = 0 \quad\Rightarrow\quad \sin t = \frac{1}{2}$ or $\sin t = -1 \quad\Rightarrow\quad t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

59. $f(x) = xe^{-x^2/8},\ [-1, 4].\quad f'(x) = x \cdot e^{-x^2/8} \cdot (-\frac{x}{4}) + e^{-x^2/8} \cdot 1 = e^{-x^2/8}(-\frac{x^2}{4} + 1)$. Since $e^{-x^2/8}$ is never 0, $f'(x) = 0 \quad\Rightarrow\quad -x^2/4 + 1 = 0 \quad\Rightarrow\quad 1 = x^2/4 \quad\Rightarrow\quad x^2 = 4 \quad\Rightarrow\quad x = \pm 2$, but -2 is not in the given interval, $[-1, 4]$. $f(-1) = -e^{-1/8} \approx -0.88$, $f(2) = 2e^{-1/2} \approx 1.21$, and $f(4) = 4e^{-2} \approx 0.54$. So $f(2) = 2e^{-1/2}$ is the absolute maximum value and $f(-1) = -e^{-1/8}$ is the absolute minimum value.

61. $f(x) = \ln(x^2 + x + 1),\ [-1, 1].\ f'(x) = \dfrac{1}{x^2 + x + 1} \cdot (2x + 1) = 0 \quad\Leftrightarrow\quad x = -\frac{1}{2}$. Since $x^2 + x + 1 > 0$ for all x, the domain of f and f' is $\mathbb{R}$. $\quad f(-1) = \ln 1 = 0$, $f\big(-\frac{1}{2}\big) = \ln\frac{3}{4} \approx -0.29$, and $f(1) = \ln 3 \approx 1.10$. So $f(1) = \ln 3 \approx 1.10$ is the absolute maximum value and $f\big(-\frac{1}{2}\big) = \ln\frac{3}{4} \approx -0.29$ is the absolute minimum value.

63. $f(x) = x^a(1-x)^b$, $0 \le x \le 1$, $a > 0$, $b > 0$.

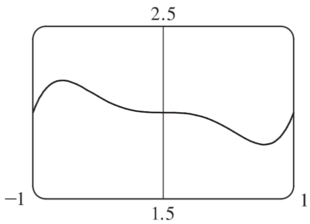
$$f'(x) = x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}[x \cdot b(-1) + (1-x) \cdot a]$$
$$= x^{a-1}(1-x)^{b-1}(a - ax - bx)$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x) = 0 \Leftrightarrow x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

65. (a)



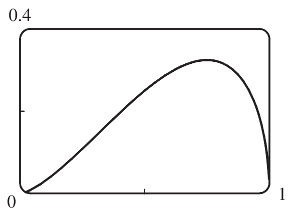
From the graph, it appears that the absolute maximum value is about $f(-0.77) = 2.19$, and the absolute minimum value is about $f(0.77) = 1.81$.

(b) $f(x) = x^5 - x^3 + 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. So $f'(x) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{\frac{3}{5}}$.

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \left(-\sqrt{\tfrac{3}{5}}\right)^5 - \left(-\sqrt{\tfrac{3}{5}}\right)^3 + 2 = -\left(\tfrac{3}{5}\right)^2\sqrt{\tfrac{3}{5}} + \tfrac{3}{5}\sqrt{\tfrac{3}{5}} + 2 = \left(\tfrac{3}{5} - \tfrac{9}{25}\right)\sqrt{\tfrac{3}{5}} + 2 = \tfrac{6}{25}\sqrt{\tfrac{3}{5}} + 2 \text{ (maximum)}$$

and similarly, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (minimum).

67. (a)



From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1-2x}{2\sqrt{x-x^2}} + \sqrt{x-x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x-x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x-x^2}}$.

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3-4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$.

$f(0) = f(1) = 0$ (minimum), and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$ (maximum).

69. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V $\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow \quad V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$.

Setting this equal to 0 and using the quadratic formula to find T, we get

$T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested

in the region $0°\text{C} \le T \le 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$;

$\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

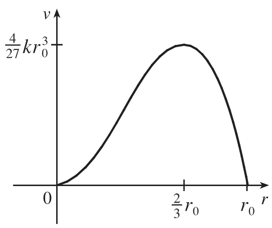
71. Let $a = -0.000\,032\,37$, $b = 0.000\,903\,7$, $c = -0.008\,956$, $d = 0.03629$, $e = -0.04458$, and $f = 0.4074$. Then $S(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f$ and $S'(t) = 5at^4 + 4bt^3 + 3ct^2 + 2dt + e$.

We now apply the Closed Interval Method to the continuous function S on the interval $0 \le t \le 10$. Since S' exists for all t, the only critical numbers of S occur when $S'(t) = 0$. We use a rootfinder on a CAS (or a graphing device) to find that $S'(t) = 0$ when $t_1 \approx 0.855$, $t_2 \approx 4.618$, $t_3 \approx 7.292$, and $t_4 \approx 9.570$. The values of S at these critical numbers are $S(t_1) \approx 0.39$, $S(t_2) \approx 0.43645$, $S(t_3) \approx 0.427$, and $S(t_4) \approx 0.43641$. The values of S at the endpoints of the interval are $S(0) \approx 0.41$ and $S(10) \approx 0.435$. Comparing the six numbers, we see that sugar was most expensive at $t_2 \approx 4.618$ (corresponding roughly to March 1998) and cheapest at $t_1 \approx 0.855$ (June 1994).

73. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \Rightarrow v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \Rightarrow kr(2r_0 - 3r) = 0 \Rightarrow$ $r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



75. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1 \ge 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

77. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

4.2 The Mean Value Theorem

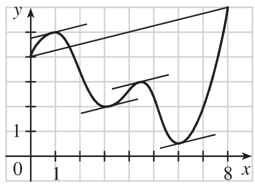
1. $f(x) = 5 - 12x + 3x^2$, $[1, 3]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Also $f(1) = -4 = f(3)$. $f'(c) = 0 \Leftrightarrow -12 + 6c = 0 \Leftrightarrow c = 2$, which is in the open interval $(1, 3)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

3. $f(x) = \sqrt{x} - \frac{1}{3}x$, $[0, 9]$. f, being the difference of a root function and a polynomial, is continuous and differentiable on $[0, \infty)$, so it is continuous on $[0, 9]$ and differentiable on $(0, 9)$. Also, $f(0) = 0 = f(9)$. $f'(c) = 0 \quad \Leftrightarrow$

$\dfrac{1}{2\sqrt{c}} - \dfrac{1}{3} = 0 \quad \Leftrightarrow \quad 2\sqrt{c} = 3 \quad \Leftrightarrow \quad \sqrt{c} = \dfrac{3}{2} \quad \Rightarrow \quad c = \dfrac{9}{4}$, which is in the open interval $(0, 9)$, so $c = \dfrac{9}{4}$ satisfies the conclusion of Rolle's Theorem.

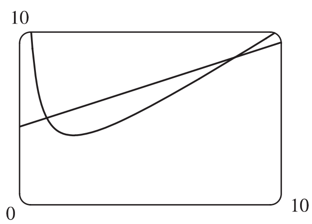
5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

7. $\dfrac{f(8) - f(0)}{8 - 0} = \dfrac{6 - 4}{8} = \dfrac{1}{4}$. The values of c which satisfy $f'(c) = \frac{1}{4}$ seem to be about $c = 0.8, 3.2, 4.4$, and 6.1.



9. (a), (b) The equation of the secant line is

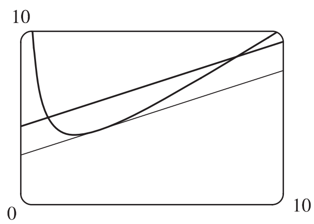
$$y - 5 = \frac{8.5 - 5}{8 - 1}(x - 1) \quad \Leftrightarrow \quad y = \tfrac{1}{2}x + \tfrac{9}{2}.$$



(c) $f(x) = x + 4/x \quad \Rightarrow \quad f'(x) = 1 - 4/x^2$.

So $f'(c) = \frac{1}{2} \quad \Rightarrow \quad c^2 = 8 \quad \Rightarrow \quad c = 2\sqrt{2}$, and $f(c) = 2\sqrt{2} + \frac{4}{2\sqrt{2}} = 3\sqrt{2}$. Thus, an equation of the tangent line is $y - 3\sqrt{2} = \frac{1}{2}\left(x - 2\sqrt{2}\right) \quad \Leftrightarrow$

$y = \frac{1}{2}x + 2\sqrt{2}$.



11. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$. f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 6c + 2 = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{10 - 6}{2} = 2 \quad \Leftrightarrow \quad 6c = 0 \quad \Leftrightarrow$

$c = 0$, which is in $(-1, 1)$.

13. $f(x) = e^{-2x}$, $[0, 3]$. f is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$.

$f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad -2e^{-2c} = \dfrac{e^{-6} - e^{0}}{3 - 0} \quad \Leftrightarrow \quad e^{-2c} = \dfrac{1 - e^{-6}}{6} \quad \Leftrightarrow \quad -2c = \ln\left(\dfrac{1 - e^{-6}}{6}\right) \quad \Leftrightarrow$

$c = -\dfrac{1}{2}\ln\left(\dfrac{1 - e^{-6}}{6}\right) \approx 0.897$, which is in $(0, 3)$.

15. $f(x) = (x-3)^{-2} \Rightarrow f'(x) = -2\,(x-3)^{-3}$. $f(4) - f(1) = f'(c)(4-1) \Rightarrow \dfrac{1}{1^2} - \dfrac{1}{(-2)^2} = \dfrac{-2}{(c-3)^3} \cdot 3 \Rightarrow$

$\dfrac{3}{4} = \dfrac{-6}{(c-3)^3} \Rightarrow (c-3)^3 = -8 \Rightarrow c - 3 = -2 \Rightarrow c = 1$, which is not in the open interval $(1, 4)$. This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$.

17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's Theorem, there is a number r in (a, b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

19. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real roots a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2, 2]$. Hence, it has at most one real root in $[-2, 2]$.

21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4-1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4-1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with

$f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

27. We use Exercise 26 with $f(x) = \sqrt{1+x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and

$f'(x) = \dfrac{1}{2\sqrt{1+x}} < \dfrac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \quad \Rightarrow \quad \sqrt{1+b} < 1 + \frac{1}{2}b$ for $b > 0$.

Another method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1+x}$ or $g(x) = \sqrt{1+x}$ on $[0, b]$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a). By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \le |\cos c|\,|b - a| \le |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \le |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.

31. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

33. Let $f(x) = \arcsin\left(\dfrac{x-1}{x+1}\right) - 2\arctan\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus,

$$f'(x) = \frac{1}{\sqrt{1 - \left(\dfrac{x-1}{x+1}\right)^2}}\,\frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x+1)} - \frac{1}{\sqrt{x}\,(x+1)} = 0.$$

Then $f(x) = C$ on $(0, \infty)$ by Theorem 5. By continuity of f, $f(x) = C$ on $[0, \infty)$. To find C, we let $x = 0 \quad \Rightarrow$ $\arcsin(-1) - 2\arctan(0) + \frac{\pi}{2} = C \quad \Rightarrow \quad -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \quad \Rightarrow$

$\arcsin\left(\dfrac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \frac{\pi}{2}$.

35. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$, where b is the finishing time. Then by the Mean Value Theorem, there is a time c, with $0 < c < b$, such that $f'(c) = \dfrac{f(b) - f(0)}{b - 0}$. But $f(b) = f(0) = 0$, so $f'(c) = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$. So at time c, both runners have the same speed $g'(c) = h'(c)$.

4.3 How Derivatives Affect the Shape of a Graph

1. (a) f is increasing on $(1, 3)$ and $(4, 6)$.
(b) f is decreasing on $(0, 1)$ and $(3, 4)$.
(c) f is concave upward on $(0, 2)$.
(d) f is concave downward on $(2, 4)$ and $(4, 6)$.
(e) The point of inflection is $(2, 3)$.

3. (a) Use the Increasing/Decreasing (I/D) Test.
(b) Use the Concavity Test.
(c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

5. (a) Since $f'(x) > 0$ on $(1, 5)$, f is increasing on this interval. Since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$. Since $f'(x) = 0$ at $x = 5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 5$.

7. There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

9. (a) $f(x) = 2x^3 + 3x^2 - 36x \quad \Rightarrow \quad f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$.

We don't need to include the "6" in the chart to determine the sign of $f'(x)$.

Interval	$x+3$	$x-2$	$f'(x)$	f
$x < -3$	$-$	$-$	$+$	increasing on $(-\infty, -3)$
$-3 < x < 2$	$+$	$-$	$-$	decreasing on $(-3, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = -3$ and from decreasing to increasing at $x = 2$. Thus, $f(-3) = 81$ is a local maximum value and $f(2) = -44$ is a local minimum value.

(c) $f'(x) = 6x^2 + 6x - 36 \quad \Rightarrow \quad f''(x) = 12x + 6$. $\quad f''(x) = 0$ at $x = -\frac{1}{2}$, $f''(x) > 0 \quad \Leftrightarrow \quad x > -\frac{1}{2}$, and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\frac{1}{2}$. Thus, f is concave upward on $\left(-\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{1}{2}\right)$. There is an inflection point at $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, \frac{37}{2}\right)$.

11. (a) $f(x) = x^4 - 2x^2 + 3 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x + 1)(x - 1)$.

Interval	$x+1$	x	$x-1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

13. (a) $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$. $f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow 1 = \dfrac{\sin x}{\cos x} \Rightarrow$ $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, $f'(x) > 0 \Leftrightarrow \cos x - \sin x > 0 \Leftrightarrow \cos x > \sin x \Leftrightarrow 0 < x < \frac{\pi}{4}$ or $\frac{5\pi}{4} < x < 2\pi$ and $f'(x) < 0 \Leftrightarrow \cos x < \sin x \Leftrightarrow \frac{\pi}{4} < x < \frac{5\pi}{4}$. So f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{\pi}{4}$ and from decreasing to increasing at $x = \frac{5\pi}{4}$. Thus, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum value and $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum value.

(c) $f''(x) = -\sin x - \cos x = 0 \Rightarrow -\sin x = \cos x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Divide the interval $(0, 2\pi)$ into subintervals with these numbers as endpoints and complete a second derivative chart.

Interval	$f''(x) = -\sin x - \cos x$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$f''\left(\frac{\pi}{2}\right) = -1 < 0$	downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$f''(\pi) = 1 > 0$	upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$f''\left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3} < 0$	downward

There are inflection points at $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

15. (a) $f(x) = e^{2x} + e^{-x} \Rightarrow f'(x) = 2e^{2x} - e^{-x}$. $f'(x) > 0 \Leftrightarrow 2e^{2x} > e^{-x} \Leftrightarrow e^{3x} > \frac{1}{2} \Leftrightarrow 3x > \ln\frac{1}{2} \Leftrightarrow$ $x > \frac{1}{3}(\ln 1 - \ln 2) \Leftrightarrow x > -\frac{1}{3}\ln 2\ [\approx -0.23]$ and $f'(x) < 0$ if $x < -\frac{1}{3}\ln 2$. So f is increasing on $\left(-\frac{1}{3}\ln 2, \infty\right)$ and f is decreasing on $\left(-\infty, -\frac{1}{3}\ln 2\right)$.

(b) f changes from decreasing to increasing at $x = -\frac{1}{3}\ln 2$. Thus,

$$f\left(-\tfrac{1}{3}\ln 2\right) = f\left(\ln\sqrt[3]{1/2}\right) = e^{2\ln\sqrt[3]{1/2}} + e^{-\ln\sqrt[3]{1/2}} = e^{\ln\sqrt[3]{1/4}} + e^{\ln\sqrt[3]{2}} = \sqrt[3]{1/4} + \sqrt[3]{2} = 2^{-2/3} + 2^{1/3}\ [\approx 1.89]$$

is a local minimum value.

(c) $f''(x) = 4e^{2x} + e^{-x} > 0$ [the sum of two positive terms]. Thus, f is concave upward on $(-\infty, \infty)$ and there is no point of inflection.

17. (a) $y = f(x) = \dfrac{\ln x}{\sqrt{x}}$. (Note that f is only defined for $x > 0$.)

$$f'(x) = \frac{\sqrt{x}\,(1/x) - \ln x\left(\frac{1}{2}x^{-1/2}\right)}{x} = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{\ln x}{2\sqrt{x}}}{x} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{2 - \ln x}{2x^{3/2}} > 0 \Leftrightarrow 2 - \ln x > 0 \Leftrightarrow$$

$\ln x < 2 \Leftrightarrow x < e^2$. Therefore f is increasing on $\left(0, e^2\right)$ and decreasing on $\left(e^2, \infty\right)$.

(b) f changes from increasing to decreasing at $x = e^2$, so $f(e^2) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$ is a local maximum value.

(c) $$f''(x) = \frac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{\left(2x^{3/2}\right)^2} = \frac{-2x^{1/2} + 3x^{1/2}(\ln x - 2)}{4x^3} = \frac{x^{1/2}(-2 + 3\ln x - 6)}{4x^3} = \frac{3\ln x - 8}{4x^{5/2}}$$

$f''(x) = 0 \Leftrightarrow \ln x = \frac{8}{3} \Leftrightarrow x = e^{8/3}$. $f''(x) > 0 \Leftrightarrow x > e^{8/3}$, so f is concave upward on $(e^{8/3}, \infty)$ and concave downward on $(0, e^{8/3})$. There is an inflection point at $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right) \approx (14.39, 0.70)$.

19. $f(x) = x^5 - 5x + 3 \quad\Rightarrow\quad f'(x) = 5x^4 - 5 = 5(x^2+1)(x+1)(x-1)$.

First Derivative Test: $f'(x) < 0 \quad\Rightarrow\quad -1 < x < 1$ and $f'(x) > 0 \quad\Rightarrow\quad x > 1$ or $x < -1$. Since f' changes from positive to negative at $x = -1$, $f(-1) = 7$ is a local maximum value; and since f' changes from negative to positive at $x = 1$, $f(1) = -1$ is a local minimum value.

Second Derivative Test: $f''(x) = 20x^3$. $f'(x) = 0 \quad\Leftrightarrow\quad x = \pm 1$. $f''(-1) = -20 < 0 \quad\Rightarrow\quad f(-1) = 7$ is a local maximum value. $f''(1) = 20 > 0 \quad\Rightarrow\quad f(1) = -1$ is a local minimum value.

Preference: For this function, the two tests are equally easy.

21. $f(x) = x + \sqrt{1-x} \quad\Rightarrow\quad f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \dfrac{1}{2\sqrt{1-x}}$. Note that f is defined for $1 - x \geq 0$; that is, for $x \leq 1$. $f'(x) = 0 \quad\Rightarrow\quad 2\sqrt{1-x} = 1 \quad\Rightarrow\quad \sqrt{1-x} = \frac{1}{2} \quad\Rightarrow\quad 1 - x = \frac{1}{4} \quad\Rightarrow\quad x = \frac{3}{4}$. f' does not exist at $x = 1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x) > 0 \quad\Rightarrow\quad x < \frac{3}{4}$ and $f'(x) < 0 \quad\Rightarrow\quad \frac{3}{4} < x < 1$. Since f' changes from positive to negative at $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

Second Derivative Test: $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) = -\dfrac{1}{4\left(\sqrt{1-x}\right)^3}$.

$f''\left(\frac{3}{4}\right) = -2 < 0 \quad\Rightarrow\quad f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum value.

Preference: The First Derivative Test may be slightly easier to apply in this case.

23. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

25. $f'(0) = f'(2) = f'(4) = 0 \quad\Rightarrow\quad$ horizontal tangents at $x = 0, 2, 4$.

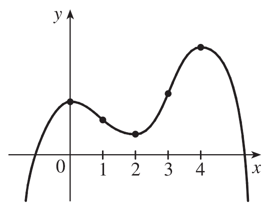
$f'(x) > 0$ if $x < 0$ or $2 < x < 4 \quad\Rightarrow\quad f$ is increasing on $(-\infty, 0)$ and $(2, 4)$.

$f'(x) < 0$ if $0 < x < 2$ or $x > 4 \quad\Rightarrow\quad f$ is decreasing on $(0, 2)$ and $(4, \infty)$.

$f''(x) > 0$ if $1 < x < 3 \quad\Rightarrow\quad f$ is concave upward on $(1, 3)$.

$f''(x) < 0$ if $x < 1$ or $x > 3 \quad\Rightarrow\quad f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$.

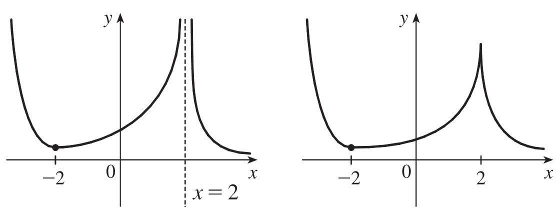
There are inflection points when $x = 1$ and 3.



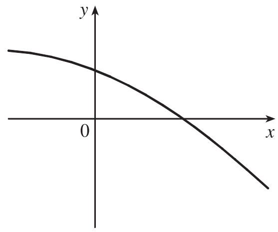
27. $f'(x) > 0$ if $|x| < 2 \quad\Rightarrow\quad f$ is increasing on $(-2, 2)$.

$f'(x) < 0$ if $|x| > 2 \quad\Rightarrow\quad f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $f'(-2) = 0 \quad\Rightarrow\quad$ horizontal tangent at $x = -2$.

$\lim\limits_{x\to 2} |f'(x)| = \infty \quad\Rightarrow\quad$ there is a vertical asymptote or vertical tangent (cusp) at $x = 2$. $f''(x) > 0$ if $x \neq 2 \quad\Rightarrow\quad$ f is concave upward on $(-\infty, 2)$ and $(2, \infty)$.



29. The function must be always decreasing (since the first derivative is always negative) and concave downward (since the second derivative is always negative).



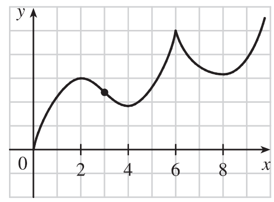
31. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



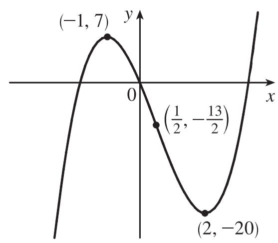
33. (a) $f(x) = 2x^3 - 3x^2 - 12x \quad \Rightarrow \quad f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$.

$f'(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < 2$. So f is increasing on $(-\infty, -1)$ and $(2, \infty)$, and f is decreasing on $(-1, 2)$.

(b) Since f changes from increasing to decreasing at $x = -1$, $f(-1) = 7$ is a local maximum value. Since f changes from decreasing to increasing at $x = 2$, $f(2) = -20$ is a local minimum value.

(c) $f''(x) = 6(2x - 1) \quad \Rightarrow \quad f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$ and $f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$. There is a change in concavity at $x = \frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2}, -\frac{13}{2}\right)$.

(d)

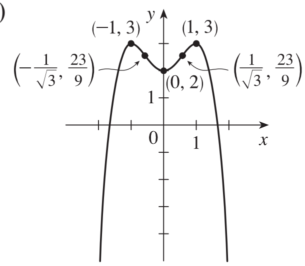


35. (a) $f(x) = 2 + 2x^2 - x^4 \quad \Rightarrow \quad f'(x) = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 + x)(1 - x)$. $\quad f'(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $0 < x < 1$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < 0$ or $x > 1$. So f is increasing on $(-\infty, -1)$ and $(0, 1)$ and f is decreasing on $(-1, 0)$ and $(1, \infty)$.

(b) f changes from increasing to decreasing at $x = -1$ and $x = 1$, so $f(-1) = 3$ and $f(1) = 3$ are local maximum values. f changes from decreasing to increasing at $x = 0$, so $f(0) = 2$ is a local minimum value.

(c) $f''(x) = 4 - 12x^2 = 4(1 - 3x^2)$. $f''(x) = 0 \iff 1 - 3x^2 = 0 \iff x^2 = \frac{1}{3} \iff x = \pm 1/\sqrt{3}$. $f''(x) > 0$ on $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$ and $f''(x) < 0$ on $\left(-\infty, -1/\sqrt{3}\right)$ and $\left(1/\sqrt{3}, \infty\right)$. So f is concave upward on $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$ and f is concave downward on $\left(-\infty, -1/\sqrt{3}\right)$ and $\left(1/\sqrt{3}, \infty\right)$. $f\left(\pm 1/\sqrt{3}\right) = 2 + \frac{2}{3} - \frac{1}{9} = \frac{23}{9}$. There are points of inflection at $\left(\pm 1/\sqrt{3}, \frac{23}{9}\right)$.

(d)

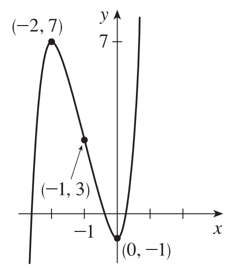


37. (a) $h(x) = (x+1)^5 - 5x - 2 \Rightarrow h'(x) = 5(x+1)^4 - 5$. $h'(x) = 0 \iff 5(x+1)^4 = 5 \iff (x+1)^4 = 1 \Rightarrow (x+1)^2 = 1 \Rightarrow x + 1 = 1$ or $x + 1 = -1 \Rightarrow x = 0$ or $x = -2$. $h'(x) > 0 \iff x < -2$ or $x > 0$ and $h'(x) < 0 \iff -2 < x < 0$. So h is increasing on $(-\infty, -2)$ and $(0, \infty)$ and h is decreasing on $(-2, 0)$.

(b) $h(-2) = 7$ is a local maximum value and $h(0) = -1$ is a local minimum value.

(c) $h''(x) = 20(x+1)^3 = 0 \iff x = -1$. $h''(x) > 0 \iff x > -1$ and $h''(x) < 0 \iff x < -1$, so h is CU on $(-1, \infty)$ and h is CD on $(-\infty, -1)$. There is a point of inflection at $(-1, h(-1)) = (-1, 3)$.

(d)



39. (a) $A(x) = x\sqrt{x+3} \Rightarrow A'(x) = x \cdot \frac{1}{2}(x+3)^{-1/2} + \sqrt{x+3} \cdot 1 = \dfrac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \dfrac{x + 2(x+3)}{2\sqrt{x+3}} = \dfrac{3x+6}{2\sqrt{x+3}}$.

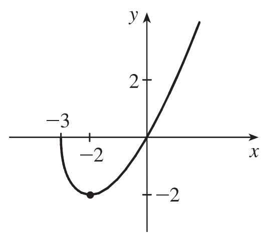
The domain of A is $[-3, \infty)$. $A'(x) > 0$ for $x > -2$ and $A'(x) < 0$ for $-3 < x < -2$, so A is increasing on $(-2, \infty)$ and decreasing on $(-3, -2)$.

(b) $A(-2) = -2$ is a local minimum value.

(c)
$$A''(x) = \frac{2\sqrt{x+3} \cdot 3 - (3x+6) \cdot \dfrac{1}{\sqrt{x+3}}}{\left(2\sqrt{x+3}\right)^2} = \frac{6(x+3) - (3x+6)}{4(x+3)^{3/2}} = \frac{3x+12}{4(x+3)^{3/2}} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$A''(x) > 0$ for all $x > -3$, so A is concave upward on $(-3, \infty)$. There is no inflection point.

(d)



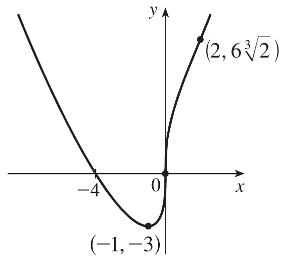
41. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \Rightarrow C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)



43. (a) $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi \quad\Rightarrow\quad f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta) = -2\sin\theta\,(1+\cos\theta)$.

$f'(\theta) = 0 \quad\Leftrightarrow\quad \theta = 0, \pi$, and 2π. $f'(\theta) > 0 \quad\Leftrightarrow\quad \pi < \theta < 2\pi$ and $f'(\theta) < 0 \quad\Leftrightarrow\quad 0 < \theta < \pi$. So f is increasing on $(\pi, 2\pi)$ and f is decreasing on $(0, \pi)$.

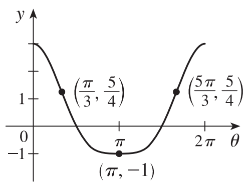
(b) $f(\pi) = -1$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta\,(1+\cos\theta) \quad\Rightarrow$

$$\begin{aligned} f''(\theta) &= -2\sin\theta\,(-\sin\theta) + (1+\cos\theta)(-2\cos\theta) = 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta \\ &= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta = -4\cos^2\theta - 2\cos\theta + 2 \\ &= -2(2\cos^2\theta + \cos\theta - 1) = -2(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \ne \pi$], $f''(\theta) > 0 \quad\Rightarrow\quad 2\cos\theta - 1 < 0 \quad\Rightarrow\quad \cos\theta < \frac{1}{2} \quad\Rightarrow\quad \frac{\pi}{3} < \theta < \frac{5\pi}{3}$ and $f''(\theta) < 0 \quad\Rightarrow\quad \cos\theta > \frac{1}{2} \quad\Rightarrow\quad 0 < \theta < \frac{\pi}{3}$ or $\frac{5\pi}{3} < \theta < 2\pi$. So f is CU on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and f is CD on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are points of inflection at $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5\pi}{3}, \frac{5}{4}\right)$.

(d)



45. $f(x) = \dfrac{x^2}{x^2-1} = \dfrac{x^2}{(x+1)(x-1)}$ has domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(a) $\displaystyle\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2/x^2}{(x^2-1)/x^2} = \lim_{x\to\pm\infty} \frac{1}{1-1/x^2} = \frac{1}{1-0} = 1$, so $y = 1$ is a HA.

$\displaystyle\lim_{x\to -1^-} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2-1) \to 0^+$ as $x \to -1^-$, so $x = -1$ is a VA.

$\displaystyle\lim_{x\to 1^+} \frac{x^2}{x^2-1} = \infty$ since $x^2 \to 1$ and $(x^2-1) \to 0^+$ as $x \to 1^+$, so $x = 1$ is a VA.

(b) $f(x) = \dfrac{x^2}{x^2-1} \quad\Rightarrow\quad f'(x) = \dfrac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \dfrac{2x[(x^2-1)-x^2]}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$. Since $(x^2-1)^2$ is positive for all x in the domain of f, the sign of the derivative is determined by the sign of $-2x$. Thus, $f'(x) > 0$ if $x < 0$ $(x \ne -1)$ and $f'(x) < 0$ if $x > 0$ $(x \ne 1)$. So f is increasing on $(-\infty, -1)$ and $(-1, 0)$, and f is decreasing on $(0, 1)$ and $(1, \infty)$.

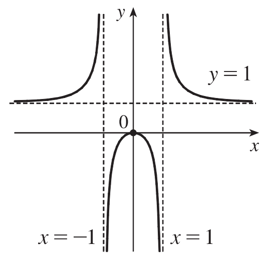
(c) $f'(x) = 0 \quad\Rightarrow\quad x = 0$ and $f(0) = 0$ is a local maximum value.

(d) $f''(x) = \dfrac{(x^2-1)^2(-2) - (-2x)\cdot 2(x^2-1)(2x)}{[(x^2-1)^2]^2}$

$= \dfrac{2(x^2-1)[-(x^2-1)+4x^2]}{(x^2-1)^4} = \dfrac{2(3x^2+1)}{(x^2-1)^3}.$

The sign of $f''(x)$ is determined by the denominator; that is, $f''(x) > 0$ if $|x| > 1$ and $f''(x) < 0$ if $|x| < 1$. Thus, f is CU on $(-\infty, -1)$ and $(1, \infty)$, and f is CD on $(-1, 1)$. There are no inflection points.

(e)



47. (a) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+1} - x\right) = \infty$ and

$\lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - x\right) = \lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - x\right)\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x^2+1}+x} = 0$, so $y = 0$ is a HA.

(b) $f(x) = \sqrt{x^2+1} - x \quad\Rightarrow\quad f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1$. Since $\dfrac{x}{\sqrt{x^2+1}} < 1$ for all x, $f'(x) < 0$, so f is decreasing on $\mathbb{R}$.

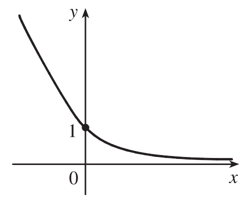
(c) No minimum or maximum

(d) $f''(x) = \dfrac{(x^2+1)^{1/2}(1) - x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2}$

$= \dfrac{(x^2+1)^{1/2} - \dfrac{x^2}{(x^2+1)^{1/2}}}{x^2+1} = \dfrac{(x^2+1)-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0,$

so f is CU on $\mathbb{R}$. No IP

(e)



49. $f(x) = \ln(1 - \ln x)$ is defined when $x > 0$ (so that $\ln x$ is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined]. The second condition is equivalent to $1 > \ln x \quad\Leftrightarrow\quad x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+$, $\ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-$, $\ln x \to 1^-$, so $1 - \ln x \to 0^+$ and $f(x) \to -\infty$. Thus, $x = 0$ and $x = e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x) = \dfrac{1}{1-\ln x}\left(-\dfrac{1}{x}\right) = -\dfrac{1}{x(1-\ln x)} < 0$ on $(0, e)$. Thus, f is decreasing on its domain, $(0, e)$.

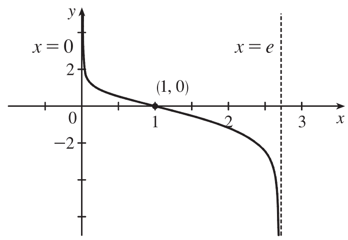
(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $f''(x) = -\dfrac{-[x(1-\ln x)]'}{[x(1-\ln x)]^2} = \dfrac{x(-1/x) + (1-\ln x)}{x^2(1-\ln x)^2}$

$= -\dfrac{\ln x}{x^2(1-\ln x)^2}$

so $f''(x) > 0 \quad\Leftrightarrow\quad \ln x < 0 \quad\Leftrightarrow\quad 0 < x < 1$. Thus, f is CU on $(0, 1)$ and CD on $(1, e)$. There is an inflection point at $(1, 0)$.

(e)



51. (a) $\lim\limits_{x\to\pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x\to-1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$,

$\lim\limits_{x\to-1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA.

(b) $f(x) = e^{-1/(x+1)} \quad\Rightarrow\quad f'(x) = e^{-1/(x+1)}\left[-(-1)\dfrac{1}{(x+1)^2}\right]$ [Reciprocal Rule] $= e^{-1/(x+1)}/(x+1)^2 \quad\Rightarrow$

$f'(x) > 0$ for all x except -1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

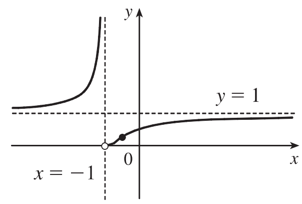
(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(x+1)^2 e^{-1/(x+1)}\left[1/(x+1)^2\right] - e^{-1/(x+1)}\left[2(x+1)\right]}{[(x+1)^2]^2}$

$= \dfrac{e^{-1/(x+1)}\left[1-(2x+2)\right]}{(x+1)^4} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \quad \Rightarrow$

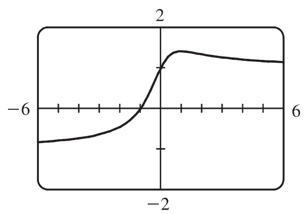
$f''(x) > 0 \quad \Leftrightarrow \quad 2x+1 < 0 \quad \Leftrightarrow \quad x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

(e)



53. The nonnegative factors $(x+1)^2$ and $(x-6)^4$ do not affect the sign of $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. So $f'(x) > 0 \quad \Rightarrow \quad (x-3)^5 > 0 \quad \Rightarrow \quad x - 3 > 0 \quad \Rightarrow \quad x > 3$. Thus, f is increasing on the interval $(3, \infty)$.

55. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

$f(x) = \dfrac{x+1}{\sqrt{x^2+1}} \quad \Rightarrow \quad f'(x) = \dfrac{1-x}{(x^2+1)^{3/2}}$.

$f'(x) = 0 \quad \Leftrightarrow \quad x = 1.\ \ f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

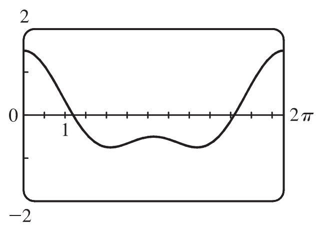
(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$f''(x) = \dfrac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \quad \Leftrightarrow \quad x = \dfrac{3 \pm \sqrt{17}}{4}$. $x = \dfrac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f'.

The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4}, \sqrt{\frac{7}{6} - \frac{\sqrt{17}}{6}}\right) \approx (-0.28, 0.69)$.

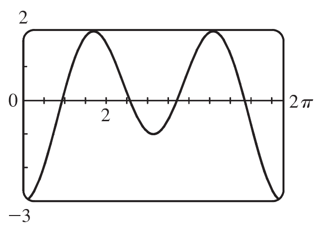
57. $f(x) = \cos x + \frac{1}{2}\cos 2x \quad \Rightarrow \quad f'(x) = -\sin x - \sin 2x \quad \Rightarrow \quad f''(x) = -\cos x - 2\cos 2x$

(a)



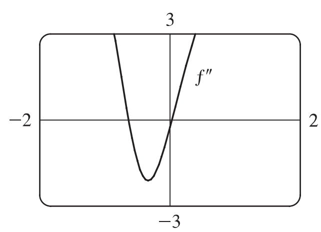
From the graph of f, it seems that f is CD on $(0, 1)$, CU on $(1, 2.5)$, CD on $(2.5, 3.7)$, CU on $(3.7, 5.3)$, and CD on $(5.3, 2\pi)$. The points of inflection appear to be at $(1, 0.4)$, $(2.5, -0.6)$, $(3.7, -0.6)$, and $(5.3, 0.4)$.

(b)



From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.94)$, CU on $(0.94, 2.57)$, CD on $(2.57, 3.71)$, CU on $(3.71, 5.35)$, and CD on $(5.35, 2\pi)$. Refined estimates of the inflection points are $(0.94, 0.44)$, $(2.57, -0.63)$, $(3.71, -0.63)$, and $(5.35, 0.44)$.

59. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-2..2);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-2,2}]`. We see that $f'' > 0$ for $x < -0.6$ and $x > 0.0$ $[\approx 0.03]$ and $f'' < 0$ for $-0.6 < x < 0.0$. So f is CU on $(-\infty, -0.6)$ and $(0.0, \infty)$ and CD on $(-0.6, 0.0)$.



61. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

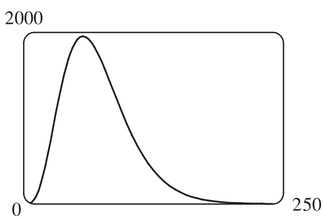
(b) The rate of increase has its maximum value at $t = 8$ hours.

(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

63. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

65. $S(t) = At^p e^{-kt}$ with $A = 0.01$, $p = 4$, and $k = 0.07$. We will find the zeros of f'' for $f(t) = t^p e^{-kt}$.



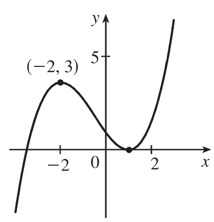
$f'(t) = t^p(-ke^{-kt}) + e^{-kt}(pt^{p-1}) = e^{-kt}(-kt^p + pt^{p-1})$

$$\begin{aligned} f''(t) &= e^{-kt}(-kpt^{p-1} + p(p-1)t^{p-2}) + (-kt^p + pt^{p-1})(-ke^{-kt}) \\ &= t^{p-2}e^{-kt}[-kpt + p(p-1) + k^2t^2 - kpt] \\ &= t^{p-2}e^{-kt}(k^2t^2 - 2kpt + p^2 - p) \end{aligned}$$

Using the given values of p and k gives us $f''(t) = t^2e^{-0.07t}(0.0049t^2 - 0.56t + 12)$. So $S''(t) = 0.01f''(t)$ and its zeros are $t = 0$ and the solutions of $0.0049t^2 - 0.56t + 12 = 0$, which are $t_1 = \frac{200}{7} \approx 28.57$ and $t_2 = \frac{600}{7} \approx 85.71$.

At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest and at t_2 minutes, the rate of decrease is the greatest.

67. $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$.



We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)$.

69. $y = \dfrac{1+x}{1+x^2} \quad \Rightarrow \quad y' = \dfrac{(1+x^2)(1) - (1+x)(2x)}{(1+x^2)^2} = \dfrac{1 - 2x - x^2}{(1+x^2)^2} \quad \Rightarrow$

$$\begin{aligned} y'' &= \frac{(1+x^2)^2(-2-2x) - (1-2x-x^2)\cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2} = \frac{2(1+x^2)[(1+x^2)(-1-x) - (1-2x-x^2)(2x)]}{(1+x^2)^4} \\ &= \frac{2(-1-x-x^2-x^3-2x+4x^2+2x^3)}{(1+x^2)^3} = \frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3} = \frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3} \end{aligned}$$

So $y'' = 0 \quad \Rightarrow \quad x = 1, -2 \pm \sqrt{3}$. Let $a = -2 - \sqrt{3}$, $b = -2 + \sqrt{3}$, and $c = 1$. We can show that $f(a) = \frac{1}{4}\left(1 - \sqrt{3}\right)$,

$f(b) = \frac{1}{4}\left(1+\sqrt{3}\right)$, and $f(c) = 1$. To show that these three points of inflection lie on one straight line, we'll show that the slopes m_{ac} and m_{bc} are equal.

$$m_{ac} = \frac{f(c)-f(a)}{c-a} = \frac{1-\frac{1}{4}\left(1-\sqrt{3}\right)}{1-\left(-2-\sqrt{3}\right)} = \frac{\frac{3}{4}+\frac{1}{4}\sqrt{3}}{3+\sqrt{3}} = \frac{1}{4}$$

$$m_{bc} = \frac{f(c)-f(b)}{c-b} = \frac{1-\frac{1}{4}\left(1+\sqrt{3}\right)}{1-\left(-2+\sqrt{3}\right)} = \frac{\frac{3}{4}-\frac{1}{4}\sqrt{3}}{3-\sqrt{3}} = \frac{1}{4}$$

71. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

73. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \;\;\Rightarrow\;\; (fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \;\;\Rightarrow$ fg is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \;\;\Rightarrow\;\; fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

75. $f(x) = \tan x - x \;\;\Rightarrow\;\; f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \;\;\Rightarrow\;\; \tan x - x > 0 \;\;\Rightarrow\;\; \tan x > x$ for $0 < x < \frac{\pi}{2}$.

77. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \Leftrightarrow x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us

$$f(x) = a[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3]$$

Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

79. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

81. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \Rightarrow$ $g''(x) = 2x(x^2)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

83. (a) $f(x) = x^4 \sin\dfrac{1}{x} \Rightarrow f'(x) = x^4 \cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) + \sin\dfrac{1}{x}\,(4x^3) = 4x^3 \sin\dfrac{1}{x} - x^2 \cos\dfrac{1}{x}$.

$g(x) = x^4\left(2 + \sin\dfrac{1}{x}\right) = 2x^4 + f(x) \Rightarrow g'(x) = 8x^3 + f'(x)$.

$h(x) = x^4\left(-2 + \sin\dfrac{1}{x}\right) = -2x^4 + f(x) \Rightarrow h'(x) = -8x^3 + f'(x)$.

It is given that $f(0) = 0$, so $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x^4 \sin\frac{1}{x} - 0}{x} = \lim\limits_{x\to 0} x^3 \sin\dfrac{1}{x}$. Since $-|x^3| \le x^3 \sin\dfrac{1}{x} \le |x^3|$ and $\lim\limits_{x\to 0} |x^3| = 0$, we see that $f'(0) = 0$ by the Squeeze Theorem. Also, $g'(0) = 8(0)^3 + f'(0) = 0$ and $h'(0) = -8(0)^3 + f'(0) = 0$, so 0 is a critical number of f, g, and h.

For $x_{2n} = \dfrac{1}{2n\pi}$ [n a nonzero integer], $\sin\dfrac{1}{x_{2n}} = \sin 2n\pi = 0$ and $\cos\dfrac{1}{x_{2n}} = \cos 2n\pi = 1$, so $f'(x_{2n}) = -x_{2n}^2 < 0$.

For $x_{2n+1} = \dfrac{1}{(2n+1)\pi}$, $\sin\dfrac{1}{x_{2n+1}} = \sin(2n+1)\pi = 0$ and $\cos\dfrac{1}{x_{2n+1}} = \cos(2n+1)\pi = -1$, so $f'(x_{2n+1}) = x_{2n+1}^2 > 0$. Thus, f' changes sign infinitely often on both sides of 0.

Next, $g'(x_{2n}) = 8x_{2n}^3 + f'(x_{2n}) = 8x_{2n}^3 - x_{2n}^2 = x_{2n}^2(8x_{2n} - 1) < 0$ for $x_{2n} < \frac{1}{8}$, but $g'(x_{2n+1}) = 8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(8x_{2n+1} + 1) > 0$ for $x_{2n+1} > -\frac{1}{8}$, so g' changes sign infinitely often on both sides of 0.

Last, $h'(x_{2n}) = -8x_{2n}^3 + f'(x_{2n}) = -8x_{2n}^3 - x_{2n}^2 = -x_{2n}^2(8x_{2n} + 1) < 0$ for $x_{2n} > -\frac{1}{8}$ and $h'(x_{2n+1}) = -8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(-8x_{2n+1} + 1) > 0$ for $x_{2n+1} < \frac{1}{8}$, so h' changes sign infinitely often on both sides of 0.

(b) $f(0) = 0$ and since $\sin \frac{1}{x}$ and hence $x^4 \sin \frac{1}{x}$ is both positive and negative inifinitely often on both sides of 0, and arbitrarily close to 0, f has neither a local maximum nor a local minimum at 0.

Since $2 + \sin \frac{1}{x} \geq 1$, $g(x) = x^4 \left(2 + \sin \frac{1}{x}\right) > 0$ for $x \neq 0$, so $g(0) = 0$ is a local minimum.

Since $-2 + \sin \frac{1}{x} \leq -1$, $h(x) = x^4 \left(-2 + \sin \frac{1}{x}\right) < 0$ for $x \neq 0$, so $h(0) = 0$ is a local maximum.

4.4 Indeterminate Forms and L'Hospital's Rule

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\stackrel{\text{H}}{=}$

1. (a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim\limits_{x\to a} \dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim\limits_{x\to a} \dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim\limits_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim\limits_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim\limits_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim\limits_{x\to a} [p(x) + q(x)] = \infty$.

5. This limit has the form $\frac{0}{0}$. We can simply factor and simplify to evaluate the limit.

$$\lim_{x\to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{x+1}{x} = \frac{1+1}{1} = 2$$

7. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1} \dfrac{x^9 - 1}{x^5 - 1} \stackrel{\text{H}}{=} \lim\limits_{x\to 1} \dfrac{9x^8}{5x^4} = \dfrac{9}{5} \lim\limits_{x\to 1} x^4 = \dfrac{9}{5}(1) = \dfrac{9}{5}$

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to(\pi/2)^+} \dfrac{\cos x}{1 - \sin x} \stackrel{\text{H}}{=} \lim\limits_{x\to(\pi/2)^+} \dfrac{-\sin x}{-\cos x} = \lim\limits_{x\to(\pi/2)^+} \tan x = -\infty$.

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0} \dfrac{e^t - 1}{t^3} \stackrel{\text{H}}{=} \lim\limits_{t\to 0} \dfrac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

13. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\tan px}{\tan qx} \overset{\text{H}}{=} \lim_{x\to 0}\frac{p\sec^2 px}{q\sec^2 qx} = \frac{p(1)^2}{q(1)^2} = \frac{p}{q}$

15. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$

17. $\displaystyle\lim_{x\to 0^+}[(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x \to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

19. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^3} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{3x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{6x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{6} = \infty$

21. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x - 1}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$

23. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\tanh x}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\operatorname{sech}^2 x}{\sec^2 x} = \frac{\operatorname{sech}^2 0}{\sec^2 0} = \frac{1}{1} = 1$

25. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{t\to 0}\frac{5^t - 3^t}{t} \overset{\text{H}}{=} \lim_{t\to 0}\frac{5^t\ln 5 - 3^t\ln 3}{1} = \ln 5 - \ln 3 = \ln\tfrac{5}{3}$

27. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1} = \lim_{x\to 0}\frac{1}{\sqrt{1-x^2}} = \frac{1}{1} = 1$

29. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sin x}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$

31. $\displaystyle\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} = \frac{0+0}{0+1} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

33. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x} \overset{\text{H}}{=} \lim_{x\to 1}\frac{-1+1/x}{-\pi\sin\pi x} \overset{\text{H}}{=} \lim_{x\to 1}\frac{-1/x^2}{-\pi^2\cos\pi x} = \frac{-1}{-\pi^2(-1)} = -\frac{1}{\pi^2}$

35. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x^a - ax + a - 1}{(x-1)^2} \overset{\text{H}}{=} \lim_{x\to 1}\frac{ax^{a-1} - a}{2(x-1)} \overset{\text{H}}{=} \lim_{x\to 1}\frac{a(a-1)x^{a-2}}{2} = \frac{a(a-1)}{2}$

37. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-\sin x + x}{4x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-\cos x + 1}{12x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sin x}{24x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cos x}{24} = \frac{1}{24}$

39. This limit has the form $\infty\cdot 0$.

$$\lim_{x\to\infty} x\sin(\pi/x) = \lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2} = \pi\lim_{x\to\infty}\cos(\pi/x) = \pi(1) = \pi$$

41. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to 0}\cot 2x\sin 6x = \lim_{x\to 0}\frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$$

43. This limit has the form $\infty\cdot 0$. $\displaystyle\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty}\frac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty}\frac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{3}{4xe^{x^2}} = 0$

45. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to1^+} \ln x\,\tan(\pi x/2) = \lim_{x\to1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

47. This limit has the form $\infty - \infty$.

$$\lim_{x\to1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to1} \frac{x\ln x - (x-1)}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to1} \frac{\ln x}{1 - (1/x) + \ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} = \lim_{x\to1} \frac{x}{1+x} = \frac{1}{1+1} = \frac{1}{2}$$

49. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1} \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

As an alternate solution, write $\sqrt{x^2+x} - x$ as $\sqrt{x^2+x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x} - 1\right)/(1/x)$, and apply l'Hospital's Rule.

51. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \text{ since } \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

53. $y = x^{x^2} \quad\Rightarrow\quad \ln y = x^2 \ln x$, so $\displaystyle\lim_{x\to0^+} \ln y = \lim_{x\to0^+} x^2\ln x = \lim_{x\to0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to0^+} \frac{1/x}{-2/x^3} = \lim_{x\to0^+}\left(-\frac{1}{2}x^2\right) = 0 \quad\Rightarrow$

$$\lim_{x\to0^+} x^{x^2} = \lim_{x\to0^+} e^{\ln y} = e^0 = 1.$$

55. $y = (1-2x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\displaystyle\lim_{x\to0} \ln y = \lim_{x\to0} \frac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim_{x\to0} \frac{-2/(1-2x)}{1} = -2 \quad\Rightarrow$

$$\lim_{x\to0}(1-2x)^{1/x} = \lim_{x\to0} e^{\ln y} = e^{-2}.$$

57. $y = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x \quad\Rightarrow\quad \ln y = x\ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right) \quad\Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\dfrac{3}{x^2} - \dfrac{10}{x^3}\right)\Big/\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + \dfrac{10}{x}}{1 + \dfrac{3}{x} + \dfrac{5}{x^2}} = 3,$$

so $\displaystyle\lim_{x\to\infty}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$

59. $y = x^{1/x} \quad\Rightarrow\quad \ln y = (1/x)\ln x \quad\Rightarrow\quad \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad\Rightarrow$

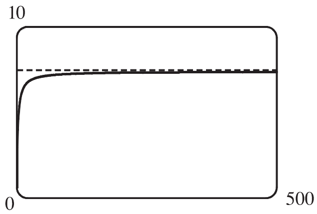
$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

61. $y = (4x+1)^{\cot x} \;\Rightarrow\; \ln y = \cot x \, \ln(4x+1)$, so $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{\sec^2 x} = 4 \;\Rightarrow$

$\displaystyle\lim_{x\to 0^+} (4x+1)^{\cot x} = \lim_{x\to 0^+} e^{\ln y} = e^4.$

63. $y = (\cos x)^{1/x^2} \;\Rightarrow\; \ln y = \dfrac{1}{x^2} \ln\cos x \;\Rightarrow\; \displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2}$

$\Rightarrow\; \displaystyle\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$

65. 

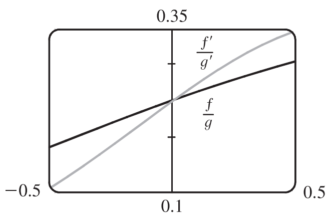
From the graph, if $x = 500$, $y \approx 7.36$. The limit has the form 1^∞.

Now $y = \left(1 + \dfrac{2}{x}\right)^x \;\Rightarrow\; \ln y = x \ln\left(1 + \dfrac{2}{x}\right) \;\Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\left(-\frac{2}{x^2}\right)}{-1/x^2}$$

$$= 2 \lim_{x\to\infty} \frac{1}{1+2/x} = 2(1) = 2 \quad\Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2 \;\; [\approx 7.39]$$

67. 

From the graph, it appears that $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0.25$.

We calculate $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{3x^2 + 4} = \frac{1}{4}$.

69. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{n!} = \infty$

71. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2/x^2 + 1/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{1} = 1$

73. First we will find $\displaystyle\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1 + \dfrac{r}{n}\right)^{nt} \;\Rightarrow\; \ln y = nt \ln\left(1 + \dfrac{r}{n}\right)$, so

$\displaystyle\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt \ln\left(1 + \frac{r}{n}\right) = t \lim_{n\to\infty} \frac{\ln(1+r/n)}{1/n} \overset{\text{H}}{=} t \lim_{n\to\infty} \frac{(-r/n^2)}{(1+r/n)(-1/n^2)} = t \lim_{n\to\infty} \frac{r}{1+i/n} = tr \;\Rightarrow$

$\displaystyle\lim_{n\to\infty} y = e^{rt}$. Thus, as $n \to \infty$, $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt} \to A_0 e^{rt}$.

75. $\lim\limits_{E\to 0^+} P(E) = \lim\limits_{E\to 0^+}\left(\dfrac{e^E+e^{-E}}{e^E-e^{-E}} - \dfrac{1}{E}\right)$

$$= \lim_{E\to 0^+}\frac{E\left(e^E+e^{-E}\right)-1\left(e^E-e^{-E}\right)}{\left(e^E-e^{-E}\right)E} = \lim_{E\to 0^+}\frac{Ee^E+Ee^{-E}-e^E+e^{-E}}{Ee^E-Ee^{-E}} \qquad \left[\text{form is } \tfrac{0}{0}\right]$$

$$\overset{\text{H}}{=} \lim_{E\to 0^+}\frac{Ee^E+e^E\cdot 1+E\left(-e^{-E}\right)+e^{-E}\cdot 1-e^E+\left(-e^{-E}\right)}{Ee^E+e^E\cdot 1-\left[E(-e^{-E})+e^{-E}\cdot 1\right]}$$

$$= \lim_{E\to 0^+}\frac{Ee^E-Ee^{-E}}{Ee^E+e^E+Ee^{-E}-e^{-E}} = \lim_{E\to 0^+}\frac{e^E-e^{-E}}{e^E+\dfrac{e^E}{E}+e^{-E}-\dfrac{e^{-E}}{E}} \qquad [\text{divide by } E]$$

$$= \frac{0}{2+L}, \quad \text{where } L = \lim_{E\to 0^+}\frac{e^E-e^{-E}}{E} \quad \left[\text{form is } \tfrac{0}{0}\right] \quad \overset{\text{H}}{=} \lim_{E\to 0^+}\frac{e^E+e^{-E}}{1} = \frac{1+1}{1} = 2$$

Thus, $\lim\limits_{E\to 0^+} P(E) = \dfrac{0}{2+2} = 0.$

77. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} \overset{\text{H}}{=} \lim_{x\to a}\frac{\frac{1}{2}(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{\frac{1}{2}(2a^3a-a^4)^{-1/2}(2a^3-4a^3)-\frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}$$

$$= \frac{(a^4)^{-1/2}(-a^3)-\frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^{3(a^4)-3/4}} = \frac{-a-\frac{1}{3}a}{-\frac{3}{4}} = \frac{4}{3}\left(\frac{4}{3}a\right) = \frac{16}{9}a$$

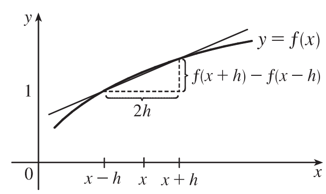
79. Since $f(2)=0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{f'(2+3x)\cdot 3+f'(2+5x)\cdot 5}{1} = f'(2)\cdot 3+f'(2)\cdot 5 = 8f'(2) = 8\cdot 7 = 56$$

81. Since $\lim\limits_{h\to 0}[f(x+h)-f(x-h)] = f(x)-f(x) = 0$ (f is differentiable and hence continuous) and $\lim\limits_{h\to 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2} = \frac{f'(x)+f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h\to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



83. (a) We show that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}} = \lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty}\frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty}\frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty}\frac{n!}{e^y} = 0 \quad \Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \lim_{x\to 0} x^n\frac{f(x)}{x^{2n}} = \lim_{x\to 0} x^n \lim_{x\to 0}\frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = 0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)\,f(x) + p_n(x)f'(x)\right] - k_n x^{k_n-1} p_n(x)\,f(x)\right]x^{-2k_n} \\ &= \left[x^{k_n} p_n'(x) + p_n(x)\left(2/x^3\right) - k_n x^{k_n-1} p_n(x)\right] f(x) x^{-2k_n} \\ &= \left[x^{k_n+3} p_n'(x) + 2p_n(x) - k_n x^{k_n+2}\, p_n(x)\right] f(x) x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

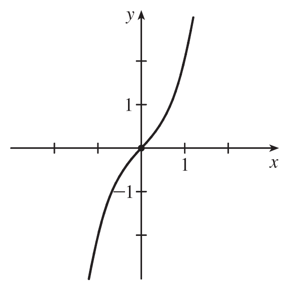
Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} \frac{p_n(x)\,f(x)/x^{k_n}}{x} = \lim_{x\to 0} \frac{p_n(x)\,f(x)}{x^{k_n+1}} \\ &= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0 \end{aligned}$$

4.5 Summary of Curve Sketching

1. $y = f(x) = x^3 + x = x(x^2 + 1)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

H.



3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x - 2)\left(x^2 - 7x + 1\right)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2} \approx 0.15, 6.85$ **C.** No symmetry **D.** No asymptote

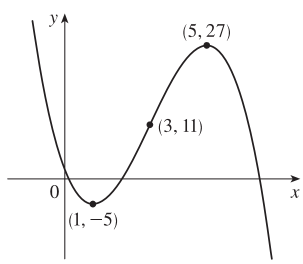
E. $f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5)$
$= -3(x - 1)(x - 5) > 0 \;\Leftrightarrow\; 1 < x < 5$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$.

F. Local maximum value $f(5) = 27$, local minimum value $f(1) = -5$

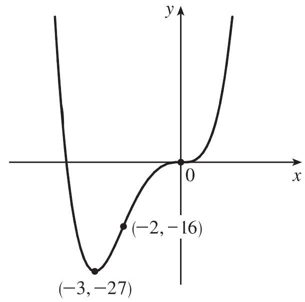
G. $f''(x) = 18 - 6x = -6(x - 3) > 0 \;\Leftrightarrow\; x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

H.



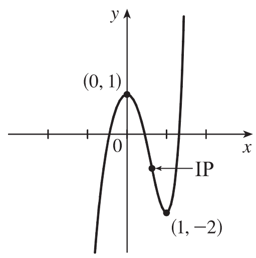
5. $y = f(x) = x^4 + 4x^3 = x^3(x+4)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \iff x = -4, 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3) > 0 \iff x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$. **F.** Local minimum value $f(-3) = -27$, no local maximum **G.** $f''(x) = 12x^2 + 24x = 12x(x+2) < 0 \iff -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$. IP at $(0, 0)$ and $(-2, -16)$

H.



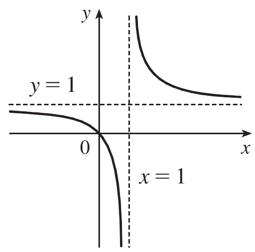
7. $y = f(x) = 2x^5 - 5x^2 + 1$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 10x^4 - 10x = 10x(x^3 - 1) = 10x(x-1)(x^2 + x + 1)$, so $f'(x) < 0 \iff 0 < x < 1$ and $f'(x) > 0 \iff x < 0$ or $x > 1$. Thus, f is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$. **F.** Local maximum value $f(0) = 1$, local minimum value $f(1) = -2$ **G.** $f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$ so $f''(x) = 0 \iff x = 1/\sqrt[3]{4}$. $f''(x) > 0 \iff x > 1/\sqrt[3]{4}$ and $f''(x) < 0 \iff x < 1/\sqrt[3]{4}$, so f is CD on $\left(-\infty, 1/\sqrt[3]{4}\right)$ and CU on $\left(1/\sqrt[3]{4}, \infty\right)$. IP at $\left(\dfrac{1}{\sqrt[3]{4}}, 1 - \dfrac{9}{2\left(\sqrt[3]{4}\right)^2}\right) \approx (0.630, -0.786)$

H.



9. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{x}{x-1} = 1$, so $y = 1$ is a HA. $\displaystyle\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$, $\displaystyle\lim_{x\to 1^+} \frac{x}{x-1} = \infty$, so $x = 1$ is a VA. **E.** $f'(x) = \dfrac{(x-1) - x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extreme values **G.** $f''(x) = \dfrac{2}{(x-1)^3} > 0 \iff x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



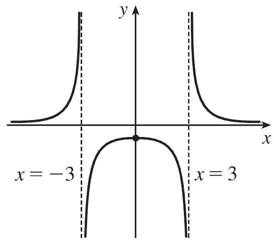
11. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about the y-axis. **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{1}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 3^-} \frac{1}{x^2-9} = -\infty$, $\displaystyle\lim_{x\to 3^+} \frac{1}{x^2-9} = \infty$, $\displaystyle\lim_{x\to -3^-} \frac{1}{x^2-9} = \infty$, $\displaystyle\lim_{x\to -3^+} \frac{1}{x^2-9} = -\infty$, so $x = 3$ and $x = -3$ are VA. **E.** $f'(x) = -\dfrac{2x}{(x^2-9)^2} > 0 \iff x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and

decreasing on $(0,3)$ and $(3,\infty)$. **F.** Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \dfrac{-2(x^2-9)^2 + (2x)2(x^2-9)(2x)}{(x^2-9)^4} = \dfrac{6(x^2+3)}{(x^2-9)^3} > 0 \quad\Leftrightarrow$

$x^2 > 9 \quad\Leftrightarrow\quad x > 3$ or $x < -3$, so f is CU on $(-\infty,-3)$ and $(3,\infty)$ and CD on $(-3,3)$. No IP

H.



13. $y = f(x) = x/(x^2+9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \quad\Leftrightarrow\quad x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty}[x/(x^2+9)] = 0$, so $y = 0$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2} = \dfrac{(3+x)(3-x)}{(x^2+9)^2} > 0 \quad\Leftrightarrow\quad -3 < x < 3$, so f is increasing on $(-3,3)$ and decreasing on $(-\infty,-3)$ and $(3,\infty)$. **F.** Local minimum value $f(-3) = -\frac{1}{6}$, local maximum value $f(3) = \frac{1}{6}$

$$f''(x) = \frac{(x^2+9)^2(-2x) - (9-x^2)\cdot 2(x^2+9)(2x)}{[(x^2+9)^2]^2} = \frac{(2x)(x^2+9)[-(x^2+9) - 2(9-x^2)]}{(x^2+9)^4} = \frac{2x(x^2-27)}{(x^2+9)^3}$$

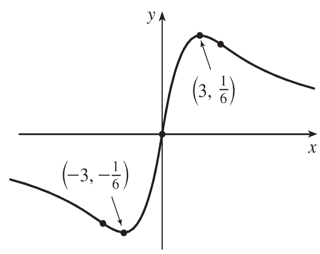
$$= 0 \quad\Leftrightarrow\quad x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$$

G. $f''(x) = \dfrac{(x^2+9)^2(-2x) - (9-x^2)\cdot 2(x^2+9)(2x)}{[(x^2+9)^2]^2} = \dfrac{(2x)(x^2+9)\left[-(x^2+9) - 2(9-x^2)\right]}{(x^2+9)^4}$

$$= \frac{2x(x^2-27)}{(x^2+9)^3} = 0 \quad\Leftrightarrow\quad x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$$

$f''(x) > 0 \quad\Leftrightarrow\quad -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on $\left(-3\sqrt{3}, 0\right)$ and $\left(3\sqrt{3},\infty\right)$, and CD on $\left(-\infty,-3\sqrt{3}\right)$ and $\left(0, 3\sqrt{3}\right)$. There are three inflection points: $(0,0)$ and $\left(\pm 3\sqrt{3}, \pm\frac{1}{12}\sqrt{3}\right)$.

H.



15. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty,0)\cup(0,\infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \quad\Leftrightarrow\quad x = 1$

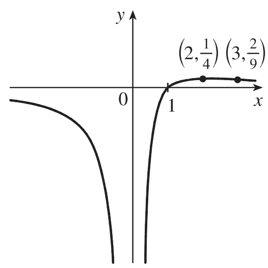
C. No symmetry **D.** $\lim\limits_{x\to\pm\infty}\dfrac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0}\dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^2\cdot 1 - (x-1)\cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \quad\Leftrightarrow\quad 0 < x < 2$ and $f'(x) < 0 \quad\Leftrightarrow$ $x < 0$ or $x > 2$. Thus, f is increasing on $(0,2)$ and decreasing on $(-\infty,0)$ and $(2,\infty)$. **F.** No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3\cdot(-1) - [-(x-2)]\cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty,0)$ and $(0,3)$ and positive on $(3,\infty)$, so f is CD on $(-\infty,0)$ and $(0,3)$ and CU on $(3,\infty)$. IP at $\left(3, \frac{2}{9}\right)$

H.



17. $y = f(x) = \dfrac{x^2}{x^2+3} = \dfrac{(x^2+3)-3}{x^2+3} = 1 - \dfrac{3}{x^2+3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$;

x-intercepts: $f(x) = 0 \iff x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{x^2+3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x) = -3 \cdot \dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$.

$f'(x) > 0 \iff x > 0$ and $f'(x) < 0 \iff x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

F. Local minimum value $f(0) = 0$, no local maximum.

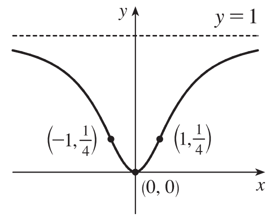
G. $$f''(x) = \frac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2}$$

$$= \frac{6(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}$$

$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$,

so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

H.



19. $y = f(x) = x\sqrt{5-x}$ **A.** The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ **B.** y-intercept: $f(0) = 0$;

x-intercepts: $f(x) = 0 \iff x = 0, 5$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = x \cdot \frac{1}{2}(5-x)^{-1/2}(-1) + (5-x)^{1/2} \cdot 1 = \frac{1}{2}(5-x)^{-1/2}[-x + 2(5-x)] = \dfrac{10-3x}{2\sqrt{5-x}} > 0 \iff$

$x < \frac{10}{3}$, so f is increasing on $\left(-\infty, \frac{10}{3}\right)$ and decreasing on $\left(\frac{10}{3}, 5\right)$.

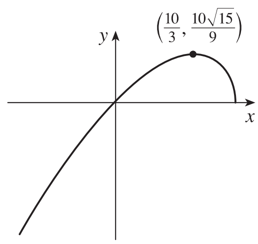
F. Local maximum value $f\left(\frac{10}{3}\right) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

G. $$f''(x) = \frac{2(5-x)^{1/2}(-3) - (10-3x) \cdot 2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2}$$

$$= \frac{(5-x)^{-1/2}[-6(5-x) + (10-3x)]}{4(5-x)} = \frac{3x-20}{4(5-x)^{3/2}}$$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP

H.



21. $y = f(x) = \sqrt{x^2+x-2} = \sqrt{(x+2)(x-1)}$ **A.** $D = \{x \mid (x+2)(x-1) \geq 0\} = (-\infty, -2] \cup [1, \infty)$

B. y-intercept: none; x-intercepts: -2 and 1 **C.** No symmetry **D.** No asymptote

E. $f'(x) = \frac{1}{2}(x^2+x-2)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x-2}}$, $f'(x) = 0$ if $x = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the domain.

$f'(x) > 0 \Rightarrow x > -\frac{1}{2}$ and $f'(x) < 0 \Rightarrow x < -\frac{1}{2}$, so (considering the domain) f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, -2)$. **F.** No local extrema

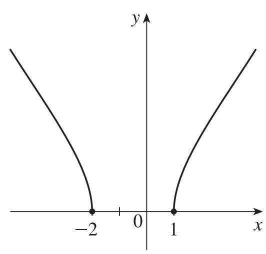
G. $$f''(x) = \frac{2(x^2+x-2)^{1/2}(2) - (2x+1) \cdot 2 \cdot \frac{1}{2}(x^2+x-2)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x-2}\right)^2}$$

$$= \frac{(x^2+x-2)^{-1/2}\left[4(x^2+x-2) - (4x^2+4x+1)\right]}{4(x^2+x-2)}$$

$$= \frac{-9}{4(x^2+x-2)^{3/2}} < 0$$

so f is CD on $(-\infty, -2)$ and $(1, \infty)$. No IP

H.



23. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/x} = \lim\limits_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+1/x^2}} = \dfrac{1}{\sqrt{1+0}} = 1$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}}$$

$$= \frac{1}{-\sqrt{1+0}} = -1 \quad \text{so } y = \pm 1 \text{ are HA.}$$

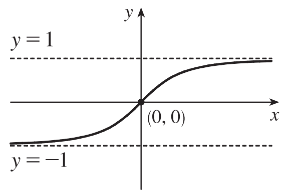
No VA.

E. $f'(x) = \dfrac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values

G. $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.

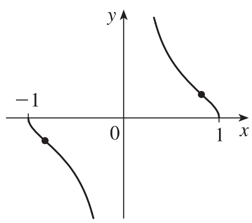


25. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \{x \mid |x| \le 1, x \ne 0\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-x^2}}{x} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{\sqrt{1-x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \Leftrightarrow -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

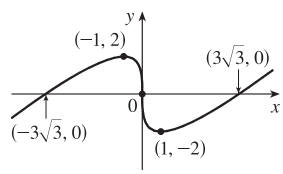
H.



27. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2 - 27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3}-1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is continuous on $(-1, 1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. IP at $(0, 0)$

H.



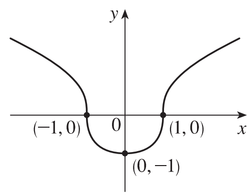
29. $y = f(x) = \sqrt[3]{x^2 - 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -1$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow$ $x = \pm 1$ **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x) = \dfrac{2x}{3\sqrt[3]{(x^2-1)^2}}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = -1$

G. $$f''(x) = \frac{2}{3} \cdot \frac{(x^2-1)^{2/3}(1) - x \cdot \frac{2}{3}(x^2-1)^{-1/3}(2x)}{[(x^2-1)^{2/3}]^2}$$
$$= \frac{2}{9} \cdot \frac{(x^2-1)^{-1/3}[3(x^2-1) - 4x^2]}{(x^2-1)^{4/3}} = -\frac{2(x^2+3)}{9(x^2-1)^{5/3}}$$

$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(\pm 1, 0)$

H.



31. $y = f(x) = 3\sin x - \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow$ $\sin x\,(3 - \sin^2 x) = 0 \Rightarrow \sin x = 0$ [since $\sin^2 x \le 1 < 3$] $\Rightarrow x = n\pi$, n an integer.

C. $f(-x) = -f(x)$, so f is odd; the graph (shown for $-2\pi \le x \le 2\pi$) is symmetric about the origin and periodic with period 2π. **D.** No asymptote **E.** $f'(x) = 3\cos x - 3\sin^2 x \cos x = 3\cos x\,(1 - \sin^2 x) = 3\cos^3 x$.

$f'(x) > 0 \Leftrightarrow \cos x > 0 \Leftrightarrow x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and $f'(x) < 0 \Leftrightarrow \cos x < 0 \Leftrightarrow$ $x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n. Thus, f is increasing on $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ for each integer n, and f is decreasing on $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$ for each integer n.

F. f has local maximum values $f(2n\pi + \frac{\pi}{2}) = 2$ and local minimum values $f(2n\pi + \frac{3\pi}{2}) = -2$.

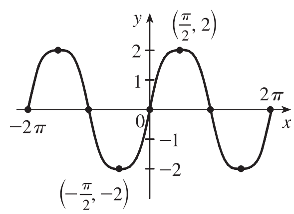
G. $f''(x) = -9\sin x \cos^2 x = -9\sin x\,(1 - \sin^2 x) = -9\sin x\,(1 - \sin x)(1 + \sin x)$.

$f''(x) < 0 \Leftrightarrow \sin x > 0$ and $\sin x \ne \pm 1 \Leftrightarrow x \in \left(2n\pi, 2n\pi + \frac{\pi}{2}\right) \cup \left(2n\pi + \frac{\pi}{2}, 2n\pi + \pi\right)$ for some integer n.

$f''(x) > 0 \Leftrightarrow \sin x < 0$ and $\sin x \ne \pm 1 \Leftrightarrow x \in \left((2n-1)\pi, (2n-1)\pi + \frac{\pi}{2}\right) \cup \left((2n-1)\pi + \frac{\pi}{2}, 2n\pi\right)$

for some integer n. Thus, f is CD on the intervals $\left(2n\pi, \left(2n + \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n + \frac{1}{2}\right)\pi, (2n+1)\,\pi\right)$ [hence CD on the intervals $(2n\pi, (2n+1)\,\pi)$] for each integer n, and f is CU on the intervals $\left((2n-1)\pi, \left(2n - \frac{1}{2}\right)\pi\right)$ and $\left(\left(2n - \frac{1}{2}\right)\pi, 2n\pi\right)$ [hence CU on the intervals $((2n-1)\pi, 2n\pi)$] for each integer n. f has inflection points at $(n\pi, 0)$ for each integer n.

H.

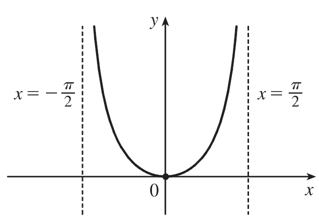


33. $y = f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to (\pi/2)^-} x\tan x = \infty$ and $\lim\limits_{x \to -(\pi/2)^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA.

E. $f'(x) = \tan x + x\sec^2 x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$. **F.** Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x\tan x \sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.

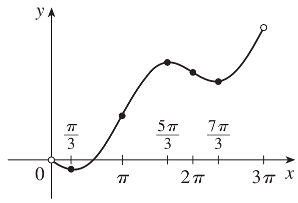


35. $y = f(x) = \frac{1}{2}x - \sin x,\ 0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y-intercept. The x-intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \Leftrightarrow \cos x < \frac{1}{2} \Leftrightarrow$ $\frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$ and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$.

F. Local minimum value $f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, local maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$, local minimum value $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$

G. $f''(x) = \sin x > 0 \Leftrightarrow 0 < x < \pi$ or $2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$. IPs at $\left(\pi, \frac{\pi}{2}\right)$ and $(2\pi, \pi)$

H.

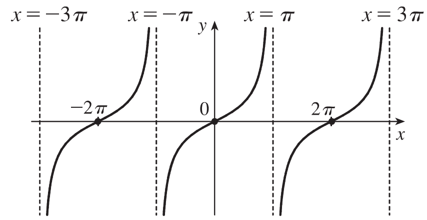


37. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \neq 1}{=} \dfrac{\sin x}{1 + \cos x} \cdot \dfrac{1 - \cos x}{1 - \cos x} = \dfrac{\sin x\,(1 - \cos x)}{\sin^2 x} = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x\right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = n\pi$, n an even integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x \to (n\pi)^-} f(x) = \infty$ and $\lim\limits_{x \to (n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n. No HA. **E.** $f'(x) = \dfrac{(1 + \cos x) \cdot \cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1 + \cos x}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

G. $f''(x) = \dfrac{\sin x}{(1 + \cos x)^2} > 0 \Rightarrow \sin x > 0 \Rightarrow$ $x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on $((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.

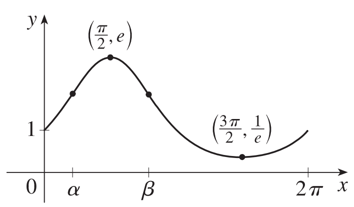


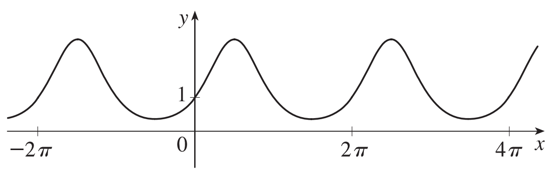
39. $y = f(x) = e^{\sin x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = e^0 = 1$; x-intercepts: none, since $e^{\sin x} > 0$ **C.** f is periodic with period 2π, so we determine **E–G** for $0 \le x \le 2\pi$. **D.** No asymptote **E.** $f'(x) = e^{\sin x} \cos x$. $f'(x) > 0 \Leftrightarrow$ $\cos x > 0 \Rightarrow x$ is in $\left(0, \frac{\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, 2\pi\right)$ [f is increasing] and $f'(x) < 0 \Rightarrow x$ is in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ [f is decreasing].

F. Local maximum value $f\left(\frac{\pi}{2}\right) = e$ and local minimum value $f\left(\frac{3\pi}{2}\right) = e^{-1}$

G. $f''(x) = e^{\sin x}(-\sin x) + \cos x\,(e^{\sin x} \cos x) = e^{\sin x}(\cos^2 x - \sin x)$. $f''(x) = 0 \Leftrightarrow \cos^2 x - \sin x = 0 \Leftrightarrow$ $1 - \sin^2 x - \sin x = 0 \Leftrightarrow \sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \alpha = \sin^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right) \approx 0.67$ and $\beta = \pi - \alpha \approx 2.48$. $f''(x) < 0$ on (α, β) [f is CD] and $f''(x) > 0$ on $(0, \alpha)$ and $(\beta, 2\pi)$ [f is CU].
The inflection points occur when $x = \alpha, \beta$.

H.

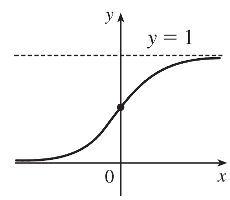




41. $y = 1/(1+e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry **D.** $\lim\limits_{x\to\infty} 1/(1+e^{-x}) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/(1+e^{-x}) = 0$ since $\lim\limits_{x\to-\infty} e^{-x} = \infty]$, so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -(1+e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1+e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$. **F.** No extreme values **G.** $f''(x) = \dfrac{(1+e^{-x})^2(-e^{-x}) - e^{-x}(2)(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} = \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$

The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{1}{2}\right)$

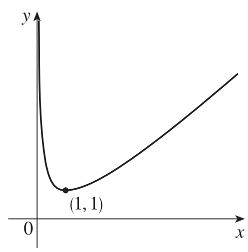
H.



43. $y = f(x) = x - \ln x$ **A.** $D = (0, \infty)$ **B.** y-intercept: none (0 is not in the domain); x-intercept: $f(x) = 0 \;\Leftrightarrow\; x = \ln x$, which has no solution, so there is no x-intercept. **C.** No symmetry **D.** $\lim\limits_{x\to 0^+}(x - \ln x) = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 1/x > 0 \;\Rightarrow\; 1 > 1/x \;\Rightarrow\; x > 1$ and $f'(x) < 0 \;\Rightarrow\; 0 < x < 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$. **F.** Local minimum value $f(1) = 1$; no local maximum value

G. $f''(x) = \dfrac{1}{x^2} > 0$ for all x, so f is CU on $(0, \infty)$. No IP

H.



45. $y = f(x) = (1+e^x)^{-2} = \dfrac{1}{(1+e^x)^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = \frac{1}{4}$. x-intercepts: none [since $f(x) > 0$] **C.** No symmetry **D.** $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 1$, so $y = 0$ and $y = 1$ are HA; no VA

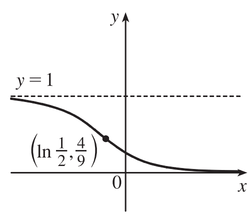
E. $f'(x) = -2(1+e^x)^{-3}e^x = \dfrac{-2e^x}{(1+e^x)^3} < 0$, so f is decreasing on $\mathbb{R}$ **F.** No local extrema

G. $f''(x) = (1+e^x)^{-3}(-2e^x) + (-2e^x)(-3)(1+e^x)^{-4}e^x$

$$= -2e^x(1+e^x)^{-4}[(1+e^x) - 3e^x] = \frac{-2e^x(1-2e^x)}{(1+e^x)^4}.$$

$f''(x) > 0 \;\Leftrightarrow\; 1 - 2e^x < 0 \;\Leftrightarrow\; e^x > \frac{1}{2} \;\Leftrightarrow\; x > \ln\frac{1}{2}$ and $f''(x) < 0 \;\Leftrightarrow\; x < \ln\frac{1}{2}$, so f is CU on $\left(\ln\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \ln\frac{1}{2}\right)$. IP at $\left(\ln\frac{1}{2}, \frac{4}{9}\right)$

H.



47. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup\limits_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi) = \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

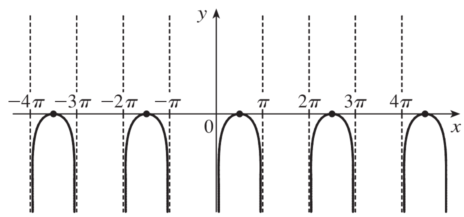
B. No y-intercept; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; \ln(\sin x) = 0 \;\Leftrightarrow\; \sin x = e^0 = 1 \;\Leftrightarrow\; x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim\limits_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim\limits_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and

decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n.

F. Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum.

G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.



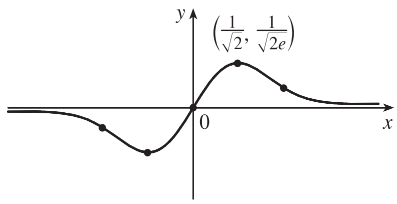
49. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty} xe^{-x^2} = \lim\limits_{x\to\pm\infty} \frac{x}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \quad\Leftrightarrow\quad x^2 < \frac{1}{2} \quad\Leftrightarrow\quad |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ **G.** $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0 \quad\Leftrightarrow$

$x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$.

IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}\, e^{-3/2}\right)$.

H.



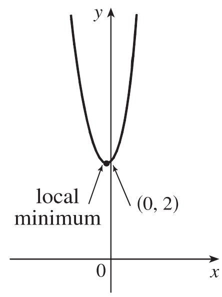
51. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptote

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \quad\Leftrightarrow\quad 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow$ $e^{5x} > \frac{2}{3} \quad\Leftrightarrow\quad 5x > \ln\frac{2}{3} \quad\Leftrightarrow\quad x > \frac{1}{5}\ln\frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \quad\Leftrightarrow$ $x < \frac{1}{5}\ln\frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$.

F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

H.



53. $m = f(v) = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. The m-intercept is $f(0) = m_0$. There are no v-intercepts. $\lim\limits_{v\to c^-} f(v) = \infty$, so $v = c$ is a VA.

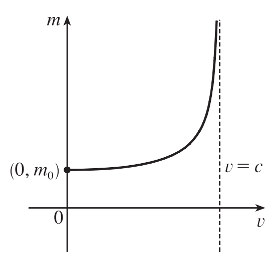
$$f'(v) = -\tfrac{1}{2}m_0(1 - v^2/c^2)^{-3/2}(-2v/c^2) = \frac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}} = \frac{m_0 v}{\dfrac{c^2(c^2 - v^2)^{3/2}}{c^3}} = \frac{m_0 cv}{(c^2 - v^2)^{3/2}} > 0, \text{ so } f \text{ is}$$

increasing on $(0, c)$. There are no local extreme values.

$$f''(v) = \frac{(c^2 - v^2)^{3/2}(m_0 c) - m_0 cv \cdot \frac{3}{2}(c^2 - v^2)^{1/2}(-2v)}{[(c^2 - v^2)^{3/2}]^2}$$

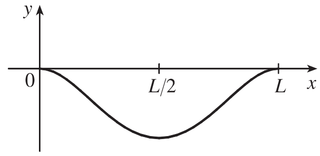
$$= \frac{m_0 c(c^2 - v^2)^{1/2}[(c^2 - v^2) + 3v^2]}{(c^2 - v^2)^3} = \frac{m_0 c(c^2 + 2v^2)}{(c^2 - v^2)^{5/2}} > 0,$$

so f is CU on $(0, c)$. There are no inflection points.



55. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2\left(x^2 - 2Lx + L^2\right)$

$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$



where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch

$f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \Leftrightarrow x(x-L)(2x-L) < 0$ [since $c < 0$] $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow 0 < x < L/2$. Thus, f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at the point $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$. $f'(x) = 2c[x(x-L)(2x-L)] \Rightarrow$

$f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

57. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r|l} & \quad\; x - 1 \\ \cline{2-2} x+1 & x^2 \qquad + 1 \\ & x^2 + x \\ \cline{2-2} & \quad -x + 1 \\ & \quad -x - 1 \\ \cline{2-2} & \qquad\quad 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1 + \frac{1}{x}}$ [for $x \ne 0$] $\to 0$ as $x \to \pm\infty$.

So the line $y = x - 1$ is a slant asymptote (SA).

59. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$. Long division gives us:

$$\begin{array}{r|l} & \qquad\qquad 2x - 2 \\ \cline{2-2} 2x^2+x-3 & 4x^3 - 2x^2 \qquad + 5 \\ & 4x^3 + 2x^2 - 6x \\ \cline{2-2} & \quad -4x^2 + 6x + 5 \\ & \quad -4x^2 - 2x + 6 \\ \cline{2-2} & \qquad\qquad 8x - 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3} = 2x - 2 + \dfrac{8x - 1}{2x^2 + x - 3}$ and $f(x) - (2x - 2) = \dfrac{8x-1}{2x^2+x-3} = \dfrac{\frac{8}{x} - \frac{1}{x^2}}{2 + \frac{1}{x} - \frac{3}{x^2}}$

[for $x \ne 0$] $\to 0$ as $x \to \pm\infty$. So the line $y = 2x - 2$ is a SA.

61. $y = f(x) = \dfrac{-2x^2 + 5x - 1}{2x - 1} = -x + 2 + \dfrac{1}{2x - 1}$ **A.** $D = \left\{x \in \mathbb{R} \mid x \ne \frac{1}{2}\right\} = \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

B. y-intercept: $f(0) = 1$; x-intercepts: $f(x) = 0 \Rightarrow -2x^2 + 5x - 1 = 0 \Rightarrow x = \dfrac{-5 \pm \sqrt{17}}{-4} \Rightarrow x \approx 0.22, 2.28$.

[continued]

C. No symmetry **D.** $\lim\limits_{x\to(1/2)^-} f(x) = -\infty$ and $\lim\limits_{x\to(1/2)^+} f(x) = \infty$, so $x = \frac{1}{2}$ is a VA.

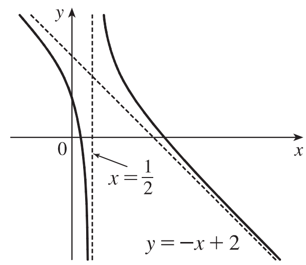
$\lim\limits_{x\to\pm\infty} [f(x) - (-x+2)] = \lim\limits_{x\to\pm\infty} \dfrac{1}{2x-1} = 0$, so the line $y = -x+2$ is a SA.

E. $f'(x) = -1 - \dfrac{2}{(2x-1)^2} < 0$ for $x \neq \frac{1}{2}$, so f is decreasing on $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$. **F.** No extreme values **G.** $f'(x) = -1 - 2(2x-1)^{-2} \Rightarrow$

$f''(x) = -2(-2)(2x-1)^{-3}(2) = \dfrac{8}{(2x-1)^3}$, so $f''(x) > 0$ when $x > \frac{1}{2}$ and

$f''(x) < 0$ when $x < \frac{1}{2}$. Thus, f is CU on $\left(\frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\right)$. No IP

H.



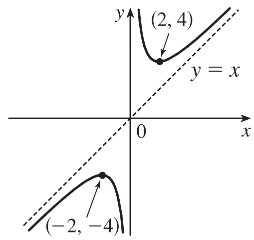
63. $y = f(x) = (x^2+4)/x = x + 4/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept

C. $f(-x) = -f(x) \Rightarrow$ symmetry about the origin **D.** $\lim\limits_{x\to\infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. $\lim\limits_{x\to 0^+} (x + 4/x) = \infty$ and

$\lim\limits_{x\to 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 4/x^2 > 0 \Leftrightarrow$

$x^2 > 4 \Leftrightarrow x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$. **F.** Local maximum value $f(-2) = -4$, local minimum value $f(2) = 4$ **G.** $f''(x) = 8/x^3 > 0 \Leftrightarrow x > 0$ so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



65. $y = f(x) = \dfrac{2x^3 + x^2 + 1}{x^2+1} = 2x + 1 + \dfrac{-2x}{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \Rightarrow$

$0 = 2x^3 + x^2 + 1 = (x+1)(2x^2 - x + 1) \Rightarrow x = -1$ **C.** No symmetry **D.** No VA

$\lim\limits_{x\to\pm\infty} [f(x) - (2x+1)] = \lim\limits_{x\to\pm\infty} \dfrac{-2x}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{-2/x}{1 + 1/x^2} = 0$, so the line $y = 2x+1$ is a slant asymptote.

E. $f'(x) = 2 + \dfrac{(x^2+1)(-2) - (-2x)(2x)}{(x^2+1)^2} = \dfrac{2(x^4 + 2x^2 + 1) - 2x^2 - 2 + 4x^2}{(x^2+1)^2} = \dfrac{2x^4 + 6x^2}{(x^2+1)^2} = \dfrac{2x^2(x^2+3)}{(x^2+1)^2}$

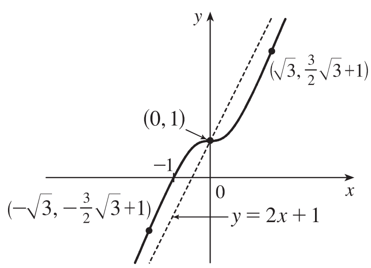
so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$.

F. No extreme values

G.
$$f''(x) = \frac{(x^2+1)^2 \cdot (8x^3 + 12x) - (2x^4 + 6x^2) \cdot 2(x^2+1)(2x)}{[(x^2+1)^2]^2}$$
$$= \frac{4x(x^2+1)[(x^2+1)(2x^2+3) - 2x^4 - 6x^2]}{(x^2+1)^4} = \frac{4x(-x^2+3)}{(x^2+1)^3}$$

so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$. There are three IPs: $(0, 1)$, $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3} + 1\right) \approx (-1.73, -1.60)$, and $\left(\sqrt{3}, \frac{3}{2}\sqrt{3} + 1\right) \approx (1.73, 3.60)$.

H.



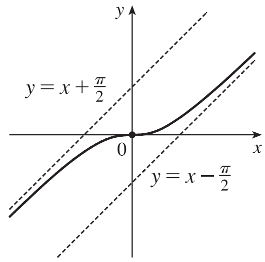
67. $y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

$f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x(1+x^2-x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

$\lim\limits_{x\to\infty}\left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim\limits_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA.

Also, $\lim\limits_{x\to-\infty}\left[f(x) - \left(x + \frac{\pi}{2}\right)\right] = \lim\limits_{x\to-\infty}\left(-\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$,

so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.



69. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2} - x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

$\lim\limits_{x\to\infty}\left[-\dfrac{b}{a}\sqrt{x^2-a^2} - \left(-\dfrac{b}{a}x\right)\right] = -\dfrac{b}{a}\cdot\lim\limits_{x\to\infty}\dfrac{-a^2}{\sqrt{x^2-a^2}+x} = 0$, so $y = -\dfrac{b}{a}x$ is a slant asymptote.

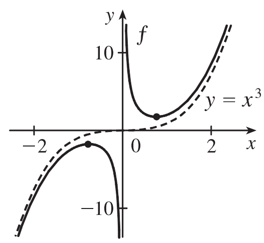
71. $\lim\limits_{x\to\pm\infty}\left[f(x) - x^3\right] = \lim\limits_{x\to\pm\infty}\dfrac{x^4+1}{x} - \dfrac{x^4}{x} = \lim\limits_{x\to\pm\infty}\dfrac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

A. $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to 0^-}\left(x^3 + \dfrac{1}{x}\right) = -\infty$ and $\lim\limits_{x\to 0^+}\left(x^3 + \dfrac{1}{x}\right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y = x^3$.

E. $f'(x) = 3x^2 - 1/x^2 > 0 \;\Leftrightarrow\; x^4 > \frac{1}{3} \;\Leftrightarrow\; |x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty, -\dfrac{1}{\sqrt[4]{3}}\right)$ and $\left(\dfrac{1}{\sqrt[4]{3}}, \infty\right)$ and decreasing on $\left(-\dfrac{1}{\sqrt[4]{3}}, 0\right)$ and $\left(0, \dfrac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\dfrac{1}{\sqrt[4]{3}}\right) = -4\cdot 3^{-5/4}$, local minimum value $f\left(\dfrac{1}{\sqrt[4]{3}}\right) = 4\cdot 3^{-5/4}$

G. $f''(x) = 6x + 2/x^3 > 0 \;\Leftrightarrow\; x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

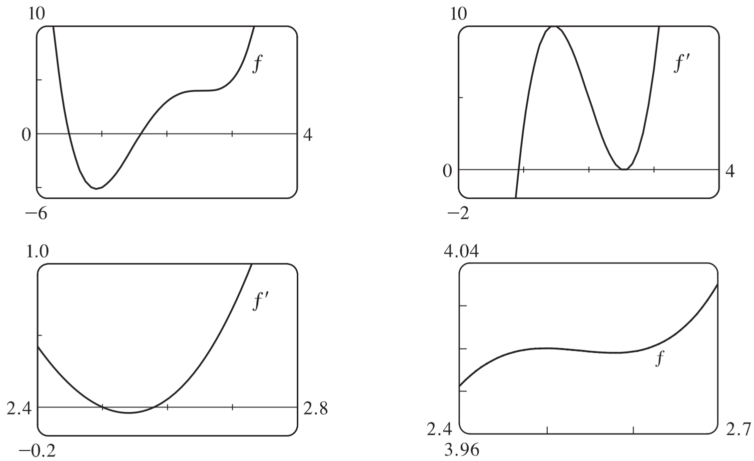
H.



4.6 Graphing with Calculus and Calculators

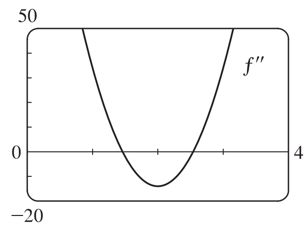
1. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29 \quad\Rightarrow\quad f'(x) = 16x^3 - 96x^2 + 178x - 95 \quad\Rightarrow\quad f''(x) = 48x^2 - 192x + 178.$

$f(x) = 0 \quad\Leftrightarrow\quad x \approx 0.5,\ 1.60$; $f'(x) = 0 \quad\Leftrightarrow\quad x \approx 0.92, 2.5, 2.58$ and $f''(x) = 0 \quad\Leftrightarrow\quad x \approx 1.46, 2.54.$



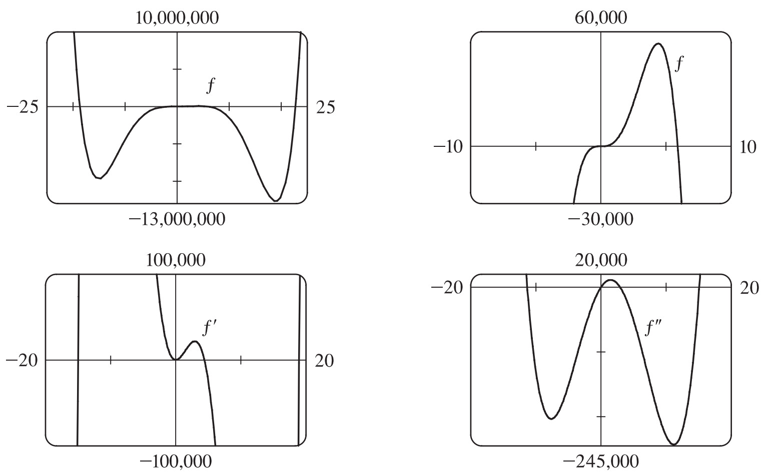
From the graphs of f', we estimate that $f' < 0$ and that f is decreasing on $(-\infty, 0.92)$ and $(2.5, 2.58)$, and that $f' > 0$ and f is increasing on $(0.92, 2.5)$ and $(2.58, \infty)$ with local minimum values $f(0.92) \approx -5.12$ and $f(2.58) \approx 3.998$ and local maximum value $f(2.5) = 4$. The graphs of f' make it clear that f has a maximum and a minimum near $x = 2.5$, shown more clearly in the fourth graph.

From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-\infty, 1.46)$ and $(2.54, \infty)$, and that $f'' < 0$ and f is CD on $(1.46, 2.54)$. There are inflection points at about $(1.46, -1.40)$ and $(2.54, 3.999)$.



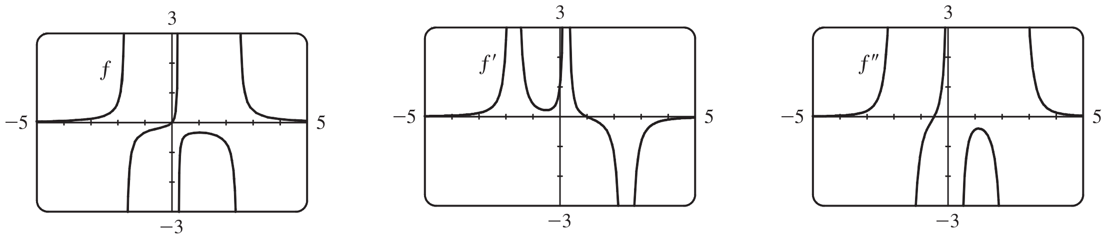
3. $f(x) = x^6 - 10x^5 - 400x^4 + 2500x^3 \quad\Rightarrow\quad f'(x) = 6x^5 - 50x^4 - 1600x^3 + 7500x^2 \quad\Rightarrow$

$f''(x) = 30x^4 - 200x^3 - 4800x^2 + 1500x$



From the graph of f', we estimate that f is decreasing on $(-\infty, -15)$, increasing on $(-15, 4.40)$, decreasing on $(4.40, 18.93)$, and increasing on $(18.93, \infty)$, with local minimum values of $f(-15) \approx -9{,}700{,}000$ and $f(18.93) \approx -12{,}700{,}000$ and local maximum value $f(4.40) \approx 53{,}800$. From the graph of f'', we estimate that f is CU on $(-\infty, -11.34)$, CD on $(-11.34, 0)$, CU on $(0, 2.92)$, CD on $(2.92, 15.08)$, and CU on $(15.08, \infty)$. There is an inflection point at $(0, 0)$ and at about $(-11.34, -6{,}250{,}000)$, $(2.92, 31{,}800)$, and $(15.08, -8{,}150{,}000)$.

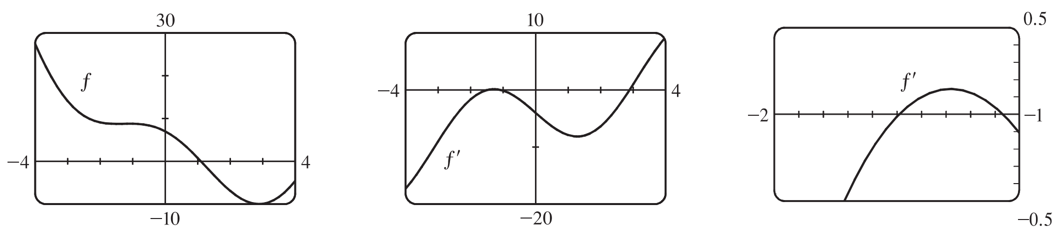
5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \quad \Rightarrow \quad f''(x) = \dfrac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3}$

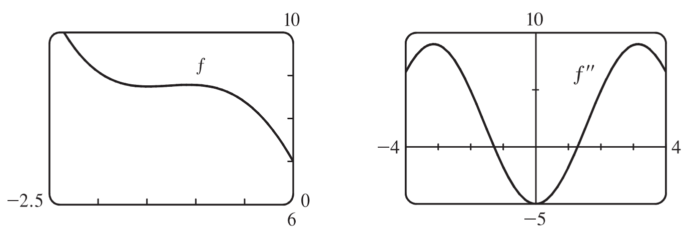


We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum value at $f(1) = -\frac{1}{3}$. From the graph of f'', we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

7. $f(x) = x^2 - 4x + 7\cos x, \quad -4 \le x \le 4. \quad f'(x) = 2x - 4 - 7\sin x \quad \Rightarrow \quad f''(x) = 2 - 7\cos x.$

$f(x) = 0 \quad \Leftrightarrow \quad x \approx 1.10$; $f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.49, -1.07$, or 2.89; $f''(x) = 0 \quad \Leftrightarrow \quad x = \pm\cos^{-1}\left(\frac{2}{7}\right) \approx \pm 1.28$.

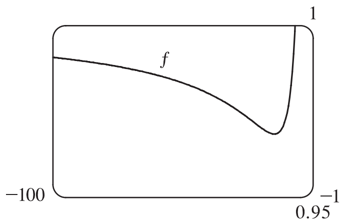


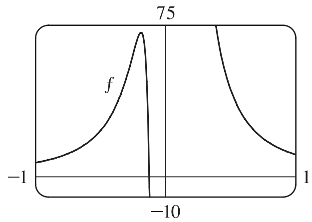


From the graphs of f', we estimate that f is decreasing ($f' < 0$) on $(-4, -1.49)$, increasing on $(-1.49, -1.07)$, decreasing on $(-1.07, 2.89)$, and increasing on $(2.89, 4)$, with local minimum values $f(-1.49) \approx 8.75$ and $f(2.89) \approx -9.99$ and local maximum value $f(-1.07) \approx 8.79$ (notice the second graph of f). From the graph of f'', we estimate that f is CU ($f'' > 0$) on $(-4, -1.28)$, CD on $(-1.28, 1.28)$, and CU on $(1.28, 4)$. There are inflection points at about $(-1.28, 8.77)$ and $(1.28, -1.48)$.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3} \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x^2} - \dfrac{16}{x^3} - \dfrac{3}{x^4} = -\dfrac{1}{x^4}(x^2 + 16x + 3) \quad \Rightarrow$

$f''(x) = \dfrac{2}{x^3} + \dfrac{48}{x^4} + \dfrac{12}{x^5} = \dfrac{2}{x^5}(x^2 + 24x + 6).$



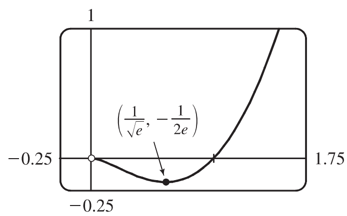


From the graphs, it appears that f increases on $(-15.8, -0.2)$ and decreases on $(-\infty, -15.8)$, $(-0.2, 0)$, and $(0, \infty)$; that f has a local minimum value of $f(-15.8) \approx 0.97$ and a local maximum value of $f(-0.2) \approx 72$; that f is CD on $(-\infty, -24)$ and $(-0.25, 0)$ and is CU on $(-24, -0.25)$ and $(0, \infty)$; and that f has IPs at $(-24, 0.97)$ and $(-0.25, 60)$.

To find the exact values, note that $f' = 0 \quad \Rightarrow \quad x = \dfrac{-16 \pm \sqrt{256 - 12}}{2} = -8 \pm \sqrt{61} \quad [\approx -0.19 \text{ and } -15.81]$.

f' is positive (f is increasing) on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$ and f' is negative (f is decreasing) on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, and $(0, \infty)$. $\quad f'' = 0 \quad \Rightarrow \quad x = \dfrac{-24 \pm \sqrt{576 - 24}}{2} = -12 \pm \sqrt{138} \quad [\approx -0.25 \text{ and } -23.75]$. $\quad f''$ is positive (f is CU) on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$ and $(0, \infty)$ and f'' is negative (f is CD) on $\left(-\infty, -12 - \sqrt{138}\right)$ and $\left(-12 + \sqrt{138}, 0\right)$.

11. (a) $f(x) = x^2 \ln x$. The domain of f is $(0, \infty)$.



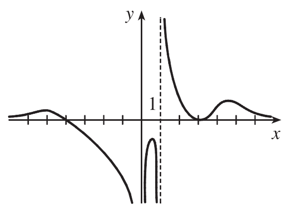
(b) $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \left(-\frac{x^2}{2}\right) = 0.$

There is a hole at $(0, 0)$.

(c) It appears that there is an IP at about $(0.2, -0.06)$ and a local minimum at $(0.6, -0.18)$. $\quad f(x) = x^2 \ln x \quad \Rightarrow$

$f'(x) = x^2(1/x) + (\ln x)(2x) = x(2 \ln x + 1) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$, so f is increasing on $\left(1/\sqrt{e}, \infty\right)$, decreasing on $\left(0, 1/\sqrt{e}\right)$. By the FDT, $f\left(1/\sqrt{e}\right) = -1/(2e)$ is a local minimum value. This point is approximately $(0.6065, -0.1839)$, which agrees with our estimate.

$f''(x) = x(2/x) + (2 \ln x + 1) = 2 \ln x + 3 > 0 \quad \Leftrightarrow \quad \ln x > -\frac{3}{2} \quad \Leftrightarrow \quad x > e^{-3/2}$, so f is CU on $(e^{-3/2}, \infty)$ and CD on $(0, e^{-3/2})$. IP is $(e^{-3/2}, -3/(2e^3)) \approx (0.2231, -0.0747)$.

13. 

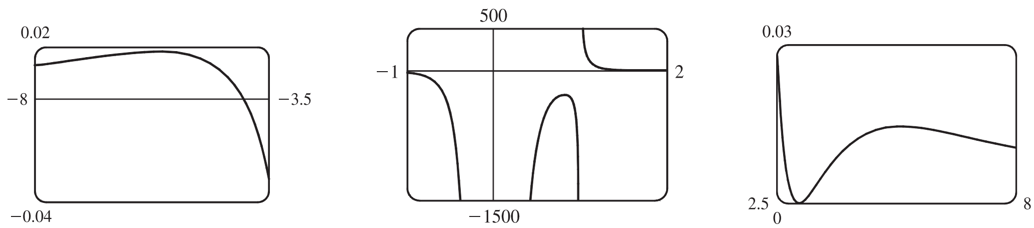
$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim\limits_{x\to 0} f(x) = -\infty$,

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

$$f(x) = \frac{\dfrac{x+4}{x}\cdot\dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3}\cdot(x-1)} \quad \left[\begin{array}{c}\text{dividing numerator}\\ \text{and denominator by } x^3\end{array}\right] = \frac{(1+4/x)(1-3/x)^2}{x(x-1)} \to 0$$

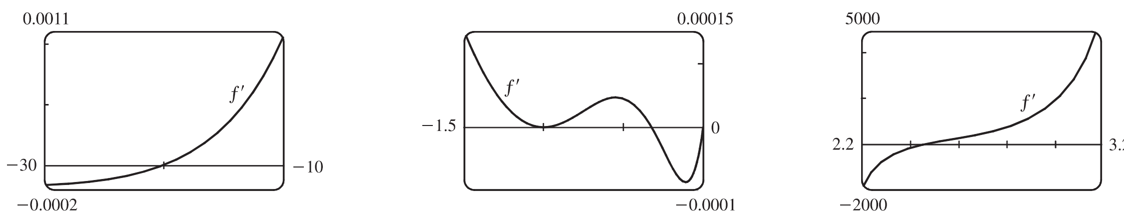
as $x \to \pm\infty$, so f is asymptotic to the x-axis.

Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \Rightarrow (x+4)(x-3)^2 = 0 \Rightarrow x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.

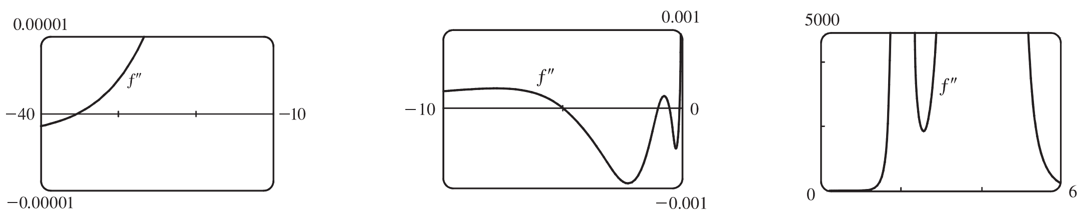


From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

15. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad\Rightarrow\quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ [from CAS].



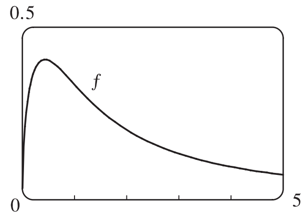
From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20$, -0.3, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.) We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.

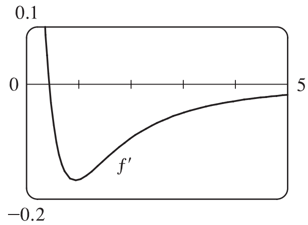


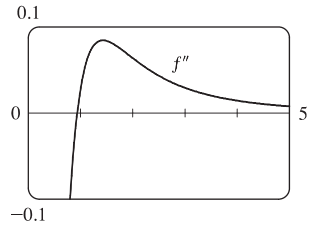
From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$ and $(4, \infty)$ and CD

on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

17. $y = f(x) = \dfrac{\sqrt{x}}{x^2 + x + 1}$. From a CAS, $y' = -\dfrac{3x^2 + x - 1}{2\sqrt{x}\,(x^2 + x + 1)^2}$ and $y'' = \dfrac{15x^4 + 10x^3 - 15x^2 - 6x - 1}{4x^{3/2}(x^2 + x + 1)^3}$.





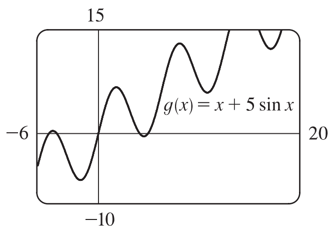


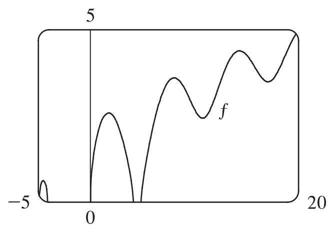
$f'(x) = 0 \;\Leftrightarrow\; x \approx 0.43$, so f is increasing on $(0, 0.43)$ and decreasing on $(0.43, \infty)$. There is a local maximum value of $f(0.43) \approx 0.41$. $f''(x) = 0 \;\Leftrightarrow\; x \approx 0.94$, so f is CD on $(0, 0.94)$ and CU on $(0.94, \infty)$. There is an inflection point at $(0.94, 0.34)$.

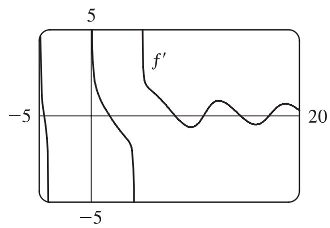
19. $y = f(x) = \sqrt{x + 5\sin x}$, $x \le 20$.

From a CAS, $y' = \dfrac{5\cos x + 1}{2\sqrt{x + 5\sin x}}$ and $y'' = -\dfrac{10\cos x + 25\sin^2 x + 10x\sin x + 26}{4(x + 5\sin x)^{3/2}}$.

We'll start with a graph of $g(x) = x + 5\sin x$. Note that $f(x) = \sqrt{g(x)}$ is only defined if $g(x) \ge 0$. $g(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx -4.91, -4.10, 4.10$, and 4.91. Thus, the domain of f is $[-4.91, -4.10] \cup [0, 4.10] \cup [4.91, 20]$.

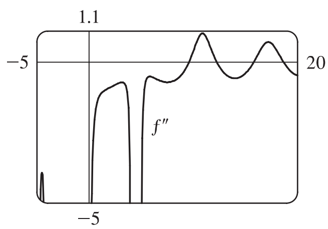




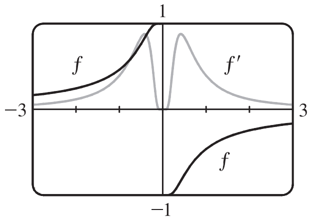


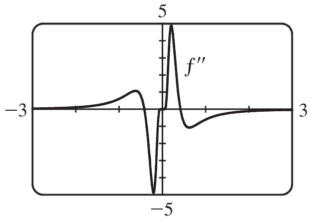
From the expression for y', we see that $y' = 0 \;\Leftrightarrow\; 5\cos x + 1 = 0 \;\Rightarrow\; x_1 = \cos^{-1}\left(-\frac{1}{5}\right) \approx 1.77$ and $x_2 = 2\pi - x_1 \approx -4.51$ (not in the domain of f). The leftmost zero of f' is $x_1 - 2\pi \approx -4.51$. Moving to the right, the zeros of f' are x_1, $x_1 + 2\pi$, $x_2 + 2\pi$, $x_1 + 4\pi$, and $x_2 + 4\pi$. Thus, f is increasing on $(-4.91, -4.51)$, decreasing on $(-4.51, -4.10)$, increasing on $(0, 1.77)$, decreasing on $(1.77, 4.10)$, increasing on $(4.91, 8.06)$, decreasing on $(8.06, 10.79)$, increasing on $(10.79, 14.34)$, decreasing on $(14.34, 17.08)$, and increasing on $(17.08, 20)$. The local maximum values are $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, and $f(14.34) \approx 4.39$. The local minimum values are $f(10.79) \approx 2.43$ and $f(17.08) \approx 3.49$.

f is CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, CU on $(9.60, 12.25)$, CD on $(12.25, 15.81)$, CU on $(15.81, 18.65)$, and CD on $(18.65, 20)$. There are inflection points at $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, and $(18.65, 4.20)$.



21. $y = f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$. From a CAS, $y' = \dfrac{2e^{1/x}}{x^2(1 + e^{1/x})^2}$ and $y'' = \dfrac{-2e^{1/x}(1 - e^{1/x} + 2x + 2xe^{1/x})}{x^4(1 + e^{1/x})^3}$.



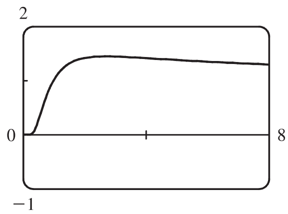


f is an odd function defined on $(-\infty, 0) \cup (0, \infty)$. Its graph has no x- or y-intercepts. Since $\lim\limits_{x \to \pm\infty} f(x) = 0$, the x-axis is a HA. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. It has no local extreme values.

$f''(x) = 0$ for $x \approx \pm 0.417$, so f is CU on $(-\infty, -0.417)$, CD on $(-0.417, 0)$, CU on $(0, 0.417)$, and CD on $(0.417, \infty)$.

f has IPs at $(-0.417, 0.834)$ and $(0.417, -0.834)$.

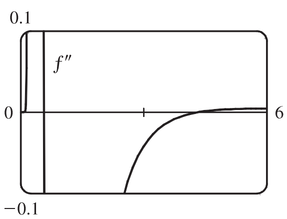
23. (a) $f(x) = x^{1/x}$



(b) Recall that $a^b = e^{b \ln a}$. $\lim\limits_{x \to 0^+} x^{1/x} = \lim\limits_{x \to 0^+} e^{(1/x)\ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x)\ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{(1/x)\ln x}$, but $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$, so $\lim\limits_{x \to \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

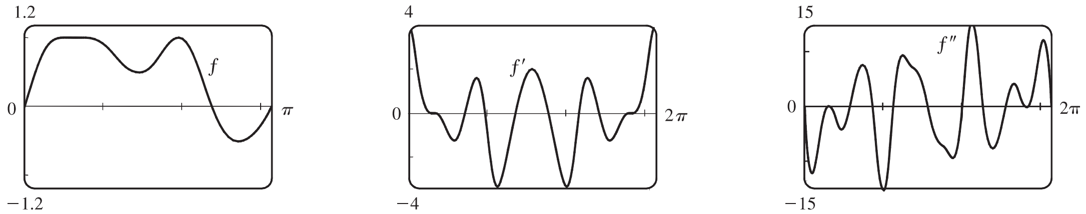
(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\,\text{lnx} \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \quad \Rightarrow \quad y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \quad \Rightarrow$ $\ln x = 1 \quad \Rightarrow \quad x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)

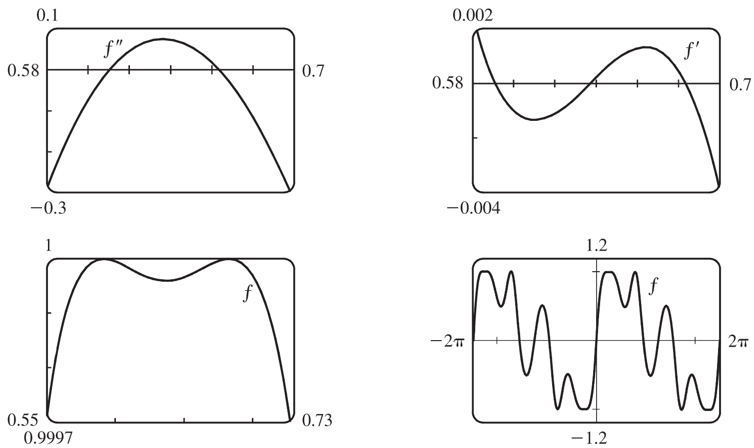


From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

25.



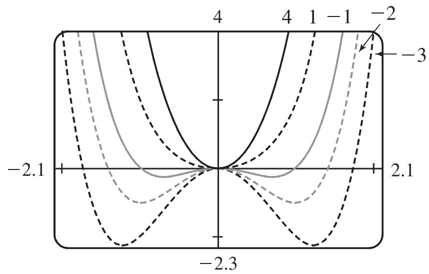
From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of $f'' = -[\sin(x + \sin 3x)]\,(1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$ is even more interesting near this x-value: it seems to just touch the x-axis.



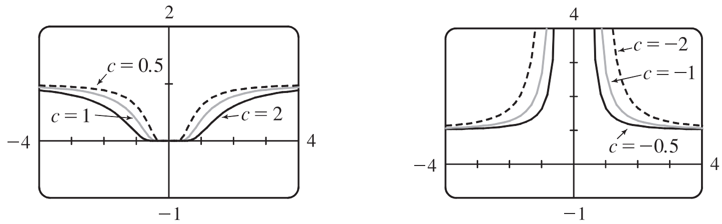
If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

27. $f(x) = x^4 + cx^2 = x^2(x^2 + c)$. Note that f is an even function. For $c \geq 0$, the only x-intercept is the point $(0, 0)$. We calculate $f'(x) = 4x^3 + 2cx = 4x\left(x^2 + \frac{1}{2}c\right) \Rightarrow f''(x) = 12x^2 + 2c$. If $c \geq 0$, $x = 0$ is the only critical point and there is no inflection point. As we can see from the examples, there is no change in the basic shape of the graph for $c \geq 0$; it merely

becomes steeper as c increases. For $c = 0$, the graph is the simple curve $y = x^4$. For $c < 0$, there are x-intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0, 0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c}, -\frac{1}{4}c^2\right)$. As $c \to -\infty$, the x-coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c}, -\frac{5}{36}c^2\right)$, which also move away from the origin as $c \to -\infty$.



29.



$c = 0$ is a transitional value—we get the graph of $y = 1$. For $c > 0$, we see that there is a HA at $y = 1$, and that the graph spreads out as c increases. At first glance there appears to be a minimum at $(0, 0)$, but $f(0)$ is undefined, so there is no minimum or maximum. For $c < 0$, we still have the HA at $y = 1$, but the range is $(1, \infty)$ rather than $(0, 1)$. We also have a VA at $x = 0$. $f(x) = e^{-c/x^2} \;\Rightarrow\; f'(x) = e^{-c/x^2}\left(\dfrac{2c}{x^3}\right) \;\Rightarrow\; f''(x) = \dfrac{2c(2c - 3x^2)}{x^6 e^{c/x^2}}$.

$f'(x) \neq 0$ and $f'(x)$ exists for all $x \neq 0$ (and 0 is not in the domain of f), so there are no maxima or minima.

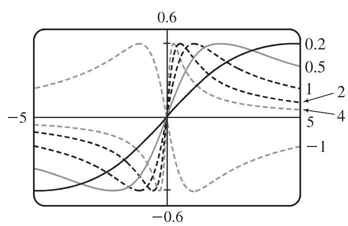
$f''(x) = 0 \;\Rightarrow\; x = \pm\sqrt{2c/3}$, so if $c > 0$, the inflection points spread out as c increases, and if $c < 0$, there are no IP.

For $c > 0$, there are IP at $\left(\pm\sqrt{2c/3}, e^{-3/2}\right)$. Note that the y-coordinate of the IP is constant.

31. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate $f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \;\Leftrightarrow\; c^2x^2 - 1 = 0 \;\Leftrightarrow\; x = \pm 1/c$.

So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

$$f''(x) = \frac{(-2c^3x)(1 + c^2x^2)^2 - (-c^3x^2 + c)[2(1 + c^2x^2)(2c^2x)]}{(1 + c^2x^2)^4}$$
$$= \frac{(-2c^3x)(1 + c^2x^2) + (c^3x^2 - c)(4c^2x)}{(1 + c^2x^2)^3} = \frac{2c^3x(c^2x^2 - 3)}{(1 + c^2x^2)^3}$$



$f''(x) = 0 \;\Leftrightarrow\; x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$. Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

33. $f(x) = cx + \sin x \quad \Rightarrow \quad f'(x) = c + \cos x \quad \Rightarrow \quad f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

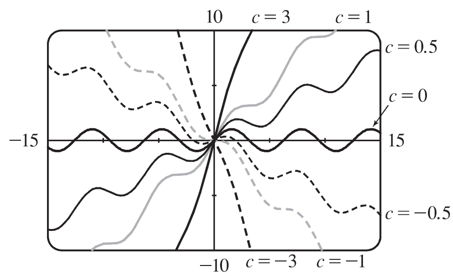
$f(x) = 0 \quad \Leftrightarrow \quad \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \le 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = 1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = -1$, $x_1 = \pi$.)

$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(2n\pi, 2n\pi c)$, where n is an integer.

If $c \ge 1$, then $f'(x) \ge 0$ for all x, so f is increasing and has no extremum. If $c \le -1$, then $f'(x) \le 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1 - c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1 - c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \ge 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.



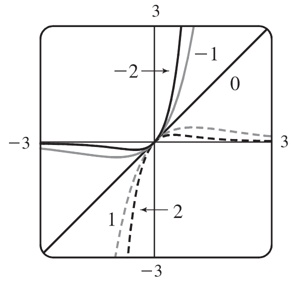
35. If $c < 0$, then $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} xe^{-cx} = \lim\limits_{x \to -\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x \to -\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x \to \infty} f(x) = \infty$.

If $c > 0$, then $\lim\limits_{x \to -\infty} f(x) = -\infty$, and $\lim\limits_{x \to \infty} f(x) \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$, respectively.

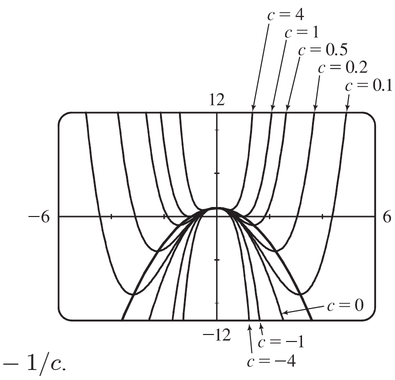
So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \quad \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \quad \Leftrightarrow \quad x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$;

and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



37. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \to 0$, the minimum points spread apart and move downward; they are below the x-axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

(b) $f'(x) = 4cx^3 - 4x = 4cx(x^2 - 1/c)$ $[c \neq 0]$. If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''\left(\pm 1/\sqrt{c}\right) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f\left(\pm 1/\sqrt{c}\right) = c(1/c^2) - 2/c + 1 = -1/c + 1$. But $\left(\pm 1/\sqrt{c}, -1/c + 1\right)$ lies on $y = 1 - x^2$ since $1 - \left(\pm 1/\sqrt{c}\right)^2 = 1 - 1/c$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then

$P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$,

we see that $P'(x) = 0 \quad \Leftrightarrow \quad x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

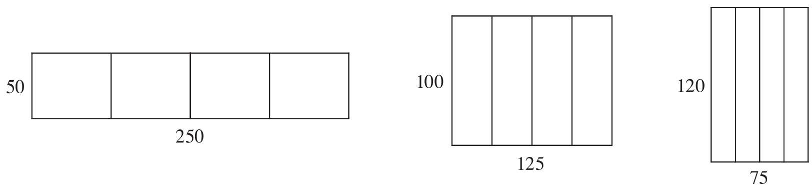
3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

5. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625\text{ m}^2$. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

7. We need to maximize Y for $N \geq 0$. $\quad Y(N) = \dfrac{kN}{1 + N^2} \quad \Rightarrow$

$Y'(N) = \dfrac{(1 + N^2)k - kN(2N)}{(1 + N^2)^2} = \dfrac{k(1 - N^2)}{(1 + N^2)^2} = \dfrac{k(1 + N)(1 - N)}{(1 + N^2)^2}$. $\quad Y'(N) > 0$ for $0 < N < 1$ and $Y'(N) < 0$ for $N > 1$. Thus, Y has an absolute maximum of $Y(1) = \frac{1}{2}k$ at $N = 1$.

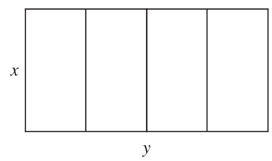
9. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.

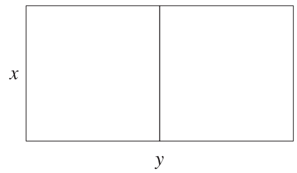


(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

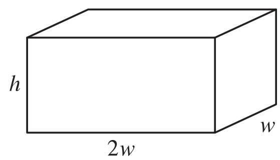
(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5\text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

11. 

$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$. $F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

13. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Rightarrow h = (1200 - b^2)/(4b)$. The volume is $V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Rightarrow V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 324). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000\ \text{cm}^3$.

15. 

$10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$. The cost is

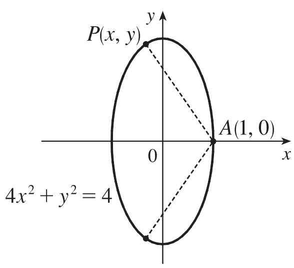
$$\begin{aligned} C(w) &= 10(2w^2) + 6[2(2wh) + 2hw] + 6(2w^2) \\ &= 32w^2 + 36wh = 32w^2 + 180/w \end{aligned}$$

$C'(w) = 64w - 180/w^2 = 4(16w^3 - 45)/w^2 \Rightarrow w = \sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32(2.8125)^{2/3} + 180/\sqrt{2.8125} \approx \$191.28$.

17. The distance from a point (x, y) on the line $y = 4x + 7$ to the origin is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. However, it is easier to work with the *square* of the distance; that is, $D(x) = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2 = x^2 + (4x + 7)^2$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D.

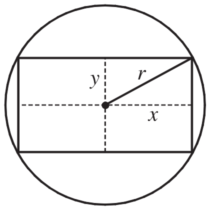
$D'(x) = 2x + 2(4x + 7)(4) = 34x + 56$, so $D'(x) = 0 \Leftrightarrow x = -\frac{28}{17}$.

$D''(x) = 34 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = -\frac{28}{17}$. The point closest to the origin is $(x, y) = \left(-\frac{28}{17}, 4\left(-\frac{28}{17}\right) + 7\right) = \left(-\frac{28}{17}, \frac{7}{17}\right)$.

19. 

From the figure, we see that there are two points that are farthest away from $A(1, 0)$. The distance d from A to an arbitrary point $P(x, y)$ on the ellipse is $d = \sqrt{(x-1)^2 + (y-0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \le x \le 1$, $S(-1) = 4$, $S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are $y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

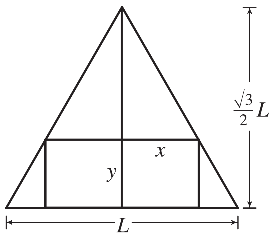
21.



The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now $A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\,\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

23.

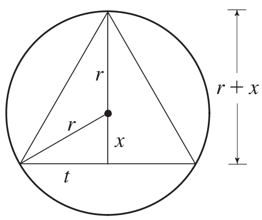


The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since $h^2 + (L/2)^2 = L^2 \;\Rightarrow\; h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \;\Rightarrow\; h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \;\Rightarrow$

$$\sqrt{3}\,x = \frac{\sqrt{3}}{2}L - y \;\Rightarrow\; y = \frac{\sqrt{3}}{2}L - \sqrt{3}\,x \;\Rightarrow\; y = \frac{\sqrt{3}}{2}(L - 2x).$$

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \;\Rightarrow\; x = \sqrt{3}\,L/\big(4\sqrt{3}\big) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

25.



The area of the triangle is

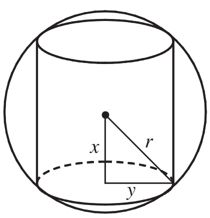
$A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}\,(r + x)$. Then

$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}}$$
$$= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \quad\Rightarrow$$

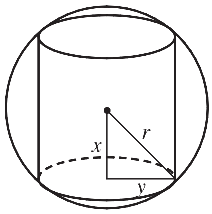
$$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \;\Rightarrow\; x^2 + rx = r^2 - x^2 \;\Rightarrow\; 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \;\Rightarrow$$

$x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \;\Rightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

27.



The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \;\Rightarrow\; y^2 = r^2 - x^2$, so $V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$. $V'(x) = 2\pi\big(r^2 - 3x^2\big) = 0 \;\Rightarrow\; x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\big(r/\sqrt{3}\big) = \pi(r^2 - r^2/3)\big(2r/\sqrt{3}\big) = 4\pi r^3/\big(3\sqrt{3}\big)$.

29. 

The cylinder has surface area

$$2(\text{area of the base}) + (\text{lateral surface area}) = 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height})$$
$$= 2\pi y^2 + 2\pi y(2x)$$

Now $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2 \quad \Rightarrow \quad y = \sqrt{r^2 - x^2}$, so the surface area is

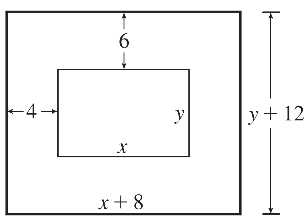
$$S(x) = 2\pi(r^2 - x^2) + 4\pi x\sqrt{r^2 - x^2}, \;\; 0 \le x \le r$$
$$= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\right)$$

Thus, $$S'(x) = 0 - 4\pi x + 4\pi\left[x \cdot \tfrac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1\right]$$
$$= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}$$

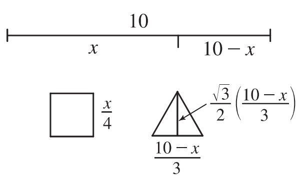
$S'(x) = 0 \quad \Rightarrow \quad x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \quad \Rightarrow \quad \left(x\sqrt{r^2 - x^2}\right)^2 = (r^2 - 2x^2)^2 \quad \Rightarrow$
$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad 5x^4 - 5r^2x^2 + r^4 = 0.$

This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10}r^2$, but we reject the root with the + sign since it doesn't satisfy ($\star$). [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\,r$. Since $S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5 - \sqrt{5}}{10}r^2 \quad \Rightarrow \quad y^2 = r^2 - \frac{5 - \sqrt{5}}{10}r^2 = \frac{5 + \sqrt{5}}{10}r^2 \quad \Rightarrow$ the surface area is

$$2\pi\left(\frac{5 + \sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\frac{5 - \sqrt{5}}{10}}\sqrt{\frac{5 + \sqrt{5}}{10}}\,r^2 = \pi r^2\left[2 \cdot \frac{5 + \sqrt{5}}{10} + 4\frac{\sqrt{(5 - \sqrt{5})(5 + \sqrt{5})}}{10}\right] = \pi r^2\left[\frac{5 + \sqrt{5}}{5} + \frac{2\sqrt{20}}{5}\right]$$
$$= \pi r^2\left[\frac{5 + \sqrt{5} + 2 \cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\frac{5 + 5\sqrt{5}}{5}\right] = \pi r^2\left(1 + \sqrt{5}\right).$$

31. 

$xy = 384 \quad \Rightarrow \quad y = 384/x$. Total area is
$A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so
$A'(x) = 12(1 - 256/x^2) = 0 \quad \Rightarrow \quad x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$.
When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

33. 

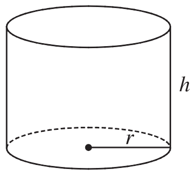
Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10 - x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10 - x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10 - x)^2, \;\; 0 \le x \le 10$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10 - x) = 0 \quad \Leftrightarrow \quad \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \quad \Leftrightarrow \quad x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}$. Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9 + 4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.35$ m.

35. 

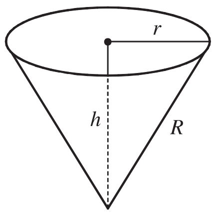
The volume is $V = \pi r^2 h$ and the surface area is

$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

$$S'(r) = 2\pi r - \frac{2V}{r^2} = 0 \quad\Rightarrow\quad 2\pi r^3 = 2V \quad\Rightarrow\quad r = \sqrt[3]{\frac{V}{\pi}}\text{ cm.}$$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$.

When $r = \sqrt[3]{\frac{V}{\pi}}$, $h = \dfrac{V}{\pi r^2} = \dfrac{V}{\pi (V/\pi)^{2/3}} = \sqrt[3]{\dfrac{V}{\pi}}$ cm.

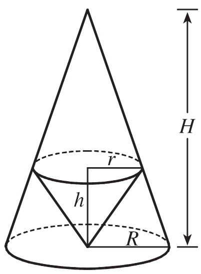
37. 

$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2 h - h^3).$

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since

$V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is

$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3.$

39. 

By similar triangles, $\dfrac{H}{R} = \dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$,

so we'll solve **(1)** for h. $\dfrac{Hr}{R} = H - h \quad\Rightarrow$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R}(R - r) \quad \textbf{(2)}.$$

Thus, $V(r) = \dfrac{\pi}{3} r^2 \cdot \dfrac{H}{R}(R - r) = \dfrac{\pi H}{3R}(Rr^2 - r^3) \quad\Rightarrow$

$$V'(r) = \frac{\pi H}{3R}(2Rr - 3r^2) = \frac{\pi H}{3R} r(2R - 3r).$$

$V'(r) = 0 \quad\Rightarrow\quad r = 0$ or $2R = 3r \quad\Rightarrow\quad r = \frac{2}{3}R$ and from **(2)**, $h = \dfrac{H}{R}\left(R - \frac{2}{3}R\right) = \dfrac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27} \cdot \frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

41. $P(R) = \dfrac{E^2 R}{(R + r)^2} \quad\Rightarrow$

$$P'(R) = \frac{(R+r)^2 \cdot E^2 - E^2 R \cdot 2(R+r)}{[(R+r)^2]^2} = \frac{(R^2 + 2Rr + r^2)E^2 - 2E^2R^2 - 2E^2Rr}{(R+r)^4}$$

$$= \frac{E^2 r^2 - E^2 R^2}{(R+r)^4} = \frac{E^2(r^2 - R^2)}{(R+r)^4} = \frac{E^2(r+R)(r-R)}{(R+r)^4} = \frac{E^2(r-R)}{(R+r)^3}$$

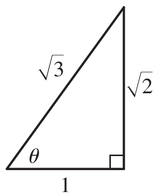
$P'(R) = 0 \quad\Rightarrow\quad R = r \quad\Rightarrow\quad P(r) = \dfrac{E^2 r}{(r+r)^2} = \dfrac{E^2 r}{4r^2} = \dfrac{E^2}{4r}.$

The expression for $P'(R)$ shows that $P'(R) > 0$ for $R < r$ and $P'(R) < 0$ for $R > r$. Thus, the maximum value of the power is $E^2/(4r)$, and this occurs when $R = r$.

43. $S = 6sh - \frac{3}{2}s^2 \cot\theta + 3s^2 \frac{\sqrt{3}}{2} \csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2 \csc^2\theta - 3s^2 \frac{\sqrt{3}}{2} \csc\theta \cot\theta$ or $\frac{3}{2}s^2 \csc\theta \left(\csc\theta - \sqrt{3}\cot\theta\right)$.

(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \;\Rightarrow\; \dfrac{1}{\sin\theta} - \sqrt{3}\,\dfrac{\cos\theta}{\sin\theta} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

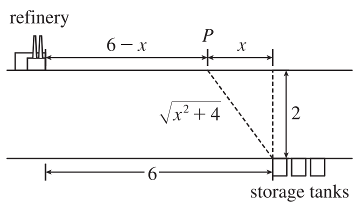
(c) 

If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$\begin{aligned} S &= 6sh - \tfrac{3}{2}s^2 \tfrac{1}{\sqrt{2}} + 3s^2 \tfrac{\sqrt{3}}{2} \tfrac{\sqrt{3}}{\sqrt{2}} = 6sh - \tfrac{3}{2\sqrt{2}}s^2 + \tfrac{9}{2\sqrt{2}}s^2 \\ &= 6sh + \tfrac{6}{2\sqrt{2}}s^2 = 6s\left(h + \tfrac{1}{2\sqrt{2}}s\right) \end{aligned}$$

45. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5 \;\Rightarrow\; T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0 \;\Leftrightarrow\; 8x = 6\sqrt{x^2+25} \;\Leftrightarrow\; 16x^2 = 9(x^2+25) \;\Leftrightarrow\; x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, he should row directly to B.

47. There are $(6-x)$ km over land and $\sqrt{x^2+4}$ km under the river.



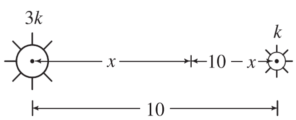
We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6-x)(4) + \left(\sqrt{x^2+4}\right)(8) \;\Rightarrow$

$C'(x) = -4 + 8 \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0 \;\Rightarrow\; 4 = \dfrac{8x}{\sqrt{x^2+4}} \;\Rightarrow\; \sqrt{x^2+4} = 2x \;\Rightarrow\; x^2 + 4 = 4x^2 \;\Rightarrow\; 4 = 3x^2 \;\Rightarrow\; x^2 = \frac{4}{3} \;\Rightarrow$

$x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0$, $2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$, $C\left(2/\sqrt{3}\right) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

49. 

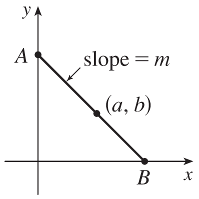
The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

$I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(10-x)^3} = 0 \;\Rightarrow\; 6k(10-x)^3 = 2kx^3 \;\Rightarrow$

$3(10-x)^3 = x^3 \;\Rightarrow\; \sqrt[3]{3}\,(10-x) = x \;\Rightarrow\; 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x \;\Rightarrow\; 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \;\Rightarrow$

$10\sqrt[3]{3} = \left(1 + \sqrt[3]{3}\right)x \;\Rightarrow\; x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

51.



Every line segment in the first quadrant passing through (a,b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x-a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\left(a - \frac{b}{m}, 0\right)$. The distance d from A to B is given by $d = \sqrt{\left[\left(a - \frac{b}{m}\right) - 0\right]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$S(m) = \left(a - \dfrac{b}{m}\right)^2 + (am - b)^2 = a^2 - \dfrac{2ab}{m} + \dfrac{b^2}{m^2} + a^2m^2 - 2abm + b^2$. Thus,

$$\begin{aligned} S'(m) &= \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3) \\ &= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3) \end{aligned}$$

Thus, $S'(m) = 0 \quad\Leftrightarrow\quad m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$.

That value is

$$\begin{aligned} S\left(-\sqrt[3]{\tfrac{b}{a}}\right) &= \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2 \\ &= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \end{aligned}$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 \quad [= (x+y)^3]$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

53. (a) If $c(x) = \dfrac{C(x)}{x}$, then, by Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when $xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad\Leftrightarrow\quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad\Leftrightarrow\quad 2x^{3/2} = 16{,}000 \quad\Leftrightarrow$ $x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate

$c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

55. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is

$\frac{10-8}{27{,}000-33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \Rightarrow$

$y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \Rightarrow R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \Rightarrow$ the price is $p(28{,}500) = \$9.50$.

57. (a) As in Example 6, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that $p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\frac{440-450}{1100-1000} = -\frac{1}{10}$, so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

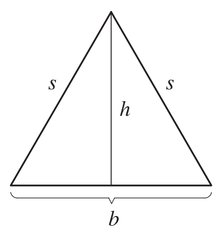
(b) $R(x) = xp(x) = -\frac{1}{10}x^2 + 550x$. $R'(x) = -\frac{1}{5}x + 550 = 0$ when $x = 5(550) = 2750$.

$p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68{,}000 + 150x \Rightarrow P(x) = R(x) - C(x) = -\frac{1}{10}x^2 + 550x - 68{,}000 - 150x = -\frac{1}{10}x^2 + 400x - 68{,}000$,

$P'(x) = -\frac{1}{5}x + 400 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be $450 - 350 = \$100$.

59.



Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$.

Let the perimeter be p, so $2s + b = p$ or $s = (p - b)/2 \Rightarrow$

$A(b) = \frac{1}{2}b\sqrt{(p-b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

$$A'(b) = \frac{\sqrt{p^2 - 2pb}}{4} - \frac{bp/4}{\sqrt{p^2 - 2pb}} = \frac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}.$$

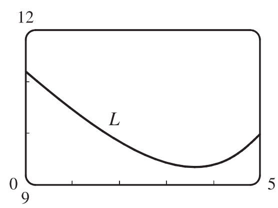
Therefore, $A'(b) = 0 \Rightarrow -3pb + p^2 = 0 \Rightarrow b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \Rightarrow s = b \Rightarrow$ the triangle is equilateral.

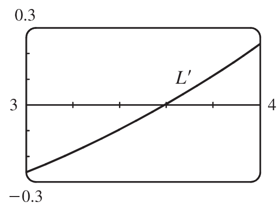
61. Note that $|AD| = |AP| + |PD| \Rightarrow 5 = x + |PD| \Rightarrow |PD| = 5 - x$.

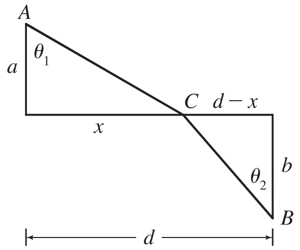
Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5-x)^2 + 2^2} + \sqrt{(5-x)^2 + 3^2}$$
$$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34} \Rightarrow$$

$L'(x) = 1 + \dfrac{x-5}{\sqrt{x^2 - 10x + 29}} + \dfrac{x-5}{\sqrt{x^2 - 10x + 34}}$. From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.





63. 

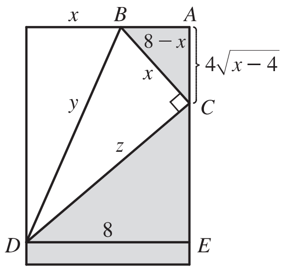
The total time is

$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d - x)^2}}{v_2}, \ 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{d - x}{v_2\sqrt{b^2 + (d - x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad \Rightarrow \quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

65. 

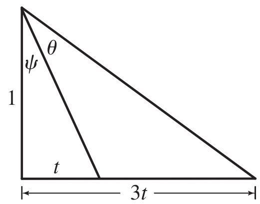
$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so

$z/8 = x/\left(4\sqrt{x - 4}\right) \quad \Rightarrow \quad z = 2x/\sqrt{x - 4}$. Thus, we minimize

$f(x) = y^2 = x^2 + 4x^2/(x - 4) = x^3/(x - 4), \ 4 < x \le 8$.

$$f'(x) = \frac{(x - 4)(3x^2) - x^3}{(x - 4)^2} = \frac{x^2[3(x - 4) - x]}{(x - 4)^2} = \frac{2x^2(x - 6)}{(x - 4)^2} = 0$$

when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

67. 

It suffices to maximize $\tan\theta$. Now

$$\frac{3t}{1} = \tan(\psi + \theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\,\tan\theta} = \frac{t + \tan\theta}{1 - t\tan\theta}. \text{ So}$$

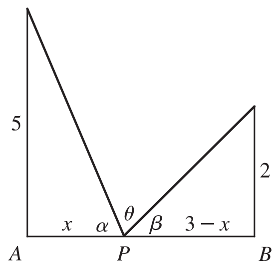
$$3t(1 - t\tan\theta) = t + \tan\theta \quad \Rightarrow \quad 2t = (1 + 3t^2)\tan\theta \quad \Rightarrow \quad \tan\theta = \frac{2t}{1 + 3t^2}.$$

Let $f(t) = \tan\theta = \dfrac{2t}{1 + 3t^2} \quad \Rightarrow \quad f'(t) = \dfrac{2(1 + 3t^2) - 2t(6t)}{(1 + 3t^2)^2} = \dfrac{2(1 - 3t^2)}{(1 + 3t^2)^2} = 0 \quad \Leftrightarrow \quad 1 - 3t^2 = 0 \quad \Leftrightarrow$

$t = \frac{1}{\sqrt{3}}$ since $t \ge 0$. Now $f'(t) > 0$ for $0 \le t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$

and $\tan\theta = \dfrac{2(1/\sqrt{3})}{1 + 3(1/\sqrt{3})^2} = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi + \theta)$ gives us

$\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \quad \Rightarrow \quad \psi = \frac{\pi}{6}$.

69. 

From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

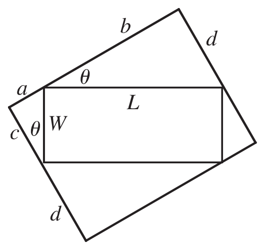
$$\alpha+\beta+\theta = 180^\circ = \pi,\ \theta = \pi - \tan^{-1}\left(\frac{5}{x}\right) - \tan^{-1}\left(\frac{2}{3-x}\right) \quad\Rightarrow$$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\dfrac{d\theta}{dx} = 0 \quad\Rightarrow\quad \dfrac{5}{x^2+25} = \dfrac{2}{x^2-6x+13} \quad\Rightarrow\quad 2x^2+50 = 5x^2-30x+65 \quad\Rightarrow$

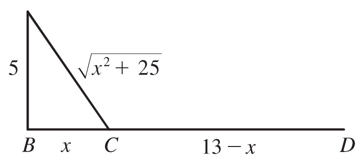
$3x^2-30x+15=0 \quad\Rightarrow\quad x^2-10x+5=0 \quad\Rightarrow\quad x = 5\pm2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5-2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5-2\sqrt{5}$, so θ is maximized when $|AP| = x = 5-2\sqrt{5} \approx 0.53$.

71. 

In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\ &= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta \\ &= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\,\cos\theta \\ &= LW(\sin^2\theta+\cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta = LW + \tfrac{1}{2}(L^2+W^2)\sin2\theta,\ \ 0 \le \theta \le \tfrac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin2\theta = 1 \quad\Leftrightarrow\quad 2\theta = \frac{\pi}{2} \quad\Rightarrow$ $\theta = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2+2LW+W^2) = \frac{1}{2}(L+W)^2$.

73. (a) 

If k = energy/km over land, then energy/km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \quad\Rightarrow\quad 1.96x^2 = x^2+25 \quad\Rightarrow\quad 0.96x^2 = 25 \quad\Rightarrow\quad x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5)+13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight.

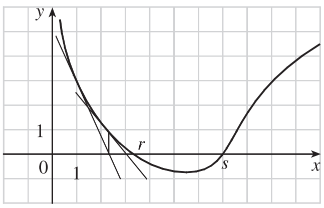
$E = W\sqrt{25+x^2} + L(13-x) \quad \Rightarrow \quad \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+} (W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1 \cdot (25+4^2)^{1/2} \quad \Rightarrow \quad c = \sqrt{41}/4 \approx 1.6$.

4.8 Newton's Method

1. (a)



The tangent line at $x = 1$ intersects the x-axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x-axis at $x \approx 3$, so $x_3 \approx 3.0$.

(b) $x_1 = 5$ would *not* be a better first approximation than $x_1 = 1$ since the tangent line is nearly horizontal. In fact, the second approximation for $x_1 = 5$ appears to be to the left of $x = 1$.

3. Since $x_1 = 3$ and $y = 5x - 4$ is tangent to $y = f(x)$ at $x = 3$, we simply need to find where the tangent line intersects the x-axis. $y = 0 \quad \Rightarrow \quad 5x_2 - 4 = 0 \quad \Rightarrow \quad x_2 = \frac{4}{5}$.

5. $f(x) = x^3 + 2x - 4 \quad \Rightarrow \quad f'(x) = 3x^2 + 2$, so $x_{n+1} = x_n - \dfrac{x_n^3 + 2x_n - 4}{3x_n^2 + 2}$. Now $x_1 = 1 \quad \Rightarrow$

$$x_2 = 1 - \frac{1+2-4}{3 \cdot 1^2 + 2} = 1 - \frac{-1}{5} = 1.2 \quad \Rightarrow \quad x_3 = 1.2 - \frac{(1.2)^3 + 2(1.2) - 4}{3(1.2)^2 + 2} \approx 1.1797.$$

7. $f(x) = x^5 - x - 1 \quad \Rightarrow \quad f'(x) = 5x^4 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n - 1}{5x_n^4 - 1}$. Now $x_1 = 1 \quad \Rightarrow$

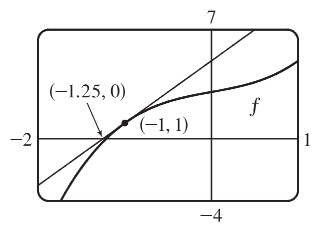
$$x_2 = 1 - \frac{1-1-1}{5-1} = 1 - \left(-\tfrac{1}{4}\right) = 1.25 \quad \Rightarrow \quad x_3 = 1.25 - \frac{(1.25)^5 - 1.25 - 1}{5(1.25)^4 - 1} \approx 1.1785.$$

9. $f(x) = x^3 + x + 3 \quad \Rightarrow \quad f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 3}{3x_n^2 + 1}$.

Now $x_1 = -1 \quad \Rightarrow$

$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1-1+3}{3+1} = -1 - \frac{1}{4} = -1.25.$$

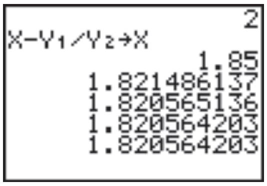
Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.

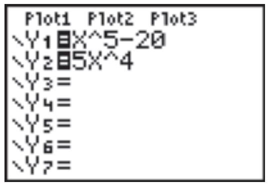


11. To approximate $x = \sqrt[5]{20}$ (so that $x^5 = 20$), we can take $f(x) = x^5 - 20$. So $f'(x) = 5x^4$, and thus,

$x_{n+1} = x_n - \dfrac{x_n^5 - 20}{5x_n^4}$. Since $\sqrt[5]{32} = 2$ and 32 is reasonably close to 20, we'll use $x_1 = 2$. We need to find approximations until they agree to eight decimal places. $x_1 = 2 \quad \Rightarrow \quad x_2 = 1.85$, $x_3 \approx 1.82148614$, $x_4 \approx 1.82056514$, $x_5 \approx 1.82056420 \approx x_6$. So $\sqrt[5]{20} \approx 1.82056420$, to eight decimal places.

Here is a quick and easy method for finding the iterations for Newton's method on a programmable calculator. (The screens shown are from the TI-84 Plus, but the method is similar on other calculators.) Assign $f(x) = x^5 - 20$ to Y_1, and $f'(x) = 5x^4$ to Y_2. Now store $x_1 = 2$ in X and then enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \to \mathrm{X}$ to get $x_2 = 1.85$. By successively pressing the ENTER key, you get the approximations $x_3, x_4, \ldots$.





In Derive, load the utility file `SOLVE`. Enter `NEWTON(x^5-20,x,2)` and then `APPROXIMATE` to get [2, 1.85, 1.82148614, 1.82056514, 1.82056420]. You can request a specific iteration by adding a fourth argument. For example, `NEWTON(x^5-20,x,2,2)` gives [2, 1.85, 1.82148614].

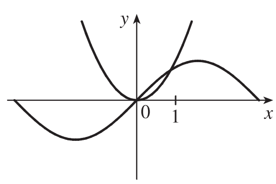
In Maple, make the assignments $f := x \to x\hat{\ }5 - 20$;, $g := x \to x - f(x)/D(f)(x)$;, and $x := 2.$;. Repeatedly execute the command $x := g(x)$; to generate successive approximations.

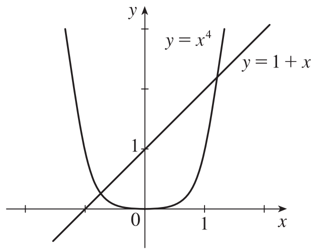
In Mathematica, make the assignments $f[x\_] := x\hat{\ }5 - 20$, $g[x\_] := x - f[x]/f'[x]$, and $x = 2$. Repeatedly execute the command $x = g[x]$ to generate successive approximations.

13. $f(x) = x^4 - 2x^3 + 5x^2 - 6 \quad \Rightarrow \quad f'(x) = 4x^3 - 6x^2 + 10x \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{x_n^4 - 2x_n^3 + 5x_n^2 - 6}{4x_n^3 - 6x_n^2 + 10x_n}$. We need to find approximations until they agree to six decimal places. We'll let x_1 equal the midpoint of the given interval, $[1, 2]$. $x_1 = 1.5 \quad \Rightarrow \quad x_2 = 1.2625$, $x_3 \approx 1.218808$, $x_4 \approx 1.217563$, $x_5 \approx 1.217562 \approx x_6$. So the root is 1.217562 to six decimal places.

15. $\sin x = x^2$, so $f(x) = \sin x - x^2 \quad \Rightarrow \quad f'(x) = \cos x - 2x \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2}{\cos x_n - 2x_n}$. From the figure, the positive root of $\sin x = x^2$ is near 1. $x_1 = 1 \quad \Rightarrow \quad x_2 \approx 0.891396$, $x_3 \approx 0.876985$, $x_4 \approx 0.876726 \approx x_5$. So the positive root is 0.876726, to six decimal places.

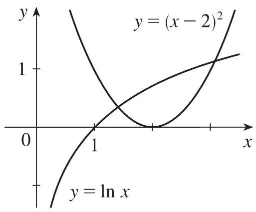


17. 

From the graph, we see that there appear to be points of intersection near $x = -0.7$ and $x = 1.2$. Solving $x^4 = 1 + x$ is the same as solving $f(x) = x^4 - x - 1 = 0$. $f(x) = x^4 - x - 1 \Rightarrow f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

$x_1 = -0.7$	$x_1 = 1.2$
$x_2 \approx -0.725253$	$x_2 \approx 1.221380$
$x_3 \approx -0.724493$	$x_3 \approx 1.220745$
$x_4 \approx -0.724492 \approx x_5$	$x_4 \approx 1.220744 \approx x_5$

To six decimal places, the roots of the equation are -0.724492 and 1.220744.

19. 

From the graph, we see that there appear to be points of intersection near $x = 1.5$ and $x = 3$. Solving $(x-2)^2 = \ln x$ is the same as solving $f(x) = (x-2)^2 - \ln x = 0$. $f(x) = (x-2)^2 - \ln x \Rightarrow$ $f'(x) = 2(x-2) - 1/x$, so $x_{n+1} = x_n - \dfrac{(x_n - 2)^2 - \ln x_n}{2(x_n - 2) - 1/x_n}$.

$x_1 = 1.5$	$x_1 = 3$
$x_2 \approx 1.406721$	$x_2 \approx 3.059167$
$x_3 \approx 1.412370$	$x_3 \approx 3.057106$
$x_4 \approx 1.412391 \approx x_5$	$x_4 \approx 3.057104 \approx x_5$

To six decimal places, the roots of the equation are 1.412391 and 3.057104.

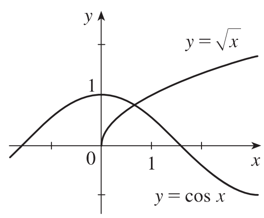
21. From the graph, there appears to be a point of intersection near $x = 0.6$.

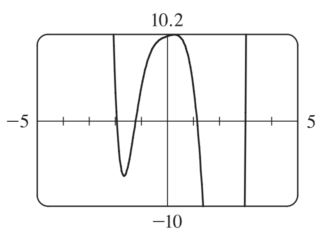
Solving $\cos x = \sqrt{x}$ is the same as solving $f(x) = \cos x - \sqrt{x} = 0$.

$f(x) = \cos x - \sqrt{x} \Rightarrow f'(x) = -\sin x - 1/\left(2\sqrt{x}\right)$, so

$x_{n+1} = x_n - \dfrac{\cos x_n - \sqrt{x_n}}{-\sin x_n - 1/\left(2\sqrt{x}\right)}$. Now $x_1 = 0.6 \Rightarrow x_2 \approx 0.641928$,

$x_3 \approx 0.641714 \approx x_4$. To six decimal places, the root of the equation is 0.641714.



23. 

$f(x) = x^6 - x^5 - 6x^4 - x^2 + x + 10 \Rightarrow$

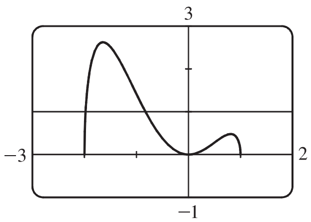
$f'(x) = 6x^5 - 5x^4 - 24x^3 - 2x + 1 \Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^6 - x_n^5 - 6x_n^4 - x_n^2 + x_n + 10}{6x_n^5 - 5x_n^4 - 24x_n^3 - 2x_n + 1}$.

From the graph of f, there appear to be roots near -1.9, -1.2, 1.1, and 3.

$x_1 = -1.9$	$x_1 = -1.2$	$x_1 = 1.1$	$x_1 = 3$
$x_2 \approx -1.94278290$	$x_2 \approx -1.22006245$	$x_2 \approx 1.14111662$	$x_2 \approx 2.99$
$x_3 \approx -1.93828380$	$x_3 \approx -1.21997997 \approx x_4$	$x_3 \approx 1.13929741$	$x_3 \approx 2.98984106$
$x_4 \approx -1.93822884$		$x_4 \approx 1.13929375 \approx x_5$	$x_4 \approx 2.98984102 \approx x_5$
$x_5 \approx -1.93822883 \approx x_6$			

To eight decimal places, the roots of the equation are -1.93822883, -1.21997997, 1.13929375, and 2.98984102.

25. 

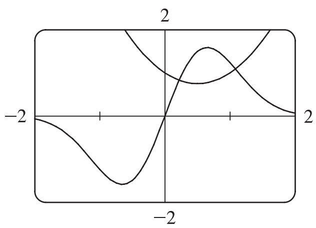
From the graph, $y = x^2\sqrt{2-x-x^2}$ and $y = 1$ intersect twice, at $x \approx -2$ and at $x \approx -1$. $f(x) = x^2\sqrt{2-x-x^2} - 1 \quad \Rightarrow$

$$f'(x) = x^2 \cdot \tfrac{1}{2}(2-x-x^2)^{-1/2}(-1-2x) + (2-x-x^2)^{1/2} \cdot 2x$$
$$= \tfrac{1}{2}x(2-x-x^2)^{-1/2}[x(-1-2x) + 4(2-x-x^2)]$$
$$= \frac{x(8-5x-6x^2)}{2\sqrt{(2+x)(1-x)}},$$

so $x_{n+1} = x_n - \dfrac{x_n^2\sqrt{2-x_n-x_n^2} - 1}{\dfrac{x_n(8-5x_n-6x_n^2)}{2\sqrt{(2+x_n)(1-x_n)}}}$. Trying $x_1 = -2$ won't work because $f'(-2)$ is undefined, so we'll try $x_1 = -1.95$.

$x_1 = -1.95$	$x_1 = -0.8$
$x_2 \approx -1.98580357$	$x_2 \approx -0.82674444$
$x_3 \approx -1.97899778$	$x_3 \approx -0.82646236$
$x_4 \approx -1.97807848$	$x_4 \approx -0.82646233 \approx x_5$
$x_5 \approx -1.97806682$	
$x_6 \approx -1.97806681 \approx x_7$	

To eight decimal places, the roots of the equation are -1.97806681 and -0.82646233.

27. 

Solving $4e^{-x^2}\sin x = x^2 - x + 1$ is the same as solving $f(x) = 4e^{-x^2}\sin x - x^2 + x - 1 = 0$.

$f'(x) = 4e^{-x^2}(\cos x - 2x\sin x) - 2x + 1 \quad \Rightarrow$

$$x_{n+1} = x_n - \frac{4e^{-x_n^2}\sin x_n - x_n^2 + x_n - 1}{4e^{-x_n^2}(\cos x_n - 2x_n\sin x_n) - 2x_n + 1}.$$

From the figure, we see that the graphs intersect at approximately $x = 0.2$ and $x = 1.1$.

$x_1 = 0.2$	$x_1 = 1.1$
$x_2 \approx 0.21883273$	$x_2 \approx 1.08432830$
$x_3 \approx 0.21916357$	$x_3 \approx 1.08422462 \approx x_4$
$x_4 \approx 0.21916368 \approx x_5$	

To eight decimal places, the roots of the equation are 0.21916368 and 1.08422462.

29. (a) $f(x) = x^2 - a \quad \Rightarrow \quad f'(x) = 2x$, so Newton's method gives

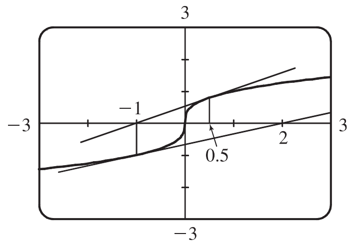
$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

31. $f(x) = x^3 - 3x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

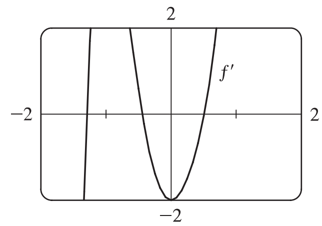
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$



Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.

35. (a) $f(x) = x^6 - x^4 + 3x^3 - 2x \quad \Rightarrow \quad f'(x) = 6x^5 - 4x^3 + 9x^2 - 2 \quad \Rightarrow$ $f''(x) = 30x^4 - 12x^2 + 18x$. To find the critical numbers of f, we'll find the zeros of f'. From the graph of f', it appears there are zeros at approximately $x = -1.3, -0.4$, and 0.5. Try $x_1 = -1.3 \quad \Rightarrow$

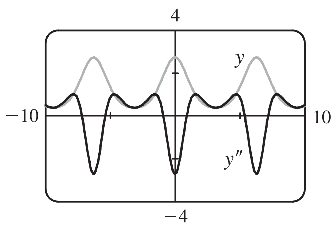
$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} \approx -1.293344 \quad \Rightarrow \quad x_3 \approx -1.293227 \approx x_4.$$



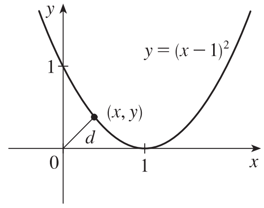
Now try $x_1 = -0.4 \quad \Rightarrow \quad x_2 \approx -0.443755 \quad \Rightarrow \quad x_3 \approx -0.441735 \quad \Rightarrow \quad x_4 \approx -0.441731 \approx x_5$. Finally try $x_1 = 0.5 \quad \Rightarrow \quad x_2 \approx 0.507937 \quad \Rightarrow \quad x_3 \approx 0.507854 \approx x_4$. Therefore, $x = -1.293227$, -0.441731, and 0.507854 are all the critical numbers correct to six decimal places.

(b) There are two critical numbers where f' changes from negative to positive, so f changes from decreasing to increasing. $f(-1.293227) \approx -2.0212$ and $f(0.507854) \approx -0.6721$, so -2.0212 is the absolute minimum value of f correct to four decimal places.

37.



From the figure, we see that $y = f(x) = e^{\cos x}$ is periodic with period 2π. To find the x-coordinates of the IP, we only need to approximate the zeros of y'' on $[0, \pi]$. $f'(x) = -e^{\cos x}\sin x \quad \Rightarrow \quad f''(x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. Since $e^{\cos x} \neq 0$, we will use Newton's method with $g(x) = \sin^2 x - \cos x$, $g'(x) = 2\sin x\,\cos x + \sin x$, and $x_1 = 1$. $x_2 \approx 0.904173$, $x_3 \approx 0.904557 \approx x_4$. Thus, $(0.904557, 1.855277)$ is the IP.



39. We need to minimize the distance from $(0,0)$ to an arbitrary point (x,y) on the curve $y=(x-1)^2$. $d=\sqrt{x^2+y^2} \Rightarrow$ $d(x)=\sqrt{x^2+[(x-1)^2]^2}=\sqrt{x^2+(x-1)^4}$. When $d'=0$, d will be minimized and equivalently, $s=d^2$ will be minimized, so we will use Newton's method with $f=s'$ and $f'=s''$.

$f(x)=2x+4(x-1)^3 \quad\Rightarrow\quad f'(x)=2+12(x-1)^2$, so $x_{n+1}=x_n-\dfrac{2x_n+4(x_n-1)^3}{2+12(x_n-1)^2}$. Try $x_1=0.5 \quad\Rightarrow$ $x_2=0.4$, $x_3\approx 0.410127$, $x_4\approx 0.410245\approx x_5$. Now $d(0.410245)\approx 0.537841$ is the minimum distance and the point on the parabola is $(0.410245, 0.347810)$, correct to six decimal places.

41. In this case, $A=18{,}000$, $R=375$, and $n=5(12)=60$. So the formula $A=\dfrac{R}{i}[1-(1+i)^{-n}]$ becomes

$18{,}000=\dfrac{375}{x}[1-(1+x)^{-60}] \quad\Leftrightarrow\quad 48x=1-(1+x)^{-60} \quad$ [multiply each term by $(1+x)^{60}$] $\quad\Leftrightarrow$ $48x(1+x)^{60}-(1+x)^{60}+1=0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x)&=48x(60)(1+x)^{59}+48(1+x)^{60}-60(1+x)^{59}\\ &=12(1+x)^{59}[4x(60)+4(1+x)-5]=12(1+x)^{59}(244x-1)\end{aligned}$$

$x_{n+1}=x_n-\dfrac{48x_n(1+x_n)^{60}-(1+x_n)^{60}+1}{12(1+x_n)^{59}(244x_n-1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x=i$. So let $x_1=1\%=0.01$, and we get $x_2\approx 0.0082202$, $x_3\approx 0.0076802$, $x_4\approx 0.0076291$, $x_5\approx 0.0076286\approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

4.9 Antiderivatives

1. $f(x)=x-3=x^1-3 \quad\Rightarrow\quad F(x)=\dfrac{x^{1+1}}{1+1}-3x+C=\frac{1}{2}x^2-3x+C$

Check: $F'(x)=\frac{1}{2}(2x)-3+0=x-3=f(x)$

3. $f(x)=\frac{1}{2}+\frac{3}{4}x^2-\frac{4}{5}x^3 \quad\Rightarrow\quad F(x)=\frac{1}{2}x+\dfrac{3}{4}\dfrac{x^{2+1}}{2+1}-\dfrac{4}{5}\dfrac{x^{3+1}}{3+1}+C=\frac{1}{2}x+\frac{1}{4}x^3-\frac{1}{5}x^4+C$

Check: $F'(x)=\frac{1}{2}+\frac{1}{4}(3x^2)-\frac{1}{5}(4x^3)+0=\frac{1}{2}+\frac{3}{4}x^2-\frac{4}{5}x^3=f(x)$

5. $f(x)=(x+1)(2x-1)=2x^2+x-1 \quad\Rightarrow\quad F(x)=2\left(\frac{1}{3}x^3\right)+\frac{1}{2}x^2-x+C=\frac{2}{3}x^3+\frac{1}{2}x^2-x+C$

7. $f(x)=5x^{1/4}-7x^{3/4} \quad\Rightarrow\quad F(x)=5\dfrac{x^{1/4+1}}{\frac{1}{4}+1}-7\dfrac{x^{3/4+1}}{\frac{3}{4}+1}+C=5\dfrac{x^{5/4}}{5/4}-7\dfrac{x^{7/4}}{7/4}+C=4x^{5/4}-4x^{7/4}+C$

9. $f(x)=6\sqrt{x}-\sqrt[6]{x}=6x^{1/2}-x^{1/6} \quad\Rightarrow$

$F(x)=6\dfrac{x^{1/2+1}}{\frac{1}{2}+1}-\dfrac{x^{1/6+1}}{\frac{1}{6}+1}+C=6\dfrac{x^{3/2}}{3/2}-\dfrac{x^{7/6}}{7/6}+C=4x^{3/2}-\frac{6}{7}x^{7/6}+C$

11. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1(b) for a similar problem.

13. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2} = \dfrac{u^4}{u^2} + \dfrac{3u^{1/2}}{u^2} = u^2 + 3u^{-3/2} \quad \Rightarrow$

$F(u) = \dfrac{u^3}{3} + 3\dfrac{u^{-3/2+1}}{-3/2+1} + C = \dfrac{1}{3}u^3 + 3\dfrac{u^{-1/2}}{-1/2} + C = \dfrac{1}{3}u^3 - \dfrac{6}{\sqrt{u}} + C$

15. $g(\theta) = \cos\theta - 5\sin\theta \quad \Rightarrow \quad G(\theta) = \sin\theta - 5(-\cos\theta) + C = \sin\theta + 5\cos\theta + C$

17. $f(x) = 5e^x - 3\cosh x \quad \Rightarrow \quad F(x) = 5e^x - 3\sinh x + C$

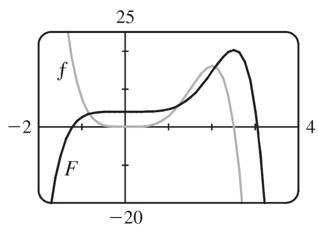
19. $f(x) = \dfrac{x^5 - x^3 + 2x}{x^4} = x - \dfrac{1}{x} + \dfrac{2}{x^3} = x - \dfrac{1}{x} + 2x^{-3} \quad \Rightarrow$

$F(x) = \dfrac{x^2}{2} - \ln|x| + 2\left(\dfrac{x^{-3+1}}{-3+1}\right) + C = \frac{1}{2}x^2 - \ln|x| - \dfrac{1}{x^2} + C$

21. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C.$

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so $F(x) = x^5 - \frac{1}{3}x^6 + 4.$

The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.



23. $f''(x) = 6x + 12x^2 \quad \Rightarrow \quad f'(x) = 6 \cdot \dfrac{x^2}{2} + 12 \cdot \dfrac{x^3}{3} + C = 3x^2 + 4x^3 + C \quad \Rightarrow$

$f(x) = 3 \cdot \dfrac{x^3}{3} + 4 \cdot \dfrac{x^4}{4} + Cx + D = x^3 + x^4 + Cx + D$ [C and D are just arbitrary constants]

25. $f''(x) = \frac{2}{3}x^{2/3} \quad \Rightarrow \quad f'(x) = \dfrac{2}{3}\left(\dfrac{x^{5/3}}{5/3}\right) + C = \frac{2}{5}x^{5/3} + C \quad \Rightarrow \quad f(x) = \dfrac{2}{5}\left(\dfrac{x^{8/3}}{8/3}\right) + Cx + D = \frac{3}{20}x^{8/3} + Cx + D$

27. $f'''(t) = e^t \quad \Rightarrow \quad f''(t) = e^t + C \quad \Rightarrow \quad f'(t) = e^t + Ct + D \quad \Rightarrow \quad f(t) = e^t + \frac{1}{2}Ct^2 + Dt + E$

29. $f'(x) = 1 - 6x \quad \Rightarrow \quad f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \quad \Rightarrow \quad C = 8$, so $f(x) = x - 3x^2 + 8$.

31. $f'(x) = \sqrt{x}(6 + 5x) = 6x^{1/2} + 5x^{3/2} \quad \Rightarrow \quad f(x) = 4x^{3/2} + 2x^{5/2} + C.$

$f(1) = 6 + C$ and $f(1) = 10 \quad \Rightarrow \quad C = 4$, so $f(x) = 4x^{3/2} + 2x^{5/2} + 4.$

33. $f'(t) = 2\cos t + \sec^2 t \quad \Rightarrow \quad f(t) = 2\sin t + \tan t + C$ because $-\pi/2 < t < \pi/2$.

$f\left(\frac{\pi}{3}\right) = 2\left(\sqrt{3}/2\right) + \sqrt{3} + C = 2\sqrt{3} + C$ and $f\left(\frac{\pi}{3}\right) = 4 \quad \Rightarrow \quad C = 4 - 2\sqrt{3}$, so $f(t) = 2\sin t + \tan t + 4 - 2\sqrt{3}$.

35. $f'(x) = x^{-1/3}$ has domain $(-\infty, 0) \cup (0, \infty)$ $\Rightarrow$ $f(x) = \begin{cases} \frac{3}{2}x^{2/3} + C_1 & \text{if } x > 0 \\ \frac{3}{2}x^{2/3} + C_2 & \text{if } x < 0 \end{cases}$

$f(1) = \frac{3}{2} + C_1$ and $f(1) = 1$ $\Rightarrow$ $C_1 = -\frac{1}{2}$. $f(-1) = \frac{3}{2} + C_2$ and $f(-1) = -1$ $\Rightarrow$ $C_2 = -\frac{5}{2}$.

Thus, $f(x) = \begin{cases} \frac{3}{2}x^{2/3} - \frac{1}{2} & \text{if } x > 0 \\ \frac{3}{2}x^{2/3} - \frac{5}{2} & \text{if } x < 0 \end{cases}$

37. $f''(x) = 24x^2 + 2x + 10$ $\Rightarrow$ $f'(x) = 8x^3 + x^2 + 10x + C$. $f'(1) = 8 + 1 + 10 + C$ and $f'(1) = -3$ $\Rightarrow$ $19 + C = -3$ $\Rightarrow$ $C = -22$, so $f'(x) = 8x^3 + x^2 + 10x - 22$ and hence, $f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + D$. $f(1) = 2 + \frac{1}{3} + 5 - 22 + D$ and $f(1) = 5$ $\Rightarrow$ $D = 22 - \frac{7}{3} = \frac{59}{3}$, so $f(x) = 2x^4 + \frac{1}{3}x^3 + 5x^2 - 22x + \frac{59}{3}$.

39. $f''(\theta) = \sin\theta + \cos\theta$ $\Rightarrow$ $f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4$ $\Rightarrow$ $C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3$ $\Rightarrow$ $D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

41. $f''(x) = 2 - 12x$ $\Rightarrow$ $f'(x) = 2x - 6x^2 + C$ $\Rightarrow$ $f(x) = x^2 - 2x^3 + Cx + D$. $f(0) = D$ and $f(0) = 9$ $\Rightarrow$ $D = 9$. $f(2) = 4 - 16 + 2C + 9 = 2C - 3$ and $f(2) = 15$ $\Rightarrow$ $2C = 18$ $\Rightarrow$ $C = 9$, so $f(x) = x^2 - 2x^3 + 9x + 9$.

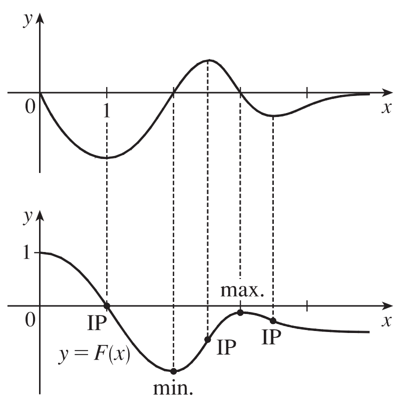
43. $f''(x) = 2 + \cos x$ $\Rightarrow$ $f'(x) = 2x + \sin x + C$ $\Rightarrow$ $f(x) = x^2 - \cos x + Cx + D$. $f(0) = -1 + D$ and $f(0) = -1$ $\Rightarrow$ $D = 0$. $f\left(\frac{\pi}{2}\right) = \pi^2/4 + \left(\frac{\pi}{2}\right)C$ and $f\left(\frac{\pi}{2}\right) = 0$ $\Rightarrow$ $\left(\frac{\pi}{2}\right)C = -\pi^2/4$ $\Rightarrow$ $C = -\frac{\pi}{2}$, so $f(x) = x^2 - \cos x - \left(\frac{\pi}{2}\right)x$.

45. $f''(x) = x^{-2}, x > 0$ $\Rightarrow$ $f'(x) = -1/x + C$ $\Rightarrow$ $f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ [since $x > 0$]. $f(1) = 0$ $\Rightarrow$ $C + D = 0$ and $f(2) = 0$ $\Rightarrow$ $-\ln 2 + 2C + D = 0$ $\Rightarrow$ $-\ln 2 + 2C - C = 0$ [since $D = -C$] $\Rightarrow$ $-\ln 2 + C = 0$ $\Rightarrow$ $C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

47. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $f(1) = 6$ $\Rightarrow$ $1^2 + 1 + C = 6$ $\Rightarrow$ $C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

49. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

51.



The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.

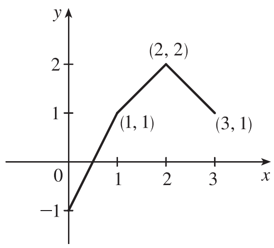
Where f is negative (positive), F is decreasing (increasing).

Where f changes from negative to positive, F will have a minimum.

Where f changes from positive to negative, F will have a maximum.

Where f is decreasing (increasing), F is concave downward (upward).

53.



$$f'(x)=\begin{cases}2 & \text{if } 0\le x<1\\ 1 & \text{if } 1<x<2\\ -1 & \text{if } 2<x\le 3\end{cases}\quad\Rightarrow\quad f(x)=\begin{cases}2x+C & \text{if } 0\le x<1\\ x+D & \text{if } 1<x<2\\ -x+E & \text{if } 2<x\le 3\end{cases}$$

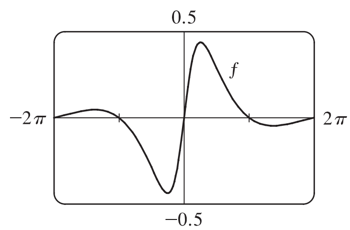
$f(0)=-1 \;\Rightarrow\; 2(0)+C=-1 \;\Rightarrow\; C=-1$. Starting at the point $(0,-1)$ and moving to the right on a line with slope 2 gets us to the point $(1,1)$.

The slope for $1<x<2$ is 1, so we get to the point $(2,2)$. Here we have used the fact that f is continuous. We can include the point $x=1$ on either the first or the second part of f. The line connecting $(1,1)$ to $(2,2)$ is $y=x$, so $D=0$. The slope for $2<x\le 3$ is -1, so we get to $(3,1)$. $f(3)=1 \;\Rightarrow\; -3+E=1 \;\Rightarrow\; E=4$. Thus

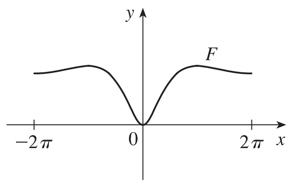
$$f(x)=\begin{cases}2x-1 & \text{if } 0\le x\le 1\\ x & \text{if } 1<x<2\\ -x+4 & \text{if } 2\le x\le 3\end{cases}$$

Note that $f'(x)$ does not exist at $x=1$ or at $x=2$.

55. $f(x)=\dfrac{\sin x}{1+x^2}$, $-2\pi\le x\le 2\pi$



Note that the graph of f is one of an odd function, so the graph of F will be one of an even function.



57. $v(t)=s'(t)=\sin t-\cos t \;\Rightarrow\; s(t)=-\cos t-\sin t+C$. $s(0)=-1+C$ and $s(0)=0 \;\Rightarrow\; C=1$, so $s(t)=-\cos t-\sin t+1$.

59. $a(t)=v'(t)=t-2 \;\Rightarrow\; v(t)=\frac{1}{2}t^2-2t+C$. $v(0)=C$ and $v(0)=3 \;\Rightarrow\; C=3$, so $v(t)=\frac{1}{2}t^2-2t+3$ and $s(t)=\frac{1}{6}t^3-t^2+3t+D$. $s(0)=D$ and $s(0)=1 \;\Rightarrow\; D=1$, and $s(t)=\frac{1}{6}t^3-t^2+3t+1$.

61. $a(t)=v'(t)=10\sin t+3\cos t \;\Rightarrow\; v(t)=-10\cos t+3\sin t+C \;\Rightarrow\; s(t)=-10\sin t-3\cos t+Ct+D$. $s(0)=-3+D=0$ and $s(2\pi)=-3+2\pi C+D=12 \;\Rightarrow\; D=3$ and $C=\frac{6}{\pi}$. Thus, $s(t)=-10\sin t-3\cos t+\frac{6}{\pi}t+3$.

63. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0)=0$ and $s(0)=450$. $v'(t)=a(t)=-9.8 \;\Rightarrow\; v(t)=-9.8t+C$. Now $v(0)=0 \;\Rightarrow\; C=0$, so $v(t)=-9.8t \;\Rightarrow$ $s(t)=-4.9t^2+D$. Last, $s(0)=450 \;\Rightarrow\; D=450 \;\Rightarrow\; s(t)=450-4.9t^2$.

(b) The stone reaches the ground when $s(t)=0$. $450-4.9t^2=0 \;\Rightarrow\; t^2=450/4.9 \;\Rightarrow\; t_1=\sqrt{450/4.9}\approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \Rightarrow 0 + C = -5 \Rightarrow$ $v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \Rightarrow D = 450 \Rightarrow s(t) = -4.9t^2 - 5t + 450$. Solving $s(t) = 0$ by using the quadratic formula gives us $t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \Rightarrow t_1 \approx 9.09$ s.

65. By Exercise 64 with $a = -9.8$, $s(t) = -4.9t^2 + v_0 t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So $[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0 t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0 t = v_0^2 - 19.6\left(-4.9t^2 + v_0 t\right)$. But $-4.9t^2 + v_0 t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

67. Using Exercise 64 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \Rightarrow -32t = -120 \Rightarrow t = 3.75$, so $0 = s(3.75) = -16(3.75)^2 + h \Rightarrow h = 16(3.75)^2 = 225$ ft.

69. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \Rightarrow$ $K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \Rightarrow C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

71. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \Rightarrow$ $v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \Rightarrow$ $s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow$ $v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow v(t) = -55$. At 55 m/s, it will take $100/55 \approx 1.8$ s to fall the last 100 m. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

73. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is 50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \Rightarrow C = 44$. Thus, $v(t) = kt + 44 \Rightarrow$ $v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \Rightarrow 5k = \frac{88}{3} \Rightarrow k = \frac{88}{15} \approx 5.87$ ft/s^2.

75. Let the acceleration be $a(t) = k$ km/h^2. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0. We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$. Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$s(t_f) = \dfrac{1}{2}k\left(-\dfrac{100}{k}\right)^2 + 100\left(-\dfrac{100}{k}\right) = 10{,}000\left(\dfrac{1}{2k} - \dfrac{1}{k}\right) = -\dfrac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is

$-\dfrac{5{,}000}{k} < 0.08 \Rightarrow -\dfrac{5{,}000}{0.08} > k$ [k is negative] $\Rightarrow k < -62{,}500$ km/h^2, or equivalently, $k < -\frac{3125}{648} \approx -4.82$ m/s^2.

77. (a) First note that $90\text{ mi/h} = 90 \times \frac{5280}{3600}\text{ ft/s} = 132\text{ ft/s}$. Then $a(t) = 4\text{ ft/s}^2 \Rightarrow v(t) = 4t + C$, but $v(0) = 0 \Rightarrow$ $C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \le t \le 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 < t \le 933$ we have $v(t) = 132\text{ ft/s} \Rightarrow s(933) = 132(900) + 2178 = 120{,}978\text{ ft} = 22.9125\text{ mi}$.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is $2178 + 110{,}088 + 2178 = 114{,}444\text{ ft} = 21.675\text{ mi}$.

(c) $45\text{ mi} = 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is $1767 + 2(33) = 1833\text{ s} = 30\text{ min }33\text{ s} = 30.55$ min.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

4 Review

CONCEPT CHECK

1. A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f, whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c. See Figure 4 in Section 4.1.

2. (a) See Theorem 4.1.3.

(b) See the Closed Interval Method before Example 8 in Section 4.1.

3. (a) See Theorem 4.1.4.

(b) See Definition 4.1.6.

4. (a) See Rolle's Theorem at the beginning of Section 4.2.

(b) See the Mean Value Theorem in Section 4.2. Geometric interpretation—there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.

5. (a) See the I/D Test before Example 1 in Section 4.3.

(b) If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I.

(c) See the Concavity Test before Example 4 in Section 4.3.

(d) An inflection point is a point where a curve changes its direction of concavity. They can be found by determining the points at which the second derivative changes sign.

6. (a) See the First Derivative Test after Example 1 in Section 4.3.

(b) See the Second Derivative Test before Example 6 in Section 4.3.

(c) See the note before Example 7 in Section 4.3.

7. (a) See l'Hospital's Rule and the three notes that follow it in Section 4.4.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

8. Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussion at the beginning of Section 4.5 and the first paragraph in Section 4.6.

9. (a) See Figure 3 in Section 4.8.

(b) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

(c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0. It also fails when $f'(x_i)$ is undefined, such as with $f(x) = 1/x - 2$ and $x_1 = 1$.

10. (a) See the definition at the beginning of Section 4.9.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I, then they differ by a constant.

TRUE-FALSE QUIZ

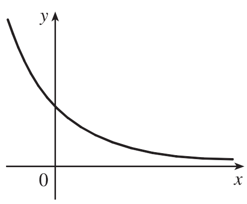
1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

5. True. This is an example of part (b) of the I/D Test.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

9. True. The graph of one such function is sketched.



11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ [since f and g are increasing on I], so $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)\,g(x) = x(x - 1)$ is not increasing on $(0, 1)$.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ [f is increasing] $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ [f is positive] $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

17. True. If f is periodic, then there is a number p such that $f(x+p) = f(p)$ for all x. Differentiating gives $f'(x) = f'(x+p) \cdot (x+p)' = f'(x+p) \cdot 1 = f'(x+p)$, so f' is periodic.

19. True. By the Mean Value Theorem, there exists a number c in $(0,1)$ such that $f(1) - f(0) = f'(c)(1-0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

EXERCISES

1. $f(x) = x^3 - 6x^2 + 9x + 1$, $[2, 4]$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. $f'(x) = 0$ $\Rightarrow$ $x = 1$ or $x = 3$, but 1 is not in the interval. $f'(x) > 0$ for $3 < x < 4$ and $f'(x) < 0$ for $2 < x < 3$, so $f(3) = 1$ is a local minimum value. Checking the endpoints, we find $f(2) = 3$ and $f(4) = 5$. Thus, $f(3) = 1$ is the absolute minimum value and $f(4) = 5$ is the absolute maximum value.

3. $f(x) = \dfrac{3x-4}{x^2+1}$, $[-2, 2]$. $f'(x) = \dfrac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2} = \dfrac{-(3x^2 - 8x - 3)}{(x^2+1)^2} = \dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$.

$f'(x) = 0$ $\Rightarrow$ $x = -\frac{1}{3}$ or $x = 3$, but 3 is not in the interval. $f'(x) > 0$ for $-\frac{1}{3} < x < 2$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{3}$, so $f\left(-\frac{1}{3}\right) = \frac{-5}{10/9} = -\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2) = -2$ and $f(2) = \frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$ is the absolute minimum value and $f(2) = \frac{2}{5}$ is the absolute maximum value.

5. $f(x) = x + \sin 2x$, $[0, \pi]$. $f'(x) = 1 + 2\cos 2x = 0$ $\Leftrightarrow$ $\cos 2x = -\frac{1}{2}$ $\Leftrightarrow$ $2x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ $\Leftrightarrow$ $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

$f''(x) = -4\sin 2x$, so $f''\left(\frac{\pi}{3}\right) = -4\sin\frac{2\pi}{3} = -2\sqrt{3} < 0$ and $f''\left(\frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 2\sqrt{3} > 0$, so $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91$ is a local maximum value and $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.23$ is a local minimum value. Also $f(0) = 0$ and $f(\pi) = \pi$, so $f(0) = 0$ is the absolute minimum value and $f(\pi) = \pi$ is the absolute maximum value.

7. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{\tan \pi x}{\ln(1+x)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\pi \sec^2 \pi x}{1/(1+x)} = \frac{\pi \cdot 1^2}{1/1} = \pi$

9. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4e^{4x} - 4}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{16e^{4x}}{2} = \lim_{x\to 0} 8e^{4x} = 8 \cdot 1 = 8$

11. This limit has the form $\infty \cdot 0$. $\displaystyle\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$

13. This limit has the form $\infty - \infty$.

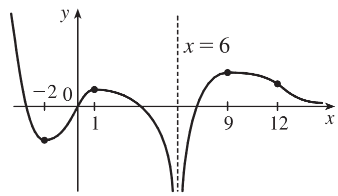
$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\left(\frac{x\ln x-x+1}{(x-1)\ln x}\right)\overset{\text{H}}{=}\lim_{x\to 1^+}\frac{x\cdot(1/x)+\ln x-1}{(x-1)\cdot(1/x)+\ln x}=\lim_{x\to 1^+}\frac{\ln x}{1-1/x+\ln x}$$

$$\overset{\text{H}}{=}\lim_{x\to 1^+}\frac{1/x}{1/x^2+1/x}=\frac{1}{1+1}=\frac{1}{2}$$

15. $f(0)=0$, $f'(-2)=f'(1)=f'(9)=0$, $\lim\limits_{x\to\infty} f(x)=0$, $\lim\limits_{x\to 6} f(x)=-\infty$,

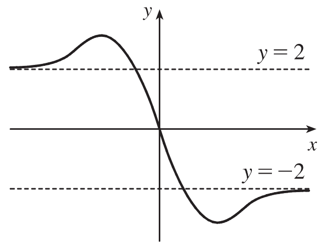
$f'(x)<0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$, $f'(x)>0$ on $(-2,1)$ and $(6,9)$,

$f''(x)>0$ on $(-\infty,0)$ and $(12,\infty)$, $f''(x)<0$ on $(0,6)$ and $(6,12)$



17. f is odd, $f'(x)<0$ for $0<x<2$, $f'(x)>0$ for $x>2$,

$f''(x)>0$ for $0<x<3$, $f''(x)<0$ for $x>3$, $\lim_{x\to\infty} f(x)=-2$



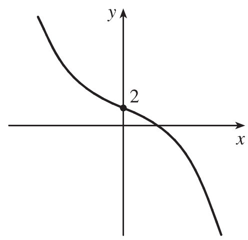
19. $y=f(x)=2-2x-x^3$ **A.** $D=\mathbb{R}$ **B.** y-intercept: $f(0)=2$.

The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x)=-2-3x^2=-(3x^2+2)<0$, so f is decreasing on $\mathbb{R}$.

F. No extreme value **G.** $f''(x)=-6x<0$ on $(0,\infty)$ and $f''(x)>0$ on $(-\infty,0)$, so f is CD on $(0,\infty)$ and CU on $(-\infty,0)$. There is an IP at $(0,2)$.

H.

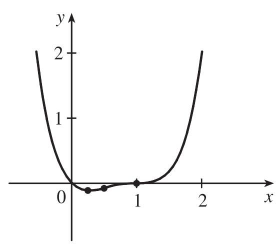


21. $y=f(x)=x^4-3x^3+3x^2-x=x(x-1)^3$ **A.** $D=\mathbb{R}$ **B.** y-intercept: $f(0)=0$; x-intercepts: $f(x)=0 \Leftrightarrow x=0$ or $x=1$ **C.** No symmetry **D.** f is a polynomial function and hence, it has no asymptote.

E. $f'(x)=4x^3-9x^2+6x-1$. Since the sum of the coefficients is 0, 1 is a root of f', so $f'(x)=(x-1)\left(4x^2-5x+1\right)=(x-1)^2(4x-1)$. $f'(x)<0 \Rightarrow x<\frac{1}{4}$, so f is decreasing on $\left(-\infty,\frac{1}{4}\right)$ and f is increasing on $\left(\frac{1}{4},\infty\right)$. **F.** $f'(x)$ does not change sign at $x=1$, so there is not a local extremum there. $f\left(\frac{1}{4}\right)=-\frac{27}{256}$ is a local minimum value.

G. $f''(x)=12x^2-18x+6=6(2x-1)(x-1)$. $f''(x)=0 \Leftrightarrow x=\frac{1}{2}$ or 1. $f''(x)<0 \Leftrightarrow \frac{1}{2}<x<1 \Rightarrow f$ is CD on $\left(\frac{1}{2},1\right)$ and CU on $\left(-\infty,\frac{1}{2}\right)$ and $(1,\infty)$. There are inflection points at $\left(\frac{1}{2},-\frac{1}{16}\right)$ and $(1,0)$.

H.



23. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts. **C.** No symmetry.

D. $\lim\limits_{x \to \pm\infty} \dfrac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 0^+} \dfrac{1}{x(x-3)^2} = \infty$, $\lim\limits_{x \to 0^-} \dfrac{1}{x(x-3)^2} = -\infty$, $\lim\limits_{x \to 3} \dfrac{1}{x(x-3)^2} = \infty$,

so $x = 0$ and $x = 3$ are VA. **E.** $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3} \Rightarrow f'(x) > 0 \Leftrightarrow 1 < x < 3$,

so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$.

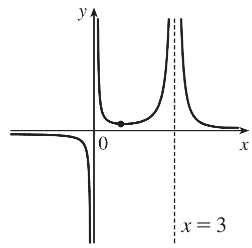
F. Local minimum value $f(1) = \frac{1}{4}$ **G.** $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$.

Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant.

So $f''(x) > 0 \Leftrightarrow x > 0 \Rightarrow f$ is CU on $(0, 3)$ and $(3, \infty)$ and

CD on $(-\infty, 0)$. No IP

H.



25. $y = f(x) = \dfrac{x^2}{x+8} = x - 8 + \dfrac{64}{x+8}$ **A.** $D = \{x \mid x \neq -8\}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim\limits_{x \to \infty} \dfrac{x^2}{x+8} = \infty$, but $f(x) - (x-8) = \dfrac{64}{x+8} \to 0$ as $x \to \infty$, so $y = x - 8$ is a slant asymptote.

$\lim\limits_{x \to -8^+} \dfrac{x^2}{x+8} = \infty$ and $\lim\limits_{x \to -8^-} \dfrac{x^2}{x+8} = -\infty$, so $x = -8$ is a VA. **E.** $f'(x) = 1 - \dfrac{64}{(x+8)^2} = \dfrac{x(x+16)}{(x+8)^2} > 0 \Leftrightarrow$

$x > 0$ or $x < -16$, so f is increasing on $(-\infty, -16)$ and $(0, \infty)$ and

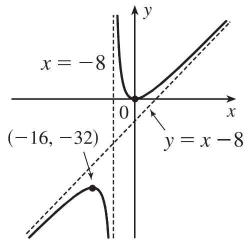
decreasing on $(-16, -8)$ and $(-8, 0)$.

F. Local maximum value $f(-16) = -32$, local minimum value $f(0) = 0$

G. $f''(x) = 128/(x+8)^3 > 0 \Leftrightarrow x > -8$, so f is CU on $(-8, \infty)$ and

CD on $(-\infty, -8)$. No IP

H.



27. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is

decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$,

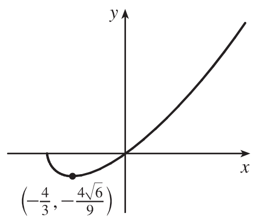
no local maximum

G. $$f''(x) = \frac{2\sqrt{2+x} \cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x) - (3x+4)}{4(2+x)^{3/2}}$$

$$= \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

H.



29. $y = f(x) = \sin^2 x - 2\cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$ **C.** $f(-x) = f(x)$, so f is symmetric with respect to the y-axis. f has period 2π. **D.** No asymptote **E.** $y' = 2\sin x\cos x + 2\sin x = 2\sin x\,(\cos x + 1)$. $y' = 0 \Leftrightarrow \sin x = 0$ or $\cos x = -1 \Leftrightarrow x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ [and so f is increasing] on $(2n\pi, (2n+1)\pi)$; $y' < 0$ [and so f is decreasing] on $((2n-1)\pi, 2n\pi)$.

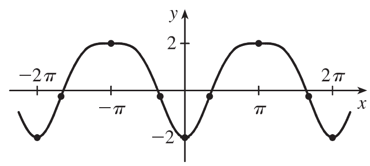
F. Local maximum values are $f((2n+1)\pi) = 2$; local minimum values are $f(2n\pi) = -2$.

G. $y' = \sin 2x + 2\sin x \Rightarrow y'' = 2\cos 2x + 2\cos x = 2(2\cos^2 x - 1) + 2\cos x = 4\cos^2 x + 2\cos x - 2$

$$= 2(2\cos^2 x + \cos x - 1) = 2(2\cos x - 1)(\cos x + 1)$$

$y'' = 0 \Leftrightarrow \cos x = \frac{1}{2}$ or $-1 \Leftrightarrow x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ [and so f is CU] on $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$; $y'' \leq 0$ [and so f is CD] on $\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3}\right)$. There are inflection points at $\left(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4}\right)$.

H.



31. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \leq 1/x \leq 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

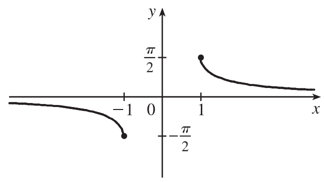
C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim\limits_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2-1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

H.



33. $y = f(x) = xe^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. No symmetry **D.** $\lim\limits_{x\to\infty} xe^{-2x} = \lim\limits_{x\to\infty} \dfrac{x}{e^{2x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{2e^{2x}} = 0$, so $y = 0$ is a HA.

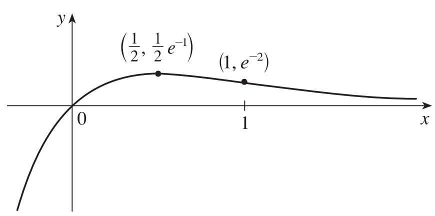
E. $f'(x) = x(-2e^{-2x}) + e^{-2x}(1) = e^{-2x}(-2x+1) > 0 \Leftrightarrow -2x + 1 > 0 \Leftrightarrow x < \frac{1}{2}$ and $f'(x) < 0 \Leftrightarrow x > \frac{1}{2}$, so f is increasing on $\left(-\infty, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1} = 1/(2e)$; no local minimum value

G. $f''(x) = e^{-2x}(-2) + (-2x+1)(-2e^{-2x})$

$$= 2e^{-2x}[-1 - (-2x+1)] = 4\,(x-1)\,e^{-2x}.$$

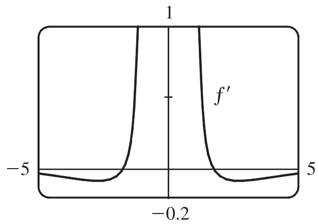
$f''(x) > 0 \Leftrightarrow x > 1$ and $f''(x) < 0 \Leftrightarrow x < 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. IP at $(1, f(1)) = (1, e^{-2})$

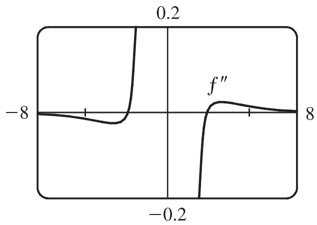
H.

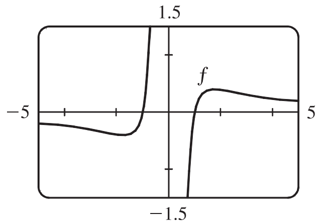


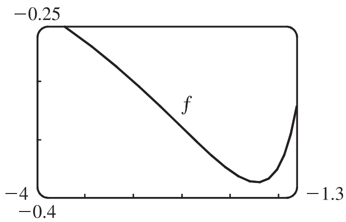
35. $f(x) = \dfrac{x^2 - 1}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \dfrac{3 - x^2}{x^4} \quad \Rightarrow$

$f''(x) = \dfrac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \dfrac{2x^2 - 12}{x^5}$









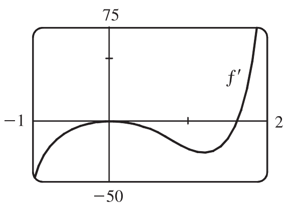
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

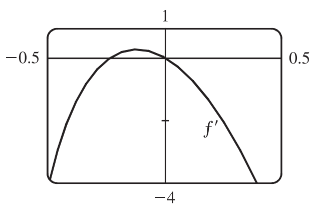
Exact: Now $f'(x) = \dfrac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$; and $f'(x)$ is negative (and so f is decreasing) on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$.

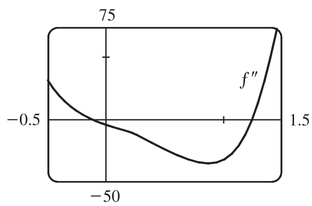
f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f(\sqrt{3}) = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2 - 12}{x^5}$ is positive (so f is CU) on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$, and negative (so f is CD) on $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$

$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$

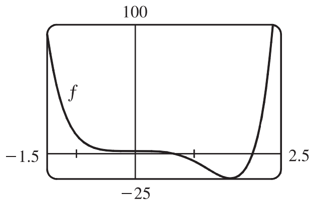


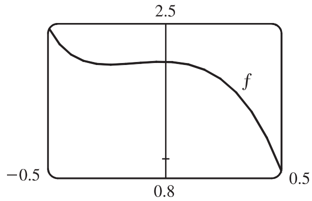


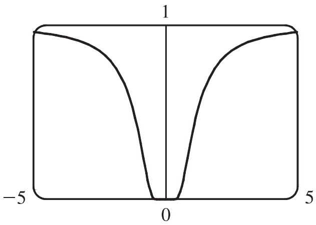


From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$;

f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.





39. 

From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$.

$f(x) = e^{-1/x^2} \quad \Rightarrow \quad f'(x) = 2x^{-3}e^{-1/x^2} \quad \Rightarrow$

$f''(x) = 2[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right)$.

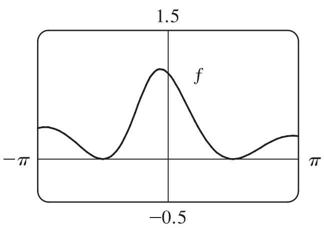
This is 0 when $2 - 3x^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

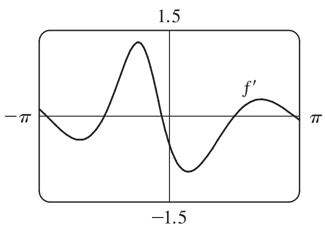
41. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}, \; -\pi \le x \le \pi \quad \Rightarrow \quad f'(x) = -\dfrac{\cos x\,[(2x+1)\cos x + 4(x^2 + x + 1)\sin x]}{2(x^2 + x + 1)^{3/2}} \quad \Rightarrow$

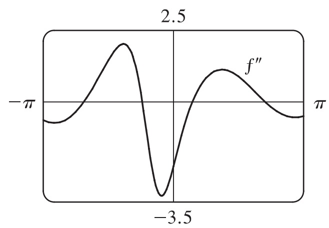
$$f''(x) = -\frac{(8x^4 + 16x^3 + 16x^2 + 8x + 9)\cos^2 x - 8(x^2 + x + 1)(2x + 1)\sin x\,\cos x - 8(x^2 + x + 1)^2\sin^2 x}{4(x^2 + x + 1)^{5/2}}$$

$f(x) = 0 \quad \Leftrightarrow \quad x = \pm\frac{\pi}{2}; \quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx -2.96, -1.57, -0.18, 1.57, 3.01;$

$f''(x) = 0 \quad \Leftrightarrow \quad x \approx -2.16, -0.75, 0.46$, and 2.21.

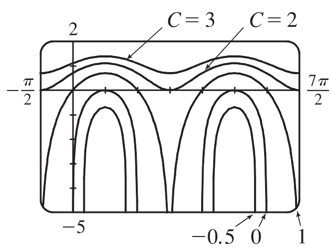






The x-coordinates of the maximum points are the values at which f' changes from positive to negative, that is, -2.96, -0.18, and 3.01. The x-coordinates of the minimum points are the values at which f' changes from negative to positive, that is, -1.57 and 1.57. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -2.16, -0.75, 0.46, and 2.21.

43. The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \le \sin x \le 1$, this happens if $C > -1$, that is, f has no graph if $C \le -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and f is continuous on $(-\infty, \infty)$. As C increases, the graph of f is shifted vertically upward and flattens out. If $-1 < C \le 1$, f is defined



where $\sin x + C > 0 \quad \Leftrightarrow \quad \sin x > -C \quad \Leftrightarrow \quad \sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is 2π, the domain of f is $\left(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\right)$, n an integer.

45. Let $f(x) = 3x + 2\cos x + 5$. Then $f(0) = 7 > 0$ and $f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 = -3(\pi - 1) < 0$, and since f is continuous on $\mathbb{R}$ (hence on $[-\pi, 0]$), the Intermediate Value Theorem assures us that there is at least one zero of f in $[-\pi, 0]$. Now $f'(x) = 3 - 2\sin x > 0$ implies that f is increasing on $\mathbb{R}$, so there is exactly one zero of f, and hence, exactly one real root of the equation $3x + 2\cos x + 5 = 0$.

47. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad \Rightarrow \quad \sqrt[5]{33} - 2 > 0 \quad \Rightarrow \quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad \Rightarrow \quad 0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad \Rightarrow \quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

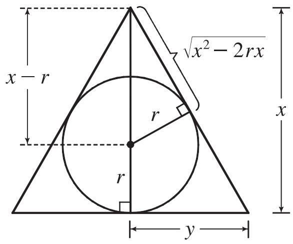
49. (a) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'(x^2) > 0$ for $x \neq 0$, so $g'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad \Rightarrow \quad g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2 f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ [since f is CU for $x > 0$], and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter [since the sign of g'' does not change there]; g is concave upward on $\mathbb{R}$.

51. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \quad \Rightarrow \quad f'(x) = 2\,(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$.

$f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is

$$\sqrt{f(x)} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

53. 

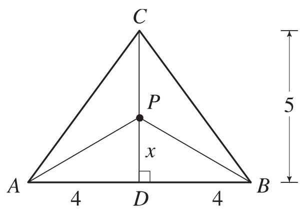
By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \quad \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2\,(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0$$

when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = \dfrac{r(9r^2)}{\sqrt{3}\,r} = 3\sqrt{3}\,r^2$.

55. 

We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2 + 16} + (5 - x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2 + 16} - 1 = 0 \quad \Leftrightarrow \quad 2x = \sqrt{x^2 + 16} \quad \Leftrightarrow$ $4x^2 = x^2 + 16 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

57. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad \Rightarrow \quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C) + (C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad \Leftrightarrow \quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad \Leftrightarrow \quad L^2 = C^2 \quad \Leftrightarrow \quad L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

59. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

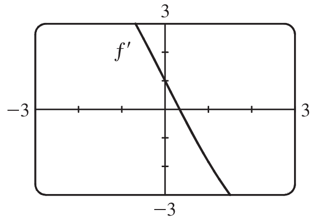
for $0 \le x \le 4$ [since the seating capacity is 15,000] $\quad \Rightarrow \quad R'(x) = -2000x + 1000 = 0 \quad \Leftrightarrow \quad x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R.

$R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

61. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 4x^3 + 6x - 3$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1.5 \quad \Rightarrow \quad x_3 \approx 1.343860 \quad \Rightarrow \quad x_4 \approx 1.300320 \quad \Rightarrow \quad x_5 \approx 1.297396 \quad \Rightarrow$ $x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is 1.297383, to six decimal places.

63. $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



65. $f'(x) = \cos x - (1 - x^2)^{-1/2} = \cos x - \dfrac{1}{\sqrt{1 - x^2}} \quad \Rightarrow \quad f(x) = \sin x - \sin^{-1} x + C$

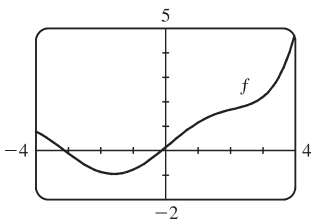
67. $f'(x) = \sqrt{x^3} + \sqrt[3]{x^2} = x^{3/2} + x^{2/3} \quad \Rightarrow \quad f(x) = \dfrac{x^{5/2}}{5/2} + \dfrac{x^{5/3}}{5/3} + C = \frac{2}{5}x^{5/2} + \frac{3}{5}x^{5/3} + C$

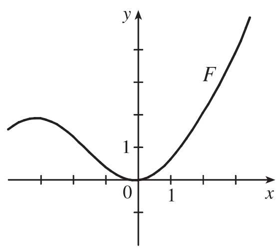
69. $f'(t) = 2t - 3\sin t \quad \Rightarrow \quad f(t) = t^2 + 3\cos t + C$.
$f(0) = 3 + C$ and $f(0) = 5 \quad \Rightarrow \quad C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

71. $f''(x) = 1 - 6x + 48x^2 \quad \Rightarrow \quad f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \quad \Rightarrow \quad C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.
$f(0) = D$ and $f(0) = 1 \quad \Rightarrow \quad D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

73. $v(t) = s'(t) = 2t - \dfrac{1}{1 + t^2} \quad \Rightarrow \quad s(t) = t^2 - \tan^{-1} t + C$.
$s(0) = 0 - 0 + C = C$ and $s(0) = 1 \quad \Rightarrow \quad C = 1$, so $s(t) = t^2 - \tan^{-1} t + 1$.

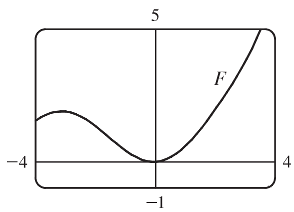
75. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.





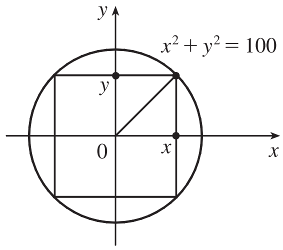
(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow$
$F(x) = 0.1e^x - \cos x + C$. $F(0) = 0 \quad \Rightarrow$
$0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so
$F(x) = 0.1e^x - \cos x + 0.9$.

(c)



77. Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \quad \Rightarrow$ $v(t) = -9.8t = s'(t) \quad \Rightarrow \quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad \Rightarrow \quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad \Rightarrow \quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad \Rightarrow \quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995 \text{ m/s}$. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

79. (a)



The cross-sectional area of the rectangular beam is

$A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \le x \le 10$, so

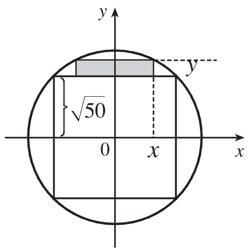
$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x) + (100 - x^2)^{1/2} \cdot 4$$

$$= \frac{-4x^2}{(100 - x^2)^{1/2}} + 4(100 - x^2)^{1/2} = \frac{4[-x^2 + (100 - x^2)]}{(100 - x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + (100 - x^2) = 0 \;\Rightarrow\; x^2 = 50 \;\Rightarrow\; x = \sqrt{50} \approx 7.07 \;\Rightarrow\; y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$.

Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \le x \le \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x)$$

$$= 2(100 - x^2)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $(100 - x^2) - \sqrt{50}\,(100 - x^2)^{1/2} - x^2 = 0 \;\Rightarrow\; 100 - 2x^2 = \sqrt{50}\,(100 - x^2)^{1/2} \;\Rightarrow$

$10{,}000 - 400x^2 + 4x^4 = 50(100 - x^2) \;\Rightarrow\; 4x^4 - 350x^2 + 5000 = 0 \;\Rightarrow\; 2x^4 - 175x^2 + 2500 = 0 \;\Rightarrow$

$x^2 = \frac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \;\Rightarrow\; x \approx 8.34$ or 4.24. But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \;\Rightarrow$

$y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is

$S = k(2x)(2y)^2 = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when

$24kx^2 = 800k \;\Rightarrow\; x^2 = \frac{100}{3} \;\Rightarrow\; x = \frac{10}{\sqrt{3}} \;\Rightarrow\; y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the

maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

81. We first show that $\frac{x}{1 + x^2} < \tan^{-1} x$ for $x > 0$. Let $f(x) = \tan^{-1} x - \frac{x}{1 + x^2}$. Then

$f'(x) = \frac{1}{1 + x^2} - \frac{1(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{(1 + x^2) - (1 - x^2)}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing

on $(0, \infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1} x - \frac{x}{1 + x^2}$. So $\frac{x}{1 + x^2} < \tan^{-1} x$ for $0 < x$. We next show

that $\tan^{-1} x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1} x$. Then $h'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0$. Hence, $h(x)$ is increasing

on $(0, \infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1} x$. Hence, $\tan^{-1} x < x$ for $x > 0$, and we conclude that

$\frac{x}{1 + x^2} < \tan^{-1} x < x$ for $x > 0$.

PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

3. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \quad \Leftrightarrow \quad \dfrac{e^x}{x} \cdot \dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \dfrac{e^x}{x} \quad \Rightarrow$ $f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \quad \Rightarrow \quad x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

5. Let $L = \displaystyle\lim_{x \to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6}$. Now L has the indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule. $L = \displaystyle\lim_{x \to 0} \frac{2ax + b\cos bx + c\cos cx + d\cos dx}{6x + 20x^3 + 42x^5}$. The denominator approaches 0 as $x \to 0$, so the numerator must also approach 0 (because the limit exists). But the numerator approaches $0 + b + c + d$, so $b + c + d = 0$. Apply l'Hospital's Rule again. $L = \displaystyle\lim_{x \to 0} \frac{2a - b^2 \sin bx - c^2 \sin cx - d^2 \sin dx}{6 + 60x^2 + 210x^4} = \frac{2a - 0}{6 + 0} = \frac{2a}{6}$, which must equal 8. $\dfrac{2a}{6} = 8 \quad \Rightarrow \quad a = 24$. Thus, $a + b + c + d = a + (b + c + d) = 24 + 0 = 24$.

7. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\,\dfrac{dy}{dx} + 2y\,\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$. At a highest or lowest point, $\dfrac{dy}{dx} = 0 \quad \Leftrightarrow \quad y = -2x$. Substituting $-2x$ for y in the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

9. $y = x^2 \quad \Rightarrow \quad y' = 2x$, so the slope of the tangent line at $P(a, a^2)$ is $2a$ and the slope of the normal line is $-\dfrac{1}{2a}$ for $a \neq 0$. An equation of the normal line is $y - a^2 = -\dfrac{1}{2a}(x - a)$. Substitute x^2 for y to find the x-coordinates of the two points of intersection of the parabola and the normal line. $\quad x^2 - a^2 = -\dfrac{x}{2a} + \dfrac{1}{2} \quad \Rightarrow \quad 2ax^2 + x - 2a^3 - a = 0 \quad \Rightarrow$

$$x = \frac{-1 \pm \sqrt{1 - 4(2a)(-2a^3 - a)}}{2(2a)} = \frac{-1 \pm \sqrt{1 + 16a^4 + 8a^2}}{4a} = \frac{-1 \pm \sqrt{(4a^2+1)^2}}{4a} = \frac{-1 \pm (4a^2+1)}{4a}$$

$$= \frac{4a^2}{4a} \text{ or } \frac{-4a^2-2}{4a}, \text{ or equivalently, } a \text{ or } -a - \frac{1}{2a}$$

So the point Q has coordinates $\left(-a - \frac{1}{2a}, \left(-a - \frac{1}{2a}\right)^2\right)$. The square S of the distance from P to Q is given by

$$S = \left(-a - \frac{1}{2a} - a\right)^2 + \left[\left(-a - \frac{1}{2a}\right)^2 - a^2\right]^2 = \left(-2a - \frac{1}{2a}\right)^2 + \left[\left(a^2 + 1 + \frac{1}{4a^2}\right) - a^2\right]^2$$

$$= \left(4a^2 + 2 + \frac{1}{4a^2}\right) + \left(1 + \frac{1}{4a^2}\right)^2 = \left(4a^2 + 2 + \frac{1}{4a^2}\right) + 1 + \frac{2}{4a^2} + \frac{1}{16a^4} = 4a^2 + 3 + \frac{3}{4a^2} + \frac{1}{16a^4}$$

$S' = 8a - \dfrac{6}{4a^3} - \dfrac{4}{16a^5} = 8a - \dfrac{3}{2a^3} - \dfrac{1}{4a^5} = \dfrac{32a^6 - 6a^2 - 1}{4a^5}$. The only real positive zero of the equation $S' = 0$ is $a = \dfrac{1}{\sqrt{2}}$. Since $S'' = 8 + \dfrac{9}{2a^4} + \dfrac{5}{4a^6} > 0$, $a = \dfrac{1}{\sqrt{2}}$ corresponds to the shortest possible length of the line segment PQ.

11. $f(x) = (a^2 + a - 6)\cos 2x + (a-2)x + \cos 1 \;\Rightarrow\; f'(x) = -(a^2 + a - 6)\sin 2x\,(2) + (a-2)$. The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \;\Leftrightarrow\; 2(a-2)(a+3)\sin 2x = a - 2 \;\Leftrightarrow$ either $a = 2$ or $2(a+3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a+3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a+3)} < -1$ or $\dfrac{1}{2(a+3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

13. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}(x_1^2 + x_2^2)(x_2 - x_1) - \tfrac{1}{2}(x_1^2 + x^2)(x - x_1) - \tfrac{1}{2}(x^2 + x_2^2)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $\quad f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

$f'(x) = 0 \;\Rightarrow\; 2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)(x_1^2 + x_2^2) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] = \tfrac{1}{8}(x_2 - x_1)\left[2(x_1^2 + x_2^2) - (x_1^2 + 2x_1x_2 + x_2^2)\right] \\ &= \tfrac{1}{8}(x_2 - x_1)(x_1^2 - 2x_1x_2 + x_2^2) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 = \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad \Rightarrow$

$x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$,

and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

15. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, > 0, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

17. Note that $f(0) = 0$, so for $x \ne 0$, $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0}\right| = \lim\limits_{x\to 0}\left|\dfrac{f(x) - f(0)}{x - 0}\right| \le \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

But $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx \quad \Rightarrow \quad f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \le 1$.

Another solution: We are given that $\left|\sum_{k=1}^n a_k \sin kx\right| \le |\sin x|$. So for x close to 0, and $x \ne 0$, we have

$\left|\sum\limits_{k=1}^n a_k \dfrac{\sin kx}{\sin x}\right| \le 1 \quad \Rightarrow \quad \lim\limits_{x\to 0}\left|\sum\limits_{k=1}^n a_k \dfrac{\sin kx}{\sin x}\right| \le 1 \quad \Rightarrow \quad \left|\sum\limits_{k=1}^n a_k \lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x}\right| \le 1$. But by l'Hospital's Rule,

$\lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x} = \lim\limits_{x\to 0}\dfrac{k\cos kx}{\cos x} = k$, so $\left|\sum\limits_{k=1}^n ka_k\right| \le 1$.

19. (a) Distance $=$ rate $\times$ time, so time $=$ distance/rate. $T_1 = \dfrac{D}{c_1}$, $T_2 = \dfrac{2\,|PR|}{c_1} + \dfrac{|RS|}{c_2} = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$,

$$T_3 = \frac{2\sqrt{h^2 + D^2/4}}{c_1} = \frac{\sqrt{4h^2 + D^2}}{c_1}.$$

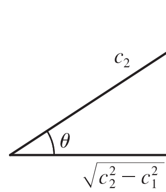
(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \quad \Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \quad \Rightarrow \quad \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \quad \Rightarrow \quad \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \;\Rightarrow\; c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2 + D^2}}{c_1} \;\Rightarrow$

$4h^2 + D^2 = T_3^2 c_1^2 \;\Rightarrow\; h = \frac{1}{2}\sqrt{T_3^2 c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2 (1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$

from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,



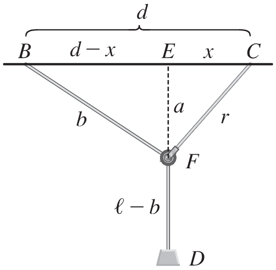
$\sin\theta = \dfrac{c_1}{c_2} \;\Rightarrow\; \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2 - c_1^2}}$, so

$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$. Using the values for T_2 [given as 0.32],

h, c_1, and D, we can graph $\mathrm{Y}_1 = T_2$ and $\mathrm{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$ and find their intersection points.

Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6°$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

21.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure. Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned}|ED| &= |EF| + |FD| = a + \ell - b = \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2}\\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2}\\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2}\end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$$f'(x) = \tfrac{1}{2}(r^2 - x^2)^{-1/2}(-2x) - \tfrac{1}{2}(d^2 + r^2 - 2dx)^{-1/2}(-2d) = \frac{-x}{\sqrt{r^2 - x^2}} + \frac{d}{\sqrt{d^2 + r^2 - 2dx}}.$$

$$f'(x) = 0 \;\Rightarrow\; \frac{x}{\sqrt{r^2 - x^2}} = \frac{d}{\sqrt{d^2 + r^2 - 2dx}} \;\Rightarrow\; \frac{x^2}{r^2 - x^2} = \frac{d^2}{d^2 + r^2 - 2dx} \;\Rightarrow$$

$$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \;\Rightarrow\; 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \;\Rightarrow$$

$$0 = 2dx^2(x - d) - r^2(x^2 - d^2) \;\Rightarrow\; 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \;\Rightarrow\; 0 = (x - d)[2dx^2 - r^2(x + d)]$$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

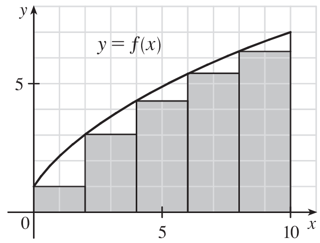
$x = \dfrac{-(-r^2) \pm \sqrt{(-r^2)^2 - 4(2d)(-dr^2)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus, $x = \dfrac{r^2 + \sqrt{r^2}\sqrt{r^2 + 8d^2}}{4d} = \dfrac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad [r > 0] \; = \dfrac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$. The maximum value of $|ED|$ occurs at this value of x.

23. $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k. Therefore, $4\pi r^2 \dfrac{dr}{dt} = k \cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t is kt, so $r = kt + C$. When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2} \cdot \frac{4}{3}\pi r_0^3 \quad \Rightarrow \quad (3k + r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow \quad 3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow$ $k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \quad \Rightarrow \quad \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes $\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.

5 □ INTEGRALS

5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



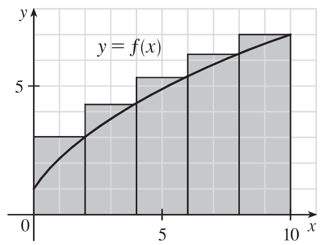
$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \qquad \left[\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2\right]$$

$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$

$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$

$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$

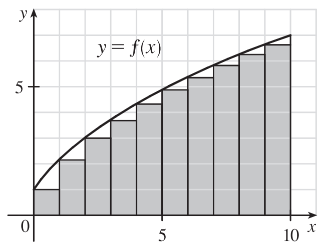
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$

$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$

$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

Comparing R_5 to L_5, we see that we have added the area of the rightmost upper rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0)\cdot 2$, from the sum.

(b) $$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \qquad \left[\Delta x = \tfrac{10-0}{10} = 1\right]$$



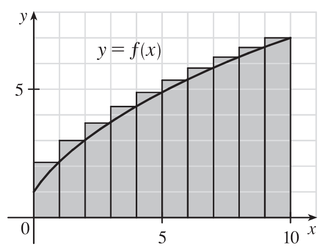
$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$

$$= f(0) + f(1) + \cdots + f(9)$$

$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$

$$= 43.2$$

$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$



$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \left[\begin{array}{c}\text{add rightmost upper rectangle,}\\ \text{subtract leftmost lower rectangle}\end{array}\right]$$

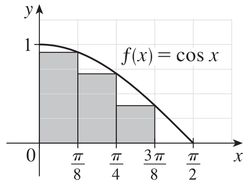
$$= 43.2 + 7 - 1 = 49.2$$

3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad \left[\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}\right] \quad = \left[\sum_{i=1}^{4} f(x_i)\right]\Delta x$

$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= \left[\cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{4\pi}{8}\right]\frac{\pi}{8}$

$\approx (0.9239 + 0.7071 + 0.3827 + 0)\frac{\pi}{8} \approx 0.7908$



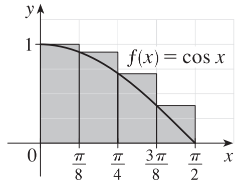
Since f is *decreasing* on $[0, \pi/2]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x = \left[\sum_{i=1}^{4} f(x_{i-1})\right]\Delta x$

$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\,\Delta x$

$= \left[\cos 0 + \cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8}\right]\frac{\pi}{8}$

$\approx (1 + 0.9239 + 0.7071 + 0.3827)\frac{\pi}{8} \approx 1.1835$



L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(0)\cdot\frac{\pi}{8} - f\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{8}$.

5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad\Rightarrow$

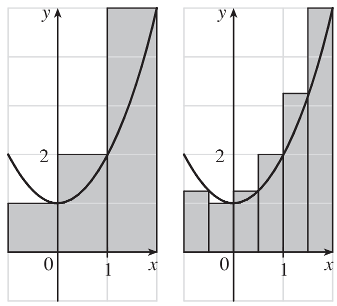
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 8.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad\Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$

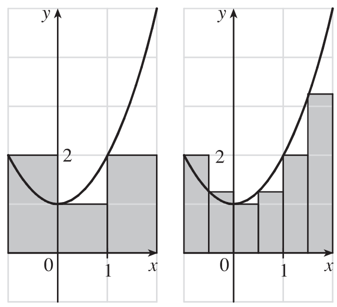


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$



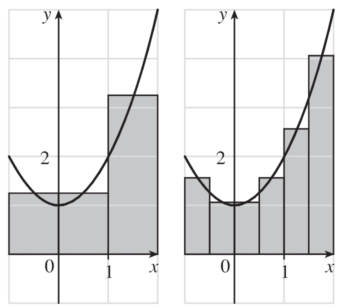
(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$

$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$
$\quad + f(0.75) + f(1.25) + f(1.75)]$

$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$= 0.5(11.875) = 5.9375$



(d) M_6 appears to be the best estimate.

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = 1, N = 10 (depending on which sum we are calculating),
DELTA_X = (X_MAX - X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT $>$ X_MAX.
2a Add (RIGHT_ENDPOINT)^4 to SUM.
2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X)·(SUM) is equal to the answer we are looking for. We find that

$R_{10} = \dfrac{1}{10}\sum\limits_{i=1}^{10}\left(\dfrac{i}{10}\right)^4 \approx 0.2533$, $R_{30} = \dfrac{1}{30}\sum\limits_{i=1}^{30}\left(\dfrac{i}{30}\right)^4 \approx 0.2170$, $R_{50} = \dfrac{1}{50}\sum\limits_{i=1}^{50}\left(\dfrac{i}{50}\right)^4 \approx 0.2101$, and

$R_{100} = \dfrac{1}{100}\sum\limits_{i=1}^{100}\left(\dfrac{i}{100}\right)^4 \approx 0.2050$. It appears that the exact area is 0.2.

The following display shows the program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assign) values for Xmin, Xmax, and N. Also, the function, x^4, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y1(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

```
prgmSUMRIGHT
Xmin=?0
Xmax=?1
N=?10
              .25333
                Done
```

9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command
`left_sum:=leftsum(1/(x^2+1),x=0..1,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(1/10)*Sum[1/(((i-1)/10)^2+1)],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

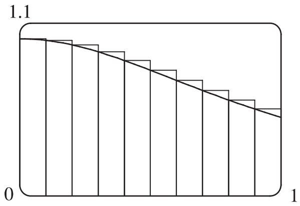
In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \dfrac{1}{x^2+1}$, $0 \le x \le 1$, the left sums are of the form $L_n = \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{1}{\left(\frac{i-1}{n}\right)^2+1}$. Specifically, $L_{10} \approx 0.8100$,

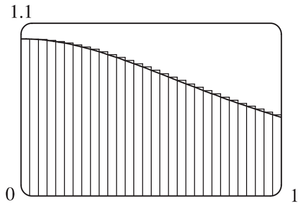
$L_{30} \approx 0.7937$, and $L_{50} \approx 0.7904$. The right sums are of the form $R_n = \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{1}{\left(\frac{i}{n}\right)^2+1}$. Specifically, $R_{10} \approx 0.7600$,

$R_{30} \approx 0.7770$, and $R_{50} \approx 0.7804$.

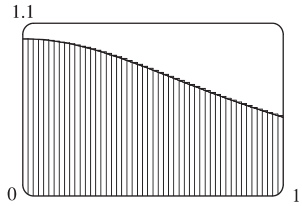
(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.



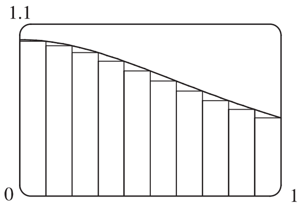
left endpoints, $n = 10$



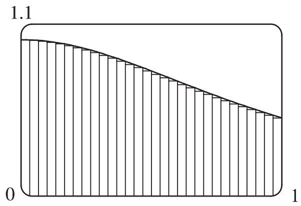
left endpoints, $n = 30$



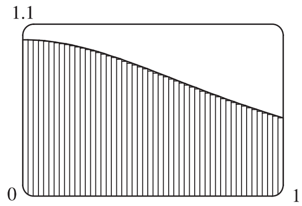
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



right endpoints, $n = 50$

(c) We know that since $y = 1/(x^2 + 1)$ is a decreasing function on $(0, 1)$, all of the left sums are larger than the actual area, and all of the right sums are smaller than the actual area. Since the left sum with $n = 50$ is about $0.7904 < 0.791$ and the right sum with $n = 50$ is about $0.7804 > 0.780$, we conclude that $0.780 < R_{50} <$ exact area $< L_{50} < 0.791$, so the exact area is between 0.780 and 0.791.

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) = 0.5(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

13. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

17. $f(x) = \sqrt[4]{x}$, $1 \le x \le 16$. $\Delta x = (16 - 1)/n = 15/n$ and $x_i = 1 + i\,\Delta x = 1 + 15i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \frac{15i}{n}} \cdot \frac{15}{n}.$$

19. $f(x) = x\cos x$, $0 \le x \le \frac{\pi}{2}$. $\Delta x = (\frac{\pi}{2} - 0)/n = \frac{\pi}{2}/n$ and $x_i = 0 + i\,\Delta x = \frac{\pi}{2}i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}.$$

21. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{\pi}{4n}\tan\dfrac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y=\tan x$ on the interval $\left[0,\frac{\pi}{4}\right]$, since for $y=\tan x$ on $\left[0,\frac{\pi}{4}\right]$ with $\Delta x=\dfrac{\pi/4-0}{n}=\dfrac{\pi}{4n}$, $x_i=0+i\,\Delta x=\dfrac{i\pi}{4n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f\left(x_i^*\right)\Delta x=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\tan\left(\dfrac{i\pi}{4n}\right)\dfrac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y=\tan(x-k\pi)$ on the interval $\left[k\pi,k\pi+\frac{\pi}{4}\right]$, where k is any integer.

23. (a) $y=f(x)=x^5$. $\Delta x=\dfrac{2-0}{n}=\dfrac{2}{n}$ and $x_i=0+i\,\Delta x=\dfrac{2i}{n}$.

$$A=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^5\cdot\frac{2}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{32i^5}{n^5}\cdot\frac{2}{n}=\lim_{n\to\infty}\frac{64}{n^6}\sum_{i=1}^{n}i^5.$$

(b) $\sum\limits_{i=1}^{n}i^5\overset{\text{CAS}}{=}\dfrac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$

(c)
$$\lim_{n\to\infty}\frac{64}{n^6}\cdot\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}=\frac{64}{12}\lim_{n\to\infty}\frac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2\cdot n^2}$$
$$=\frac{16}{3}\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(2+\frac{2}{n}-\frac{1}{n^2}\right)=\tfrac{16}{3}\cdot1\cdot2=\tfrac{32}{3}$$

25. $y=f(x)=\cos x$. $\Delta x=\dfrac{b-0}{n}=\dfrac{b}{n}$ and $x_i=0+i\,\Delta x=\dfrac{bi}{n}$.

$$A=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{bi}{n}\right)\cdot\frac{b}{n}\overset{\text{CAS}}{=}\lim_{n\to\infty}\left[\frac{b\sin\left(b\left(\dfrac{1}{2n}+1\right)\right)}{2n\sin\left(\dfrac{b}{2n}\right)}-\frac{b}{2n}\right]\overset{\text{CAS}}{=}\sin b$$

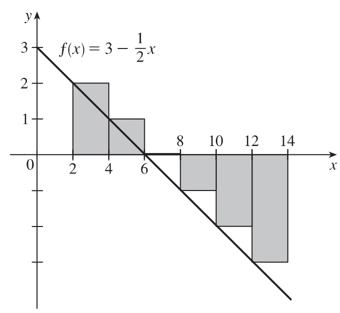
If $b=\frac{\pi}{2}$, then $A=\sin\frac{\pi}{2}=1$.

5.2 The Definite Integral

1. $f(x)=3-\frac{1}{2}x$, $2\le x\le14$. $\Delta x=\dfrac{b-a}{n}=\dfrac{14-2}{6}=2$.

Since we are using left endpoints, $x_i^*=x_{i-1}$.

$$\begin{aligned}L_6&=\sum_{i=1}^{6}f(x_{i-1})\,\Delta x\\&=(\Delta x)\left[f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right]\\&=2[f(2)+f(4)+f(6)+f(8)+f(10)+f(12)]\\&=2[2+1+0+(-1)+(-2)+(-3)]=2(-3)=-6\end{aligned}$$

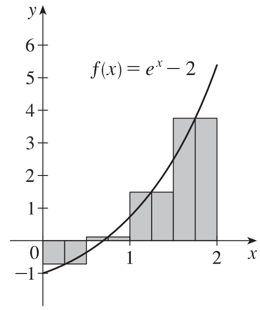


The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

3. $f(x) = e^x - 2,\ 0 \le x \le 2.\ \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{4} = \dfrac{1}{2}.$

Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = (\Delta x)\ [f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)] \\ &= \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right] \\ &= \tfrac{1}{2}\left[(e^{1/4} - 2) + (e^{3/4} - 2) + (e^{5/4} - 2) + (e^{7/4} - 2)\right] \\ &\approx 2.322986 \end{aligned}$$



The Riemann sum represents the sum of the areas of the three rectangles above the x-axis minus the area of the rectangle below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

5. $\Delta x = (b-a)/n = (8-0)/4 = 8/4 = 2$.

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2[f(1) + f(3) + f(5) + f(7)] \approx 2[3 + 2 + 1 + (-1)] = 10.$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

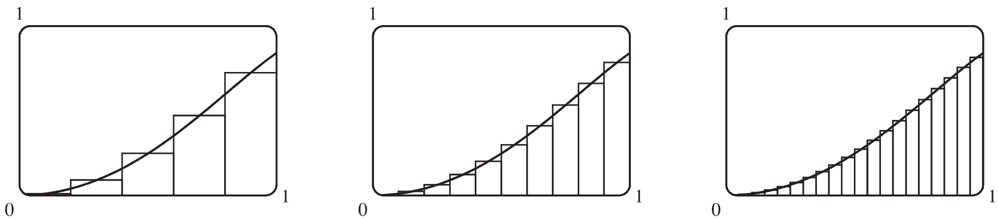
$$\begin{aligned} \text{Lower estimate} = L_5 &= \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5[f(0) + f(5) + f(10) + f(15) + f(20)] \\ &= 5(-42 - 37 - 25 - 6 + 15) = 5(-95) = -475 \\ \text{Upper estimate} = R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x = 5[f(5) + f(10) + f(15) + f(20) + f(25)] \\ &= 5(-37 - 25 - 6 + 15 + 36) = 5(-17) = -85 \end{aligned}$$

9. $\Delta x = (10-2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives $\int_2^{10} \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\left(\sqrt{3^3+1} + \sqrt{5^3+1} + \sqrt{7^3+1} + \sqrt{9^3+1}\right) \approx 124.1644$.

11. $\Delta x = (1-0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

$$\int_0^1 \sin(x^2)\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 0.2\left[\sin(0.1)^2 + \sin(0.3)^2 + \sin(0.5)^2 + \sin(0.7)^2 + \sin(0.9)^2\right] \approx 0.3084.$$

13. In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sin(x^2),x=0..1,5);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command `middlebox(sin(x^2),x=0..1,5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.



15. We'll create the table of values to approximate $\int_0^{\pi} \sin x\, dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5$, 10, 50, and 100.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

The values of R_n appear to be approaching 2.

17. On $[2, 6]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} x_i \ln(1 + x_i^2)\, \Delta x = \int_2^6 x \ln(1 + x^2)\, dx$.

19. On $[1, 8]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\, \Delta x = \int_1^8 \sqrt{2x + x^2}\, dx$.

21. Note that $\Delta x = \dfrac{5 - (-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\begin{aligned}
\int_{-1}^{5} (1 + 3x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + 3\left(-1 + \frac{6i}{n}\right)\right] \frac{6}{n} = \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left[-2 + \frac{18i}{n}\right] \\
&= \lim_{n\to\infty} \frac{6}{n} \left[\sum_{i=1}^{n} (-2) + \sum_{i=1}^{n} \frac{18i}{n}\right] = \lim_{n\to\infty} \frac{6}{n} \left[-2n + \frac{18}{n} \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \frac{6}{n} \left[-2n + \frac{18}{n} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[-12 + \frac{108}{n^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[-12 + 54\frac{n+1}{n}\right] = \lim_{n\to\infty} \left[-12 + 54\left(1 + \frac{1}{n}\right)\right] = -12 + 54 \cdot 1 = 42
\end{aligned}$$

23. Note that $\Delta x = \dfrac{2 - 0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\begin{aligned}
\int_{0}^{2} \left(2 - x^2\right) dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n} \left[\sum_{i=1}^{n} 2 - \frac{4}{n^2} \sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left(2n - \frac{4}{n^2} \sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty} \left[4 - \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left(4 - \frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}\right) = \lim_{n\to\infty} \left[4 - \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] = 4 - \tfrac{4}{3} \cdot 1 \cdot 2 = \tfrac{4}{3}
\end{aligned}$$

25. Note that $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + i\,\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\begin{aligned}
\int_1^2 x^3\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^n \left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left(\frac{n+i}{n}\right)^3 \\
&= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^n\left(n^3+3n^2i+3ni^2+i^3\right) = \lim_{n\to\infty}\frac{1}{n^4}\left[\sum_{i=1}^n n^3 + \sum_{i=1}^n 3n^2 i + \sum_{i=1}^n 3ni^2 + \sum_{i=1}^n i^3\right] \\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3 + 3n^2\sum_{i=1}^n i + 3n\sum_{i=1}^n i^2 + \sum_{i=1}^n i^3\right] \\
&= \lim_{n\to\infty}\left[1 + \frac{3}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty}\left[1 + \frac{3}{2}\cdot\frac{n+1}{n} + \frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + \frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right] \\
&= \lim_{n\to\infty}\left[1 + \frac{3}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1 + \frac{3}{2} + \tfrac{1}{2}\cdot 2 + \tfrac{1}{4} = 3.75
\end{aligned}$$

27.
$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a + \frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^n i\right] \\
&= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n + \frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a\,(b-a) + \lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right) \\
&= a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)\left(a + \tfrac{1}{2}b - \tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}\left(b^2-a^2\right)
\end{aligned}$$

29. $f(x) = \dfrac{x}{1+x^5}$, $a = 2$, $b = 6$, and $\Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}$. Using Theorem 4, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \dfrac{4i}{n}$,

so $$\int_2^6 \frac{x}{1+x^5}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^n \frac{2+\dfrac{4i}{n}}{1+\left(2+\dfrac{4i}{n}\right)^5}\cdot\frac{4}{n}.$$

31. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x\,dx = \lim_{n\to\infty}\sum_{i=1}^n(\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^n\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

33. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)2 = 4$.

(b)
$$\begin{aligned}
\int_0^5 f(x)\,dx &= \underset{\text{trapezoid}}{\int_0^2 f(x)\,dx} + \underset{\text{rectangle}}{\int_2^3 f(x)\,dx} + \underset{\text{triangle}}{\int_3^5 f(x)\,dx} \\
&= \tfrac{1}{2}(1+3)2 + 3\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 3 = 4 + 3 + 3 = 10
\end{aligned}$$

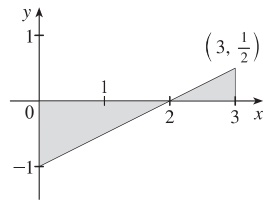
(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,

$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2.$

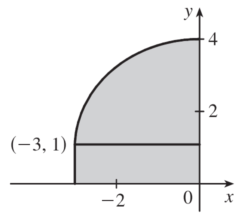
35. $\int_0^3 \left(\frac{1}{2}x - 1\right) dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,

$\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.

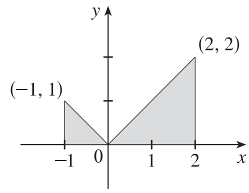


37. $\int_{-3}^{0}\left(1 + \sqrt{9 - x^2}\right) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9 - x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so

$\int_{-3}^{0}\left(1 + \sqrt{9 - x^2}\right) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$.



39. $\int_{-1}^{2} |x| \, dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



41. $\int_{\pi}^{\pi} \sin^2 x \cos^4 x \, dx = 0$ since the limits of intergration are equal.

43. $\int_0^1 (5 - 6x^2)\, dx = \int_0^1 5\, dx - 6\int_0^1 x^2\, dx = 5(1 - 0) - 6\left(\frac{1}{3}\right) = 5 - 2 = 3$

45. $\int_1^3 e^{x+2}\, dx = \int_1^3 e^x \cdot e^2\, dx = e^2 \int_1^3 e^x\, dx = e^2(e^3 - e) = e^5 - e^3$

47. $\int_{-2}^{2} f(x)\, dx + \int_2^5 f(x)\, dx - \int_{-2}^{-1} f(x)\, dx = \int_{-2}^{5} f(x)\, dx + \int_{-1}^{-2} f(x)\, dx$ [by Property 5 and reversing limits]

$= \int_{-1}^{5} f(x)\, dx$ [Property 5]

49. $\int_0^9 [2f(x) + 3g(x)]\, dx = 2\int_0^9 f(x)\, dx + 3\int_0^9 g(x)\, dx = 2(37) + 3(16) = 122$

51. Using Integral Comparison Property 8, $m \le f(x) \le M \;\Rightarrow\; m(2 - 0) \le \int_0^2 f(x)\, dx \le M(2 - 0) \;\Rightarrow$ $2m \le \int_0^2 f(x)\, dx \le 2M$.

53. If $-1 \le x \le 1$, then $0 \le x^2 \le 1$ and $1 \le 1 + x^2 \le 2$, so $1 \le \sqrt{1 + x^2} \le \sqrt{2}$ and $1[1 - (-1)] \le \int_{-1}^{1} \sqrt{1 + x^2}\, dx \le \sqrt{2}\,[1 - (-1)]$ [Property 8]; that is, $2 \le \int_{-1}^{1} \sqrt{1 + x^2}\, dx \le 2\sqrt{2}$.

55. If $1 \le x \le 4$, then $1 \le \sqrt{x} \le 2$, so $1(4 - 1) \le \int_1^4 \sqrt{x}\, dx \le 2(4 - 1)$; that is, $3 \le \int_1^4 \sqrt{x}\, dx \le 6$.

57. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\, dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\, dx \le \frac{\pi}{12}\sqrt{3}$.

59. The only critical number of $f(x) = xe^{-x}$ on $[0, 2]$ is $x = 1$. Since $f(0) = 0$, $f(1) = e^{-1} \approx 0.368$, and $f(2) = 2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0, 2]$ is 0, and the absolute maximum is e^{-1}. By Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2 \;\Rightarrow\; 0(2 - 0) \le \int_0^2 xe^{-x}\, dx \le e^{-1}(2 - 0) \;\Rightarrow\; 0 \le \int_0^2 xe^{-x}\, dx \le 2/e$.

61. $\sqrt{x^4 + 1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4 + 1}\, dx \ge \int_1^3 x^2\, dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

63. Using right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n cf(x_i)\,\Delta x = \lim_{n\to\infty} c\sum_{i=1}^n f(x_i)\,\Delta x = c\lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x = c\int_a^b f(x)\,dx.$$

65. Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad\Rightarrow\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \quad\Rightarrow\quad |b| \le a$ for all real numbers b and nonnegative numbers a.

67. To show that f is integrable on $[0,1]$, we must show that $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n f(x_i^*)\,\Delta x$ exists. Let n denote a positive integer and divide the interval $[0,1]$ into n equal subintervals $\left[0,\frac{1}{n}\right], \left[\frac{1}{n},\frac{2}{n}\right], \ldots, \left[\frac{n-1}{n},1\right]$. If we choose x_i^* to be a rational number in the ith subinterval, then we obtain the Riemann sum $\sum\limits_{i=1}^n f(x_i^*)\cdot\frac{1}{n} = 0$, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 0 = 0$. Now suppose we choose x_i^* to be an irrational number. Then we get $\sum\limits_{i=1}^n f(x_i^*)\cdot\frac{1}{n} = \sum\limits_{i=1}^n 1\cdot\frac{1}{n} = n\cdot\frac{1}{n} = 1$ for each n, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n f(x_i^*)\cdot\frac{1}{n} = \lim\limits_{n\to\infty} 1 = 1$. Since the value of $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n f(x_i^*)\,\Delta x$ depends on the choice of the sample points x_i^*, the limit does not exist, and f is not integrable on $[0,1]$.

69. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n \frac{i^4}{n^5} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^n \frac{i^4}{n^4}\cdot\frac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^n \left(\frac{i}{n}\right)^4\frac{1}{n}$. At this point, we need to recognize the limit as being of the form $\lim\limits_{n\to\infty}\sum\limits_{i=1}^n f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral is $\int_0^1 x^4\,dx$.

71. Choose $x_i = 1 + \frac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} n\sum_{i=1}^n \frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^n\left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \quad [\text{by the hint}] \quad = \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i}-\sum_{i=1}^n\frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right]-\left[\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$

5.3 The Fundamental Theorem of Calculus

1. One process undoes what the other one does. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it on page 387.

3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$

$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

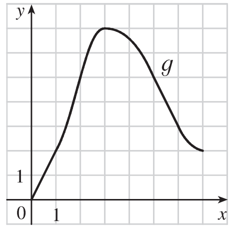
$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the x-axis]

$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

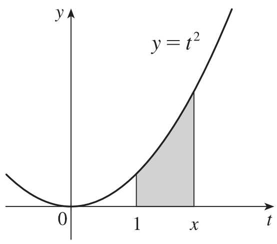
(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

(c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

(d)



5.



(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \Rightarrow$

$g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \Rightarrow g'(x) = x^2$.

7. $f(t) = \dfrac{1}{t^3+1}$ and $g(x) = \displaystyle\int_1^x \frac{1}{t^3+1}\,dt$, so by FTC1, $g'(x) = f(x) = \dfrac{1}{x^3+1}$. Note that the lower limit, 1, could be any real number greater than -1 and not affect this answer.

9. $f(t) = t^2 \sin t$ and $g(y) = \int_2^y t^2 \sin t\,dt$, so by FTC1, $g'(y) = f(y) = y^2 \sin y$.

11. $$F(x) = \int_x^\pi \sqrt{1+\sec t}\,dt = -\int_\pi^x \sqrt{1+\sec t}\,dt \Rightarrow F'(x) = -\frac{d}{dx}\int_\pi^x \sqrt{1+\sec t}\,dt = -\sqrt{1+\sec x}$$

13. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \arctan t\,dt = \frac{d}{du}\int_2^u \arctan t\,dt \cdot \frac{du}{dx} = \arctan u\,\frac{du}{dx} = -\frac{\arctan(1/x)}{x^2}.$$

15. Let $u = \tan x$. Then $\dfrac{du}{dx} = \sec^2 x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt = \frac{d}{du}\int_0^u \sqrt{t+\sqrt{t}}\,dt \cdot \frac{du}{dx} = \sqrt{u+\sqrt{u}}\,\frac{du}{dx} = \sqrt{\tan x + \sqrt{\tan x}}\,\sec^2 x.$$

17. Let $w = 1 - 3x$. Then $\dfrac{dw}{dx} = -3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{dw}\dfrac{dw}{dx}$, so

$$y' = \frac{d}{dx}\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du = \frac{d}{dw}\int_w^1 \frac{u^3}{1+u^2}du \cdot \frac{dw}{dx} = -\frac{d}{dw}\int_1^w \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3(1-3x)^3}{1+(1-3x)^2}$$

19. $$\int_{-1}^2 (x^3 - 2x)\,dx = \left[\frac{x^4}{4} - x^2\right]_{-1}^2 = \left(\frac{2^4}{4} - 2^2\right) - \left(\frac{(-1)^4}{4} - (-1)^2\right) = (4-4) - \left(\tfrac{1}{4} - 1\right) = 0 - \left(-\tfrac{3}{4}\right) = \tfrac{3}{4}$$

21. $\int_1^4 (5-2t+3t^2)\,dt = \left[5t - t^2 + t^3\right]_1^4 = (20-16+64)-(5-1+1) = 68-5 = 63$

23. $\int_0^1 x^{4/5}\,dx = \left[\frac{5}{9}x^{9/5}\right]_0^1 = \frac{5}{9} - 0 = \frac{5}{9}$

25. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt = 3\int_1^2 t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^2 = \frac{3}{-3}\left[\frac{1}{t^3}\right]_1^2 = -1\left(\frac{1}{8}-1\right) = \frac{7}{8}$

27. $\int_0^2 x(2+x^5)\,dx = \int_0^2 (2x+x^6)\,dx = \left[x^2 + \frac{1}{7}x^7\right]_0^2 = \left(4+\frac{128}{7}\right) - (0+0) = \frac{156}{7}$

29. $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx = \left[\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right]_1^9$

$= \left(\frac{2}{3}\cdot 27 - 2\cdot 3\right) - \left(\frac{2}{3} - 2\right) = 12 - \left(-\frac{4}{3}\right) = \frac{40}{3}$

31. $\int_0^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

33. $\int_1^2 (1+2y)^2\,dy = \int_1^2 (1+4y+4y^2)\,dy = \left[y + 2y^2 + \frac{4}{3}y^3\right]_1^2 = \left(2+8+\frac{32}{3}\right) - \left(1+2+\frac{4}{3}\right) = \frac{62}{3} - \frac{13}{3} = \frac{49}{3}$

35. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \frac{1}{2}\Big[\ln|x|\Big]_1^9 = \frac{1}{2}(\ln 9 - \ln 1) = \frac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

37. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\left[\sin^{-1} t\right]_{1/2}^{\sqrt{3}/2} = 6\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right] = 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 6\left(\frac{\pi}{6}\right) = \pi$

39. $\int_{-1}^1 e^{u+1}\,du = \left[e^{u+1}\right]_{-1}^1 = e^2 - e^0 = e^2 - 1$ [or start with $e^{u+1} = e^u e^1$]

41. If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$$\int_0^\pi f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^\pi \cos x\,dx = \left[-\cos x\right]_0^{\pi/2} + \left[\sin x\right]_{\pi/2}^\pi = -\cos\frac{\pi}{2} + \cos 0 + \sin\pi - \sin\frac{\pi}{2}$$

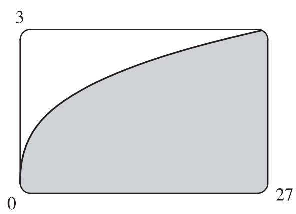
$$= -0 + 1 + 0 - 1 = 0$$

Note that f is integrable by Theorem 3 in Section 5.2.

43. $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\int_{-2}^1 x^{-4}\,dx$ does not exist.

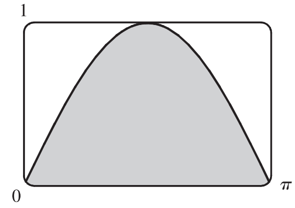
45. $f(\theta) = \sec\theta\,\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^\pi \sec\theta\,\tan\theta\,d\theta$ does not exist.

47. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3} dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4}\cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

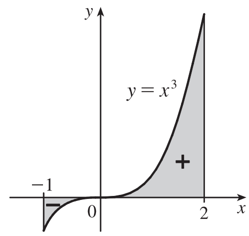


49. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is

$\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2.$



51. $\int_{-1}^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^2 = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



53. $g(x) = \displaystyle\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du \quad\Rightarrow$

$g'(x) = -\dfrac{(2x)^2-1}{(2x)^2+1}\cdot\dfrac{d}{dx}(2x) + \dfrac{(3x)^2-1}{(3x)^2+1}\cdot\dfrac{d}{dx}(3x) = -2\cdot\dfrac{4x^2-1}{4x^2+1} + 3\cdot\dfrac{9x^2-1}{9x^2+1}$

55. $y = \int_{\sqrt{x}}^{x^3}\sqrt{t}\sin t\,dt = \int_{\sqrt{x}}^{1}\sqrt{t}\sin t\,dt + \int_1^{x^3}\sqrt{t}\sin t\,dt = -\int_1^{\sqrt{x}}\sqrt{t}\sin t\,dt + \int_1^{x^3}\sqrt{t}\sin t\,dt \quad\Rightarrow$

$y' = -\sqrt[4]{x}\,(\sin\sqrt{x})\cdot\dfrac{d}{dx}(\sqrt{x}) + x^{3/2}\sin(x^3)\cdot\dfrac{d}{dx}(x^3) = -\dfrac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}} + x^{3/2}\sin(x^3)(3x^2)$

$= 3x^{7/2}\sin(x^3) - \dfrac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

57. $F(x) = \displaystyle\int_1^x f(t)\,dt \quad\Rightarrow\quad F'(x) = f(x) = \int_1^{x^2}\frac{\sqrt{1+u^4}}{u}\,du \quad \left[\text{since } f(t) = \int_1^{t^2}\frac{\sqrt{1+u^4}}{u}\,du\right] \quad\Rightarrow$

$F''(x) = f'(x) = \dfrac{\sqrt{1+(x^2)^4}}{x^2}\cdot\dfrac{d}{dx}(x^2) = \dfrac{\sqrt{1+x^8}}{x^2}\cdot 2x = \dfrac{2\sqrt{1+x^8}}{x}$. So $F''(2) = \sqrt{1+2^8} = \sqrt{257}$.

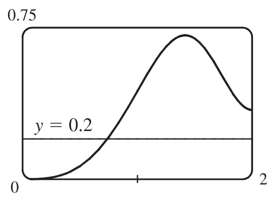
59. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \quad\Rightarrow\quad f(4) = 17 + 12 = 29$.

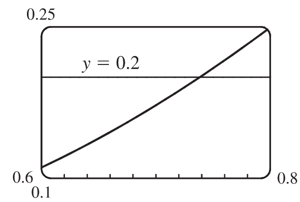
61. (a) The Fresnel function $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0 = S'(x) = \sin\left(\frac{\pi}{2}t^2\right)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x^2 = 2(2n-1) \quad\Leftrightarrow\quad x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow\quad x^2 = 4n \quad\Leftrightarrow\quad x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right) = \pi x\cos\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, $S''(x) > 0$ where $\cos\left(\frac{\pi}{2}x^2\right) > 0 \quad\Leftrightarrow\quad 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi$, n any integer $\quad\Leftrightarrow\quad 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos\left(\frac{\pi}{2}x^2\right) < 0 \quad\Leftrightarrow\quad \left(2n - \frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n - \frac{1}{2}\right)\pi$, n any integer $\quad\Leftrightarrow$ $4n - 3 < x^2 < 4n - 1 \quad\Leftrightarrow\quad \sqrt{4n-3} < |x| < \sqrt{4n-1} \quad\Rightarrow\quad \sqrt{4n-3} < -x < \sqrt{4n-1} \quad\Rightarrow$

$-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $\left(-\sqrt{3},-1\right)$, $\left(\sqrt{3},\sqrt{5}\right)$, $\left(-\sqrt{7},-\sqrt{5}\right)$, $\left(\sqrt{7},3\right)$, ...

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt = 0.2$ at $x \approx 0.74$.



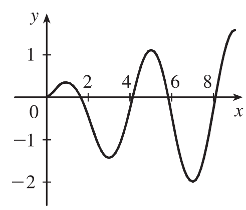


63. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4,6)$ and $(8,9)$. So g is concave downward on these intervals.

(d)



65. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

67. Suppose $h < 0$. Since f is continuous on $[x+h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x+h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_x^{x+h} f(t)\,dt \le f(v)(-h)$.

Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\dfrac{g(x+h)-g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$ for $h \ne 0$, and hence $f(u) \le \dfrac{g(x+h)-g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

69. (a) Let $f(x)=\sqrt{x} \;\Rightarrow\; f'(x)=1/(2\sqrt{x})>0$ for $x>0 \;\Rightarrow\; f$ is increasing on $(0,\infty)$. If $x\geq 0$, then $x^3\geq 0$, so $1+x^3\geq 1$ and since f is increasing, this means that $f\left(1+x^3\right)\geq f(1) \;\Rightarrow\; \sqrt{1+x^3}\geq 1$ for $x\geq 0$. Next let $g(t)=t^2-t \;\Rightarrow\; g'(t)=2t-1 \;\Rightarrow\; g'(t)>0$ when $t\geq 1$. Thus, g is increasing on $(1,\infty)$. And since $g(1)=0$, $g(t)\geq 0$ when $t\geq 1$. Now let $t=\sqrt{1+x^3}$, where $x\geq 0$. $\sqrt{1+x^3}\geq 1$ (from above) $\Rightarrow\; t\geq 1 \;\Rightarrow\; g(t)\geq 0 \;\Rightarrow$ $\left(1+x^3\right)-\sqrt{1+x^3}\geq 0$ for $x\geq 0$. Therefore, $1\leq\sqrt{1+x^3}\leq 1+x^3$ for $x\geq 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx\leq\int_0^1\sqrt{1+x^3}\,dx\leq\int_0^1\left(1+x^3\right)dx \;\Leftrightarrow$
$\left[x\right]_0^1\leq\int_0^1\sqrt{1+x^3}\,dx\leq\left[x+\frac{1}{4}x^4\right]_0^1 \;\Leftrightarrow\; 1\leq\int_0^1\sqrt{1+x^3}\,dx\leq 1+\frac{1}{4}=1.25.$

71. $0<\dfrac{x^2}{x^4+x^2+1}<\dfrac{x^2}{x^4}=\dfrac{1}{x^2}$ on $[5,10]$, so

$$0\leq\int_5^{10}\frac{x^2}{x^4+x^2+1}\,dx<\int_5^{10}\frac{1}{x^2}\,dx=\left[-\frac{1}{x}\right]_5^{10}=-\frac{1}{10}-\left(-\frac{1}{5}\right)=\frac{1}{10}=0.1.$$

73. Using FTC1, we differentiate both sides of $6+\displaystyle\int_a^x\frac{f(t)}{t^2}\,dt=2\sqrt{x}$ to get $\dfrac{f(x)}{x^2}=2\dfrac{1}{2\sqrt{x}} \;\Rightarrow\; f(x)=x^{3/2}$.

To find a, we substitute $x=a$ in the original equation to obtain $6+\displaystyle\int_a^a\frac{f(t)}{t^2}\,dt=2\sqrt{a} \;\Rightarrow\; 6+0=2\sqrt{a} \;\Rightarrow$ $3=\sqrt{a} \;\Rightarrow\; a=9.$

75. (a) Let $F(t)=\int_0^t f(s)\,ds$. Then, by FTC1, $F'(t)=f(t)=$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0,t]$.

(b) $C(t)=\dfrac{1}{t}\left[A+\displaystyle\int_0^t f(s)\,ds\right]=\dfrac{A+F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0,t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t)=\dfrac{1}{t}\left[A+\displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t)=-\dfrac{1}{t^2}\left[A+\displaystyle\int_0^t f(s)\,ds\right]+\dfrac{1}{t}f(t)$.

$$C'(t)=0 \;\Rightarrow\; t\,f(t)=A+\int_0^t f(s)\,ds \;\Rightarrow\; f(t)=\frac{1}{t}\left[A+\int_0^t f(s)\,ds\right]=C(t).$$

5.4 Indefinite Integrals and the Net Change Theorem

1. $\dfrac{d}{dx}\left[\sqrt{x^2+1}+C\right]=\dfrac{d}{dx}\left[\left(x^2+1\right)^{1/2}+C\right]=\frac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x+0=\dfrac{x}{\sqrt{x^2+1}}$

3. $$\begin{aligned}\frac{d}{dx}\left[\sin x-\tfrac{1}{3}\sin^3 x+C\right]=\frac{d}{dx}\left[\sin x-\tfrac{1}{3}(\sin x)^3+C\right]&=\cos x-\tfrac{1}{3}\cdot 3(\sin x)^2(\cos x)+0\\&=\cos x(1-\sin^2 x)=\cos x(\cos^2 x)=\cos^3 x\end{aligned}$$

5. $\displaystyle\int\left(x^2+x^{-2}\right)dx=\frac{x^3}{3}+\frac{x^{-1}}{-1}+C=\frac{1}{3}x^3-\frac{1}{x}+C$

7. $\int \left(x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\right) dx = \dfrac{x^5}{5} - \dfrac{1}{2}\dfrac{x^4}{4} + \dfrac{1}{4}\dfrac{x^2}{2} - 2x + C = \frac{1}{5}x^5 - \frac{1}{8}x^4 + \frac{1}{8}x^2 - 2x + C$

9. $\int (1-t)(2+t^2)\, dt = \int (2 - 2t + t^2 - t^3)\, dt = 2t - 2\dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + C = 2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$

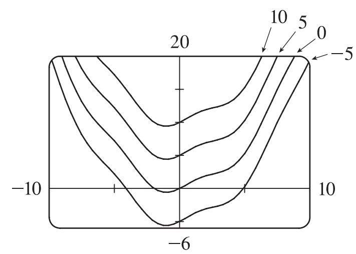
11. $\displaystyle\int \frac{x^3 - 2\sqrt{x}}{x}\, dx = \int \left(\frac{x^3}{x} - \frac{2x^{1/2}}{x}\right) dx = \int (x^2 - 2x^{-1/2})\, dx = \frac{x^3}{3} - 2\,\frac{x^{1/2}}{1/2} + C = \tfrac{1}{3}x^3 - 4\sqrt{x} + C$

13. $\int (\sin x + \sinh x)\, dx = -\cos x + \cosh x + C$

15. $\int (\theta - \csc\theta\,\cot\theta)\, d\theta = \frac{1}{2}\theta^2 + \csc\theta + C$

17. $\int (1 + \tan^2\alpha)\, d\alpha = \int \sec^2\alpha\, d\alpha = \tan\alpha + C$

19. $\int \left(\cos x + \frac{1}{2}x\right) dx = \sin x + \frac{1}{4}x^2 + C$. The members of the family in the figure correspond to $C = -5, 0, 5,$ and 10.



21. $\int_0^2 (6x^2 - 4x + 5)\, dx = \left[6 \cdot \frac{1}{3}x^3 - 4 \cdot \frac{1}{2}x^2 + 5x\right]_0^2 = \left[2x^3 - 2x^2 + 5x\right]_0^2 = (16 - 8 + 10) - 0 = 18$

23. $\int_{-1}^0 (2x - e^x)\, dx = \left[x^2 - e^x\right]_{-1}^0 = (0 - 1) - \left(1 - e^{-1}\right) = -2 + 1/e$

25. $\int_{-2}^2 (3u+1)^2\, du = \int_{-2}^2 \left(9u^2 + 6u + 1\right)\, du = \left[9 \cdot \frac{1}{3}u^3 + 6 \cdot \frac{1}{2}u^2 + u\right]_{-2}^2 = \left[3u^3 + 3u^2 + u\right]_{-2}^2$

$= (24 + 12 + 2) - (-24 + 12 - 2) = 38 - (-14) = 52$

27. $\int_1^4 \sqrt{t}\,(1+t)\, dt = \int_1^4 (t^{1/2} + t^{3/2})\, dt = \left[\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}\right]_1^4 = \left(\frac{16}{3} + \frac{64}{5}\right) - \left(\frac{2}{3} + \frac{2}{5}\right) = \frac{14}{3} + \frac{62}{5} = \frac{256}{15}$

29. $\displaystyle\int_{-2}^{-1} \left(4y^3 + \frac{2}{y^3}\right) dy = \left[4 \cdot \tfrac{1}{4}y^4 + 2 \cdot \frac{1}{-2}y^{-2}\right]_{-2}^{-1} = \left[y^4 - \frac{1}{y^2}\right]_{-2}^{-1} = (1 - 1) - \left(16 - \tfrac{1}{4}\right) = -\tfrac{63}{4}$

31. $\int_0^1 x\left(\sqrt[3]{x} + \sqrt[4]{x}\right)\, dx = \int_0^1 (x^{4/3} + x^{5/4})\, dx = \left[\frac{3}{7}x^{7/3} + \frac{4}{9}x^{9/4}\right]_0^1 = \left(\frac{3}{7} + \frac{4}{9}\right) - 0 = \frac{55}{63}$

33. $\int_1^4 \sqrt{5/x}\, dx = \sqrt{5}\int_1^4 x^{-1/2}\, dx = \sqrt{5}\left[2\sqrt{x}\right]_1^4 = \sqrt{5}\,(2 \cdot 2 - 2 \cdot 1) = 2\sqrt{5}$

35. $\int_0^\pi (4\sin\theta - 3\cos\theta)\, d\theta = \left[-4\cos\theta - 3\sin\theta\right]_0^\pi = (4 - 0) - (-4 - 0) = 8$

37. $\displaystyle\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta = \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\right) d\theta = \int_0^{\pi/4} (\sec^2\theta + 1)\, d\theta$

$= \left[\tan\theta + \theta\right]_0^{\pi/4} = \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (0 + 0) = 1 + \frac{\pi}{4}$

39. $\displaystyle\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx = \int_1^{64}\left(\frac{1}{x^{1/2}} + \frac{x^{1/3}}{x^{1/2}}\right)dx = \int_1^{64}\left(x^{-1/2} + x^{(1/3)-(1/2)}\right)dx = \int_1^{64}\left(x^{-1/2}+x^{-1/6}\right)dx$

$\displaystyle = \left[2x^{1/2} + \tfrac{6}{5}x^{5/6}\right]_1^{64} = \left(16 + \tfrac{192}{5}\right) - \left(2 + \tfrac{6}{5}\right) = 14 + \tfrac{186}{5} = \tfrac{256}{5}$

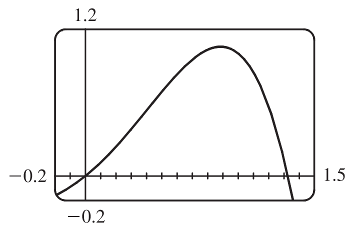
41. $\displaystyle\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt = \int_0^{1/\sqrt{3}} \frac{t^2-1}{(t^2+1)(t^2-1)}\,dt = \int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\,dt = \left[\arctan t\right]_0^{1/\sqrt{3}} = \arctan\left(1/\sqrt{3}\right) - \arctan 0$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

43. $\int_{-1}^{2}(x - 2\,|x|)\,dx = \int_{-1}^{0}[x - 2(-x)]\,dx + \int_0^2 [x - 2(x)]\,dx = \int_{-1}^{0} 3x\,dx + \int_0^2 (-x)\,dx = 3\left[\frac{1}{2}x^2\right]_{-1}^{0} - \left[\frac{1}{2}x^2\right]_0^2$

$= 3\left(0 - \frac{1}{2}\right) - (2 - 0) = -\frac{7}{2} = -3.5$

45. The graph shows that $y = x + x^2 - x^4$ has x-intercepts at $x = 0$ and at $x = a \approx 1.32$. So the area of the region that lies under the curve and above the x-axis is



$\int_0^a (x + x^2 - x^4)\,dx = \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^a$

$= \left(\frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{5}a^5\right) - 0 \approx 0.84$

47. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

49. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

51. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

53. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

55. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters. (A newton-meter is abbreviated N·m and is called a joule.)

57. (a) Displacement $= \int_0^3 (3t - 5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) Distance traveled $= \int_0^3 |3t - 5|\,dt = \int_0^{5/3}(5 - 3t)\,dt + \int_{5/3}^{3}(3t - 5)\,dt$

$= \left[5t - \frac{3}{2}t^2\right]_0^{5/3} + \left[\frac{3}{2}t^2 - 5t\right]_{5/3}^{3} = \frac{25}{3} - \frac{3}{2}\cdot\frac{25}{9} + \frac{27}{2} - 15 - \left(\frac{3}{2}\cdot\frac{25}{9} - \frac{25}{3}\right) = \frac{41}{6}$ m

59. (a) $v'(t) = a(t) = t + 4 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + C \;\Rightarrow\; v(0) = C = 5 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) Distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10}\left|\frac{1}{2}t^2 + 4t + 5\right|dt = \int_0^{10}\left(\frac{1}{2}t^2 + 4t + 5\right)dt = \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10}$

$= \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

61. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 \left(9 + 2\sqrt{x}\right) dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

63. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour.

So the distance traveled is

$\int_0^{100} v(t)\,dt \approx \frac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \frac{1}{180}(38 + 58 + 51 + 53 + 47) = \frac{247}{180} \approx 1.4$ miles.

65. From the Net Change Theorem, the increase in cost if the production level is raised

from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} \left(3 - 0.01x + 0.000006x^2\right) dx = \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000} = 60{,}000 - 2{,}000 = \$58{,}000$

67. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\begin{aligned}\text{coefficient of inequality} &= \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx} \\ &= \frac{\int_0^1 [x - L(x)]\,dx}{\left[x^2/2\right]_0^1} = \frac{\int_0^1 [x - L(x)]\,dx}{1/2} = 2\int_0^1 [x - L(x)]\,dx\end{aligned}$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \quad \Rightarrow \quad L(50\%) = L\left(\frac{1}{2}\right) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.3958\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned}\text{coefficient of inequality} &= 2\int_0^1 [x - L(x)]\,dx = 2\int_0^1 \left(x - \tfrac{5}{12}x^2 - \tfrac{7}{12}x\right) dx = 2\int_0^1 \left(\tfrac{5}{12}x - \tfrac{5}{12}x^2\right) dx \\ &= 2\int_0^1 \tfrac{5}{12}\left(x - x^2\right) dx = \tfrac{5}{6}\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{5}{6}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{5}{6}\left(\tfrac{1}{6}\right) = \tfrac{5}{36}\end{aligned}$$

5.5 The Substitution Rule

1. Let $u = -x$. Then $du = -\,dx$, so $dx = -\,du$. Thus, $\int e^{-x}dx = \int e^u(-du) = -e^u + C = -e^{-x} + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$\frac{d}{dx}(-e^{-x} + C) = -[e^{-x}(-1)] = e^{-x}$, the desired result.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3 + 1}\,dx = \int \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3 + 1)^{3/2} + C.$$

5. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$$\int \cos^3\theta\,\sin\theta\,d\theta = \int u^3\,(-du) = -\frac{u^4}{4} + C = -\tfrac{1}{4}\cos^4\theta + C.$$

7. Let $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so $\int x\sin(x^2)\,dx = \int \sin u\left(\frac{1}{2}\,du\right) = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C$.

9. Let $u = 3x - 2$. Then $du = 3\,dx$ and $dx = \frac{1}{3}\,du$, so $\int (3x-2)^{20}\,dx = \int u^{20}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{21}u^{21} + C = \frac{1}{63}(3x-2)^{21} + C$.

11. Let $u = 2x + x^2$. Then $du = (2+2x)\,dx = 2(1+x)\,dx$ and $(x+1)\,dx = \frac{1}{2}\,du$, so

$$\int (x+1)\sqrt{2x+x^2}\,dx = \int \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}\left(2x+x^2\right)^{3/2} + C.$$

Or: Let $u = \sqrt{2x+x^2}$. Then $u^2 = 2x + x^2 \;\Rightarrow\; 2u\,du = (2+2x)\,dx \;\Rightarrow\; u\,du = (1+x)\,dx$, so

$\int (x+1)\sqrt{2x+x^2}\,dx = \int u\cdot u\,du = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(2x+x^2)^{3/2} + C$.

13. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5-3x} = \int \frac{1}{u}\left(-\tfrac{1}{3}\,du\right) = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5-3x| + C.$$

15. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}\,du$, so $\int \sin \pi t\,dt = \int \sin u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}(-\cos u) + C = -\frac{1}{\pi}\cos\pi t + C$.

17. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a+bx^2)\,dx$, so

$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx = \int \frac{\frac{1}{3}\,du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}\,du = \tfrac{1}{3}\cdot 2u^2 + C = \tfrac{2}{3}\sqrt{3ax+bx^3} + C.$$

19. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$.

21. Let $u = \sqrt{t}$. Then $du = \dfrac{dt}{2\sqrt{t}}$ and $\dfrac{1}{\sqrt{t}}\,dt = 2\,du$, so $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int \cos u\,(2\,du) = 2\sin u + C = 2\sin\sqrt{t} + \dot{C}$.

23. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \cos\theta\,\sin^6\theta\,d\theta = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}\sin^7\theta + C$.

25. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

27. Let $u = 1 + z^3$. Then $du = 3z^2\,dz$ and $z^2\,dz = \frac{1}{3}\,du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz = \int u^{-1/3}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}u^{2/3} + C = \tfrac{1}{2}(1+z^3)^{2/3} + C.$$

29. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int e^{\tan x}\sec^2 x\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

31. Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$

[or $-\csc x + C$].

33. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\,\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

35. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$2I = -2\displaystyle\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$

Or: Let $u = 1+\cos^2 x$.

37. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

39. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\,\tan x)\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C.$

41. Let $u = \sin^{-1} x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1} x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|\sin^{-1} x\right| + C$.

43. Let $u = 1+x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int \frac{1+x}{1+x^2}\,dx &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C \\ &= \tan^{-1} x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].\end{aligned}$$

45. Let $u = x+2$. Then $du = dx$, so

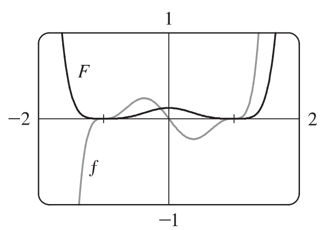
$$\begin{aligned}\int \frac{x}{\sqrt[4]{x+2}}\,dx &= \int \frac{u-2}{\sqrt[4]{u}}\,du = \int (u^{3/4} - 2u^{-1/4})\,du = \tfrac{4}{7}u^{7/4} - 2\cdot\tfrac{4}{3}u^{3/4} + C \\ &= \tfrac{4}{7}(x+2)^{7/4} - \tfrac{8}{3}(x+2)^{3/4} + C\end{aligned}$$

In Exercises 47–50, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

47. $f(x) = x(x^2-1)^3$. $\quad u = x^2 - 1 \quad\Rightarrow\quad du = 2x\,dx$, so

$\int x(x^2-1)^3\,dx = \int u^3\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{8}u^4 + C = \tfrac{1}{8}(x^2-1)^4 + C$

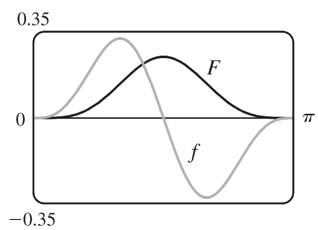
Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).



49. $f(x) = \sin^3 x\,\cos x$. $u = \sin x \quad\Rightarrow\quad du = \cos x\,dx$, so

$\int \sin^3 x\,\cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$

Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



51. Let $u = x-1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 7(b), since $f(u) = u^{25}$ is an odd function.

53. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$\int_0^1 x^2\left(1+2x^3\right)^5\,dx = \int_1^3 u^5\left(\frac{1}{6}\,du\right) = \frac{1}{6}\left[\frac{1}{6}u^6\right]_1^3 = \frac{1}{36}(3^6 - 1^6) = \frac{1}{36}(729-1) = \frac{728}{36} = \frac{182}{9}$.

55. Let $u = t/4$, so $du = \frac{1}{4}\,dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$\int_0^\pi \sec^2(t/4)\,dt = \int_0^{\pi/4} \sec^2 u\,(4\,du) = 4\big[\tan u\big]_0^{\pi/4} = 4\left(\tan\frac{\pi}{4} - \tan 0\right) = 4(1-0) = 4$.

57. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta = 0$ by Theorem 7(b), since $f(\theta) = \tan^3\theta$ is an odd function.

59. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

61. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3-1) = 3.$$

63. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x\sqrt{x^2+a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2} = \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2}-1\right)a^3$$

65. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

67. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

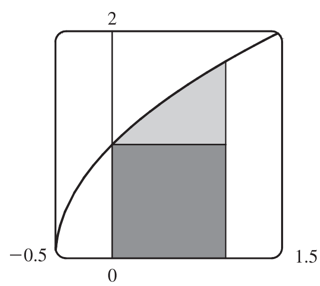
$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

69. Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

71. From the graph, it appears that the area under the curve is about $1 + \left(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$A = \int_1^3 \sqrt{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^3 = \frac{1}{3}\left(3\sqrt{3}-1\right) = \sqrt{3} - \frac{1}{3} \approx 1.399$.



73. First write the integral as a sum of two integrals:

$I = \int_{-2}^2 (x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2}\,dx + \int_{-2}^2 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 7(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

75. **First Figure** Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 ue^u\,du.$

Third Figure Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$A_3 = \int_0^{\pi/2} e^{\sin x}\sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du.$

Since $A_1 = A_2 = A_3$, all three areas are equal.

77. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$$\int_0^{60} r(t)\,dt = \int_0^{60} 100e^{-0.01t}\,dt \qquad [u = -0.01t, du = -0.01dt]$$
$$= 100\int_0^{-0.6} e^u(-100\,du) = -10{,}000\left[e^u\right]_0^{-0.6} = -10{,}000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ liters}$$

79. The volume of inhaled air in the lungs at time t is

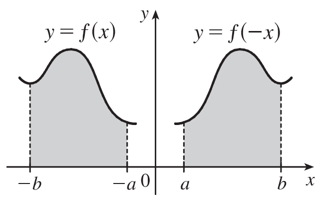
$$V(t) = \int_0^t f(u)\,du = \int_0^t \tfrac{1}{2}\sin\left(\tfrac{2\pi}{5}u\right)du = \int_0^{2\pi t/5} \tfrac{1}{2}\sin v\left(\tfrac{5}{2\pi}\,dv\right) \qquad \left[\text{substitute } v = \tfrac{2\pi}{5}u,\, dv = \tfrac{2\pi}{5}\,du\right]$$
$$= \tfrac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \tfrac{5}{4\pi}\left[-\cos\left(\tfrac{2\pi}{5}t\right) + 1\right] = \tfrac{5}{4\pi}\left[1 - \cos\left(\tfrac{2\pi}{5}t\right)\right] \text{ liters}$$

81. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5.$

83. Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



85. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a\,u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$

87. $\dfrac{x\sin x}{1+\cos^2 x} = x\cdot\dfrac{\sin x}{2-\sin^2 x} = x\,f(\sin x)$, where $f(t) = \dfrac{t}{2-t^2}$. By Exercise 86,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^1$$
$$= \frac{\pi}{2}[\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

5 Review

CONCEPT CHECK

1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f.

x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph beside it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph by it (contains "**net area**").

3. See the Fundamental Theorem of Calculus after Example 9 in Section 5.3.

4. (a) See the Net Change Theorem after Example 5 in Section 5.4.

(b) $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Net Change Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous.

7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

3. True by Property 3 of the Integral in Section 5.2.

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

7. True by Comparison Property 7 of the Integral in Section 5.2.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.7(b).

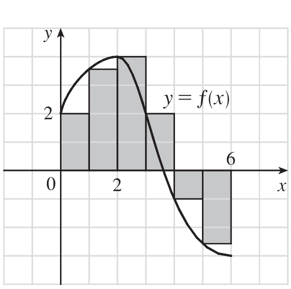
11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

15. False. $\int_a^b f(x)\,dx$ is a constant, so $\dfrac{d}{dx}\left(\int_a^b f(x)\,dx\right) = 0$, not $f(x)$ [unless $f(x) = 0$]. Compare the given statement carefully with FTC1, in which the upper limit in the integral is x.

EXERCISES

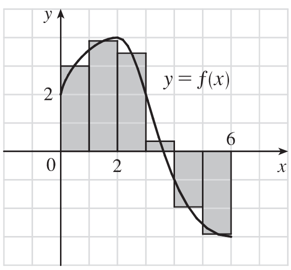
1. (a)



$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad \left[\Delta x = \tfrac{6-0}{6} = 1\right]$$
$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$
$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

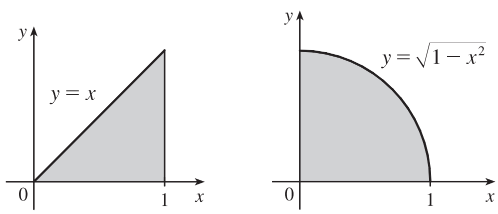
The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

(b)



$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad \left[\Delta x = \tfrac{6-0}{6} = 1\right]$$
$$= f(\overline{x}_1)\cdot 1 + f(\overline{x}_2)\cdot 1 + f(\overline{x}_3)\cdot 1 + f(\overline{x}_4)\cdot 1 + f(\overline{x}_5)\cdot 1 + f(\overline{x}_6)\cdot 1$$
$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$
$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

3. $\int_0^1 \left(x + \sqrt{1 - x^2}\,\right) dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1 - x^2}\,dx = I_1 + I_2$.
I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle.
Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.



5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad \Rightarrow \quad 10 = 7 + \int_4^6 f(x)\,dx \quad \Rightarrow \quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

9. $\int_1^2 \left(8x^3 + 3x^2\right) dx = \left[8\cdot\frac{1}{4}x^4 + 3\cdot\frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = \left(2\cdot 2^4 + 2^3\right) - (2 + 1) = 40 - 3 = 37$

11. $\int_0^1 \left(1 - x^9\right) dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = \left(1 - \frac{1}{10}\right) - 0 = \frac{9}{10}$

13. $\displaystyle\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du = \int_1^9 (u^{-1/2}-2u)\,du = \left[2u^{1/2}-u^2\right]_1^9 = (6-81)-(2-1) = -76$

15. Let $u = y^2+1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y=0$, $u=1$; when $y=1$, $u=2$. Thus,

$\int_0^1 y(y^2+1)^5\ dy = \int_1^2 u^5\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64-1) = \frac{63}{12} = \frac{21}{4}$.

17. $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \dfrac{1}{(t-4)^2}$ has an infinite discontinuity at $t=4$; that is, f is discontinuous on the interval $[1,5]$.

19. Let $u = v^3$, so $du = 3v^2\,dv$. When $v=0$, $u=0$; when $v=1$, $u=1$. Thus,

$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1$.

21. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4\tan t}{2+\cos t}\,dt = 0$ by Theorem 5.5.7(b), since $f(t) = \dfrac{t^4\tan t}{2+\cos t}$ is an odd function.

23. $\displaystyle\int\left(\frac{1-x}{x}\right)^2 dx = \int\left(\frac{1}{x}-1\right)^2 dx = \int\left(\frac{1}{x^2}-\frac{2}{x}+1\right) dx = -\frac{1}{x} - 2\ln|x| + x + C$

25. Let $u = x^2+4x$. Then $du = (2x+4)\,dx = 2(x+2)\,dx$, so

$\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2+4x} + C.$

27. Let $u = \sin\pi t$. Then $du = \pi\ \cos\pi t\,dt$, so $\int \sin\pi t\ \cos\pi t\,dt = \int u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}\cdot\frac{1}{2}u^2 + C = \frac{1}{2\pi}(\sin\pi t)^2 + C$.

29. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

31. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx$, so

$\int \tan x\ \ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C$.

33. Let $u = 1+x^4$. Then $du = 4x^3\,dx$, so $\displaystyle\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \tfrac{1}{4}\ln\left(1+x^4\right) + C$.

35. Let $u = 1+\sec\theta$. Then $du = \sec\theta\ \tan\theta\,d\theta$, so

$\displaystyle\int \frac{\sec\theta\ \tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{1+\sec\theta}\,(\sec\theta\ \tan\theta\,d\theta) = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$

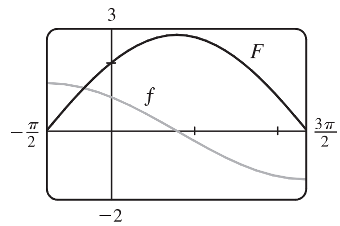
37. Since $x^2-4<0$ for $0\le x<2$ and $x^2-4>0$ for $2<x\le 3$, we have $\left|x^2-4\right| = -(x^2-4) = 4-x^2$ for $0\le x<2$ and $\left|x^2-4\right| = x^2-4$ for $2<x\le 3$. Thus,

$$\begin{aligned}\int_0^3 \left|x^2-4\right|dx &= \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \left[4x-\frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3}-4x\right]_2^3 \\ &= \left(8-\tfrac{8}{3}\right) - 0 + (9-12) - \left(\tfrac{8}{3}-8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}\end{aligned}$$

In Exercises 39 and 40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

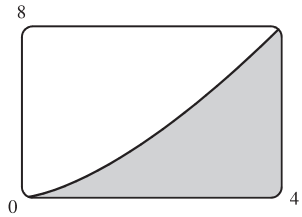
39. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$



41. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{2}{5}(4)^{5/2} = \frac{64}{5} = 12.8.$



43. $F(x) = \displaystyle\int_0^x \frac{t^2}{1+t^3}\,dt \quad\Rightarrow\quad F'(x) = \frac{d}{dx}\int_0^x \frac{t^2}{1+t^3}\,dt = \frac{x^2}{1+x^3}$

45. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_0^{x^4} \cos(t^2)\,dt = \frac{d}{du}\int_0^u \cos(t^2)\,dt \cdot \frac{du}{dx} = \cos(u^2)\,\frac{du}{dx} = 4x^3\cos(x^8).$$

47. $y = \displaystyle\int_{\sqrt{x}}^x \frac{e^t}{t}\,dt = \int_{\sqrt{x}}^1 \frac{e^t}{t}\,dt + \int_1^x \frac{e^t}{t}\,dt = -\int_1^{\sqrt{x}} \frac{e^t}{t}\,dt + \int_1^x \frac{e^t}{t}\,dt \quad\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{d}{dx}\left(\displaystyle\int_1^{\sqrt{x}} \frac{e^t}{t}\,dt\right) + \dfrac{d}{dx}\left(\displaystyle\int_1^x \frac{e^t}{t}\,dt\right)$. Let $u = \sqrt{x}$. Then

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^t}{t}\,dt = \frac{d}{dx}\int_1^u \frac{e^t}{t}\,dt = \frac{d}{du}\left(\int_1^u \frac{e^t}{t}\,dt\right)\frac{du}{dx} = \frac{e^u}{u}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x},$$

so $\dfrac{dy}{dx} = -\dfrac{e^{\sqrt{x}}}{2x} + \dfrac{e^x}{x}$.

49. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \quad\Rightarrow\quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so

$2(3-1) \le \int_1^3 \sqrt{x^2+3}\,dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

51. $0 \le x \le 1 \quad\Rightarrow\quad 0 \le \cos x \le 1 \quad\Rightarrow\quad x^2\cos x \le x^2 \quad\Rightarrow\quad \int_0^1 x^2\cos x\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

53. $\cos x \le 1 \quad\Rightarrow\quad e^x\cos x \le e^x \quad\Rightarrow\quad \int_0^1 e^x\cos x\,dx \le \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$

55. $\Delta x = (3-0)/6 = \frac{1}{2}$, so the endpoints are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, and 3, and the midpoints are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}$, and $\frac{11}{4}$.

The Midpoint Rule gives

$$\int_0^3 \sin(x^3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{1}{2}\left[\sin\left(\tfrac{1}{4}\right)^3 + \sin\left(\tfrac{3}{4}\right)^3 + \sin\left(\tfrac{5}{4}\right)^3 + \sin\left(\tfrac{7}{4}\right)^3 + \sin\left(\tfrac{9}{4}\right)^3 + \sin\left(\tfrac{11}{4}\right)^3\right] \approx 0.280981.$$

57. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem,

$\int_0^8 r(t)\,dt = b(8) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2008.

59. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

$$\begin{aligned}\int_0^{24} r(t)\,dt \approx M_6 &= 4[r(2) + r(6) + r(10) + r(14) + r(18) + r(22)] \\ &\approx 4[50 + 1000 + 7000 + 8550 + 1350 + 150] = 4(18{,}100) = 72{,}400\end{aligned}$$

61. Let $u = 2\sin\theta$. Then $du = 2\cos\theta\,d\theta$ and when $\theta = 0$, $u = 0$; when $\theta = \frac{\pi}{2}$, $u = 2$. Thus,

$\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta = \int_0^2 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^2 f(u)\,du = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}(6) = 3.$

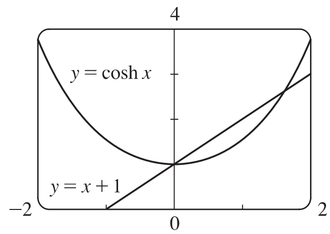
63. Area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1 $\Rightarrow$

$\int_0^1 \sinh cx\,dx = 1 \;\Rightarrow\; \frac{1}{c}\left[\cosh cx\right]_0^1 = 1 \;\Rightarrow\; \frac{1}{c}(\cosh c - 1) = 1 \;\Rightarrow$

$\cosh c - 1 = c \;\Rightarrow\; \cosh c = c + 1$. From the graph, we get $c = 0$ and

$c \approx 1.6161$, but $c = 0$ isn't a solution for this problem since the curve

$y = \sinh cx$ becomes $y = 0$ and the area under it is 0. Thus, $c \approx 1.6161$.



65. Using FTC1, we differentiate both sides of the given equation, $\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$, and get

$$f(x) = e^{2x} + 2xe^{2x} + e^{-x} f(x) \;\Rightarrow\; f(x)\left(1 - e^{-x}\right) = e^{2x} + 2xe^{2x} \;\Rightarrow\; f(x) = \frac{e^{2x}(1+2x)}{1 - e^{-x}}.$$

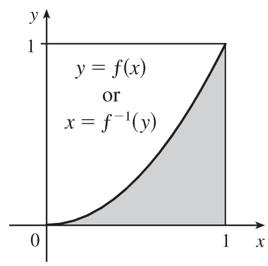
67. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

69. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

71. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$

gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$.

So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



☐ PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\, dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2x f(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

3. Differentiating the given equation, $\int_0^x f(t)\, dt = [f(x)]^2$, using FTC1 gives $f(x) = 2f(x)\, f'(x) \quad \Rightarrow$ $f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. Since $f(x)$ is never 0, we must have $f'(x) = \frac{1}{2}$ and $f'(x) = \frac{1}{2} \quad \Rightarrow$ $f(x) = \frac{1}{2}x + C$. To find C, we substitute into the given equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \quad \Leftrightarrow$ $\frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C^2 = 0$, so $C = 0$, and $f(x) = \frac{1}{2}x$.

5. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\, dt$, where $g(x) = \displaystyle\int_0^{\cos x} [1 + \sin(t^2)]\, dt$. Using FTC1 and the Chain Rule (twice) we have

$f'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\, g'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\, [1 + \sin(\cos^2 x)](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \displaystyle\int_0^0 [1 + \sin(t^2)]\, dt = 0$, so

$f'\left(\frac{\pi}{2}\right) = \dfrac{1}{\sqrt{1 + 0}}\, (1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

7. By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y) = e^y$,

$$\lim_{x \to 0} \frac{\int_0^x (1 - \tan 2t)^{1/t}\, dt}{x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{(1 - \tan 2x)^{1/x}}{1} = \exp\left(\lim_{x \to 0} \frac{\ln(1 - \tan 2x)}{x}\right)$$
$$\overset{\text{H}}{=} \exp\left(\lim_{x \to 0} \frac{-2\sec^2 2x}{1 - \tan 2x}\right) = \exp\left(\frac{-2 \cdot 1^2}{1 - 0}\right) = e^{-2}$$

9. $f(x) = 2 + x - x^2 = (-x + 2)(x + 1) = 0 \quad \Leftrightarrow \quad x = 2$ or $x = -1$. $\quad f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\, dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

11. (a) We can split the integral $\int_0^n [\![x]\!]\, dx$ into the sum $\displaystyle\sum_{i=1}^{n} \left[\int_{i-1}^{i} [\![x]\!]\, dx\right]$. But on each of the intervals $[i - 1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i - 1)[i - (i - 1)] = (i - 1)$. So the original integral is equal to $\displaystyle\sum_{i=1}^{n} (i - 1) = \sum_{i=1}^{n-1} i = \frac{(n - 1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!]-1)\,[\![b]\!]$,

by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!]\,, b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)\,[\![b]\!] + [\![b]\!]\,(b-[\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly $\int_0^a [\![x]\!]\,dx = \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

13. Let $Q(x) = \displaystyle\int_0^x P(t)\,dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3 + \frac{d}{4}t^4\right]_0^x = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4$. Then $Q(0) = 0$, and $Q(1) = 0$ by the given condition, $a + \dfrac{b}{2} + \dfrac{c}{3} + \dfrac{d}{4} = 0$. Also, $Q'(x) = P(x) = a + bx + cx^2 + dx^3$ by FTC1. By Rolle's Theorem, applied to Q on $[0,1]$, there is a number r in $(0,1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$. Thus, the equation $P(x) = 0$ has a root between 0 and 1.

More generally, if $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and if $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$, then the equation $P(x) = 0$ has a root between 0 and 1. The proof is the same as before:

Let $Q(x) = \displaystyle\int_0^x P(t)\,dt = a_0x + \frac{a_1}{2}\,x^2 + \frac{a_2}{3}\,x^3 + \cdots + \frac{a_n}{n+1}\,x^n$. Then $Q(0) = Q(1) = 0$ and $Q'(x) = P(x)$. By Rolle's Theorem applied to Q on $[0,1]$, there is a number r in $(0,1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$.

15. Note that $\dfrac{d}{dx}\left(\displaystyle\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \displaystyle\int_0^x f(t)\,dt$ by FTC1, while

$$\begin{aligned}\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] &= \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right] \\ &= \textstyle\int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du\end{aligned}$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

17.
$$\begin{aligned}\lim_{n\to\infty}&\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}}\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}}\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}}\right) \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n f\!\left(\frac{i}{n}\right) \qquad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}}\right] \\ &= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = \left[2\sqrt{1+x}\right]_0^1 = 2\left(\sqrt{2}-1\right)\end{aligned}$$

6 □ APPLICATIONS OF INTEGRATION

6.1 Areas Between Curves

1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

3. $A = \int_{y=-1}^{y=1} (x_R - x_L)\,dy = \int_{-1}^{1} \left[e^y - (y^2 - 2)\right] dy = \int_{-1}^{1} \left(e^y - y^2 + 2\right) dy$

$= \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1} = \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - \dfrac{1}{e} + \dfrac{10}{3}$

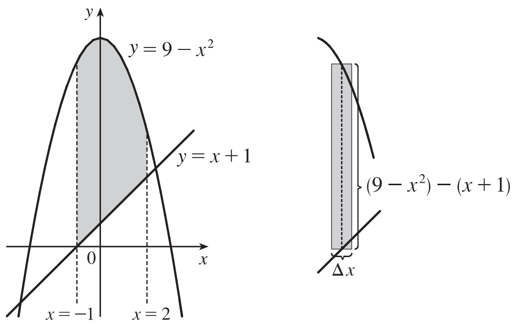
5. $A = \int_{-1}^{2} \left[(9 - x^2) - (x + 1)\right] dx$

$= \int_{-1}^{2} (8 - x - x^2)\,dx$

$= \left[8x - \dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-1}^{2}$

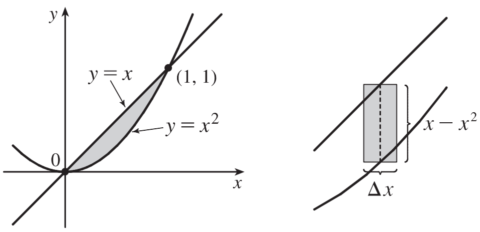
$= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right)$

$= 22 - 3 + \frac{1}{2} = \frac{39}{2}$



7. The curves intersect when $x = x^2 \quad\Leftrightarrow\quad x^2 - x = 0 \quad\Leftrightarrow$ $x(x - 1) = 0 \quad\Leftrightarrow\quad x = 0$ or 1.

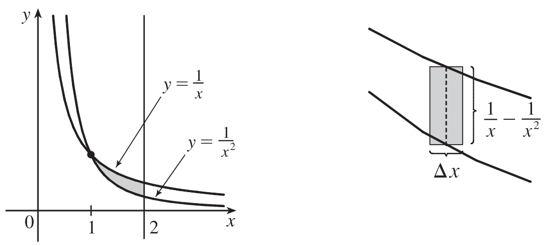
$A = \int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$



9. $A = \int_1^2 \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) dx = \left[\ln x + \dfrac{1}{x}\right]_1^2$

$= \left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1)$

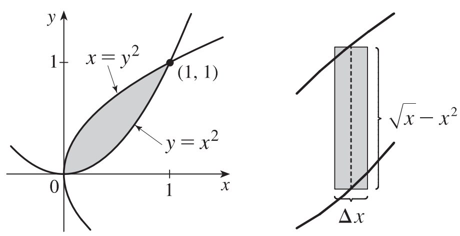
$= \ln 2 - \frac{1}{2} \approx 0.19$



11. $A = \int_0^1 \left(\sqrt{x} - x^2\right) dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$



13. $12 - x^2 = x^2 - 6 \quad \Leftrightarrow \quad 2x^2 = 18 \quad \Leftrightarrow$

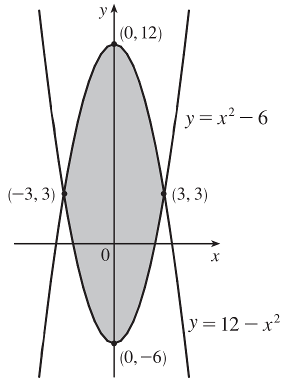
$x^2 = 9 \quad \Leftrightarrow \quad x = \pm 3$, so

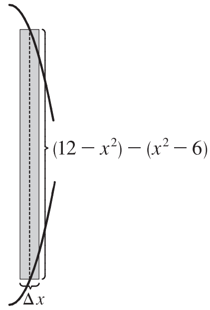
$$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6)\right] dx$$

$$= 2\int_0^3 \left(18 - 2x^2\right) dx \qquad \text{[by symmetry]}$$

$$= 2\left[18x - \tfrac{2}{3}x^3\right]_0^3 = 2\left[(54 - 18) - 0\right]$$

$$= 2(36) = 72$$





15. The curves intersect when $\tan x = 2\sin x \quad (\text{on } [-\pi/3, \pi/3]) \quad \Leftrightarrow \quad \sin x = 2\sin x\,\cos x \quad \Leftrightarrow$

$2\sin x\,\cos x - \sin x = 0 \quad \Leftrightarrow \quad \sin x\,(2\cos x - 1) = 0 \quad \Leftrightarrow \quad \sin x = 0$ or $\cos x = \frac{1}{2} \quad \Leftrightarrow \quad x = 0$ or $x = \pm\frac{\pi}{3}$.

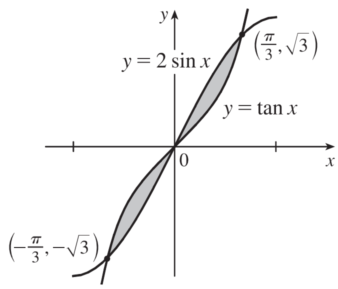
$$A = \int_{-\pi/3}^{\pi/3} (2\sin x - \tan x)\,dx$$

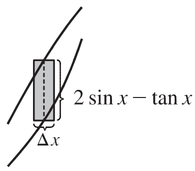
$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\,dx \qquad \text{[by symmetry]}$$

$$= 2\Big[-2\cos x - \ln|\sec x|\Big]_0^{\pi/3}$$

$$= 2\left[(-1 - \ln 2) - (-2 - 0)\right]$$

$$= 2(1 - \ln 2) = 2 - 2\ln 2$$

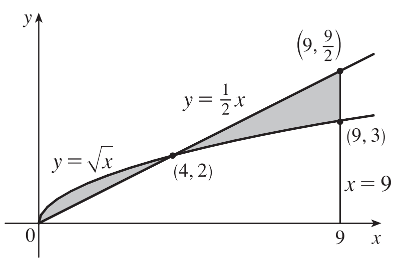


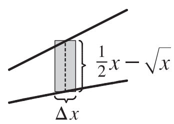


17. $\frac{1}{2}x = \sqrt{x} \quad \Rightarrow \quad \frac{1}{4}x^2 = x \quad \Rightarrow \quad x^2 - 4x = 0 \quad \Rightarrow \quad x(x - 4) = 0 \quad \Rightarrow \quad x = 0$ or 4, so

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$

$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$

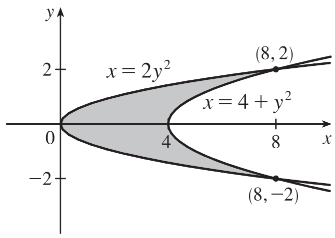


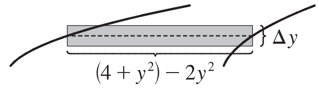


For $4 < x < 9$

19. $2y^2 = 4 + y^2 \quad \Leftrightarrow \quad y^2 = 4 \quad \Leftrightarrow \quad y = \pm 2$, so

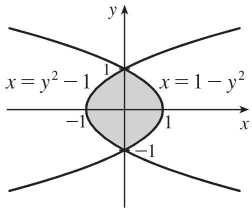
$$\begin{aligned} A &= \int_{-2}^{2} \left[(4 + y^2) - 2y^2\right] dy \\ &= 2\int_0^2 (4 - y^2)\, dy \qquad \text{[by symmetry]} \\ &= 2\left[4y - \tfrac{1}{3}y^3\right]_0^2 = 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3} \end{aligned}$$

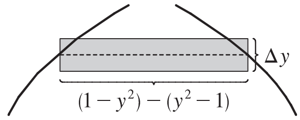




21. The curves intersect when $1 - y^2 = y^2 - 1 \quad \Leftrightarrow \quad 2 = 2y^2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm 1$.

$$\begin{aligned} A &= \int_{-1}^{1} \left[(1 - y^2) - (y^2 - 1)\right] dy \\ &= \int_{-1}^{1} 2(1 - y^2)\, dy \\ &= 2 \cdot 2\int_0^1 (1 - y^2)\, dy \\ &= 4\left[y - \tfrac{1}{3}y^3\right]_0^1 = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3} \end{aligned}$$



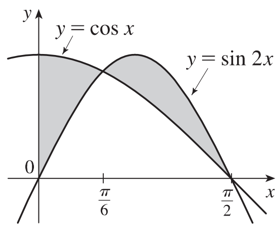


23. Notice that $\cos x = \sin 2x = 2\sin x \cos x \quad \Leftrightarrow$

$2\sin x \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$

$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$\begin{aligned} A &= \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx \\ &= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2} \end{aligned}$$

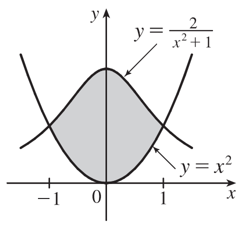


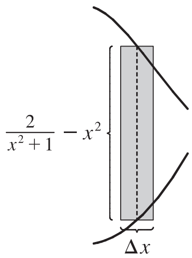
25. The curves intersect when $x^2 = \dfrac{2}{x^2 + 1} \quad \Leftrightarrow$

$x^4 + x^2 = 2 \quad \Leftrightarrow \quad x^4 + x^2 - 2 = 0 \quad \Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0 \quad \Leftrightarrow \quad x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$.

$$\begin{aligned} A &= \int_{-1}^{1} \left(\frac{2}{x^2 + 1} - x^2\right) dx = 2\int_0^1 \left(\frac{2}{x^2 + 1} - x^2\right) dx \\ &= 2\left[2\tan^{-1} x - \tfrac{1}{3}x^3\right]_0^1 = 2\left(2 \cdot \tfrac{\pi}{4} - \tfrac{1}{3}\right) = \pi - \tfrac{2}{3} \approx 2.47 \end{aligned}$$

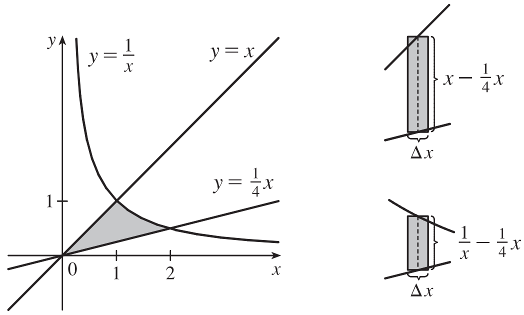




27. $1/x = x \quad \Leftrightarrow \quad 1 = x^2 \quad \Leftrightarrow \quad x = \pm 1$ and $1/x = \frac{1}{4}x \quad \Leftrightarrow$

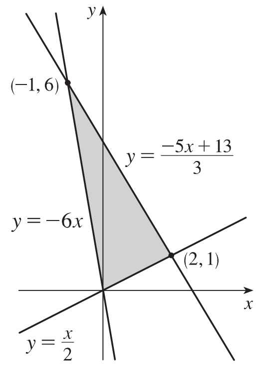
$4 = x^2 \quad \Leftrightarrow \quad x = \pm 2$, so for $x > 0$,

$$A = \int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$
$$= \int_0^1 \left(\frac{3}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$
$$= \left[\tfrac{3}{8}x^2\right]_0^1 + \left[\ln|x| - \tfrac{1}{8}x^2\right]_1^2$$
$$= \tfrac{3}{8} + \left(\ln 2 - \tfrac{1}{2}\right) - \left(0 - \tfrac{1}{8}\right) = \ln 2$$



29. An equation of the line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$; through $(0, 0)$ and $(-1, 6)$ is $y = -6x$; through $(2, 1)$ and $(-1, 6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.

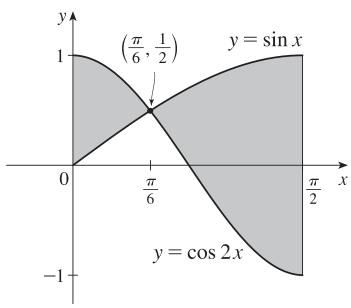
$$A = \int_{-1}^0 \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right] dx + \int_0^2 \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right] dx$$
$$= \int_{-1}^0 \left(\tfrac{13}{3}x + \tfrac{13}{3}\right) dx + \int_0^2 \left(-\tfrac{13}{6}x + \tfrac{13}{3}\right) dx$$
$$= \tfrac{13}{3}\int_{-1}^0 (x+1)\, dx + \tfrac{13}{3}\int_0^2 \left(-\tfrac{1}{2}x + 1\right) dx$$
$$= \tfrac{13}{3}\left[\tfrac{1}{2}x^2 + x\right]_{-1}^0 + \tfrac{13}{3}\left[-\tfrac{1}{4}x^2 + x\right]_0^2$$
$$= \tfrac{13}{3}\left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}\left[(-1+2) - 0\right] = \tfrac{13}{3}\cdot\tfrac{1}{2} + \tfrac{13}{3}\cdot 1 = \tfrac{13}{2}$$



31. The curves intersect when $\sin x = \cos 2x \quad (\text{on } [0, \pi/2]) \quad \Leftrightarrow \quad \sin x = 1 - 2\sin^2 x \quad \Leftrightarrow \quad 2\sin^2 x + \sin x - 1 = 0 \quad \Leftrightarrow$

$(2\sin x - 1)(\sin x + 1) = 0 \quad \Rightarrow \quad \sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}$.

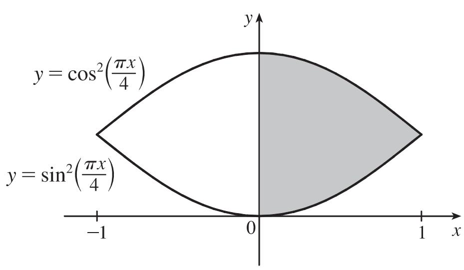
$$A = \int_0^{\pi/2} |\sin x - \cos 2x|\, dx$$
$$= \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x)\, dx$$
$$= \left[\tfrac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \tfrac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2}$$
$$= \left(\tfrac{1}{4}\sqrt{3} + \tfrac{1}{2}\sqrt{3}\right) - (0 + 1) + (0 - 0) - \left(-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\sqrt{3}\right)$$
$$= \tfrac{3}{2}\sqrt{3} - 1$$



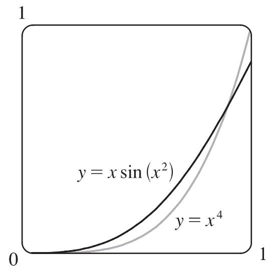
33. Let $f(x) = \cos^2\left(\dfrac{\pi x}{4}\right) - \sin^2\left(\dfrac{\pi x}{4}\right)$ and $\Delta x = \dfrac{1-0}{4}$.

The shaded area is given by

$$A = \int_0^1 f(x)\, dx \approx M_4$$
$$= \tfrac{1}{4}\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right)\right]$$
$$\approx 0.6407$$



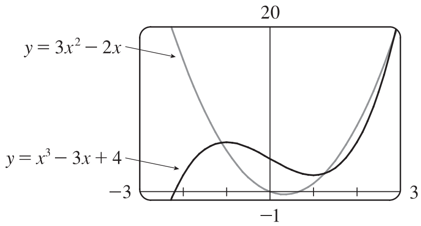
35.



From the graph, we see that the curves intersect at $x = 0$ and $x = a \approx 0.896$, with $x\sin(x^2) > x^4$ on $(0, a)$. So the area A of the region bounded by the curves is

$$A = \int_0^a \left[x\sin(x^2) - x^4\right] dx = \left[-\tfrac{1}{2}\cos(x^2) - \tfrac{1}{5}x^5\right]_0^a$$
$$= -\tfrac{1}{2}\cos(a^2) - \tfrac{1}{5}a^5 + \tfrac{1}{2} \approx 0.037$$

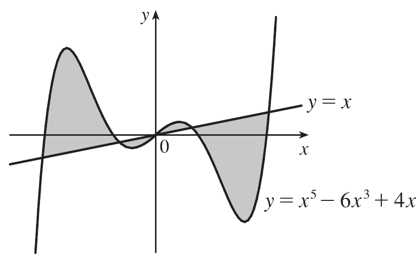
37.



From the graph, we see that the curves intersect at $x = a \approx -1.11$, $x = b \approx 1.25$, and $x = c \approx 2.86$, with $x^3 - 3x + 4 > 3x^2 - 2x$ on (a, b) and $3x^2 - 2x > x^3 - 3x + 4$ on (b, c). So the area of the region bounded by the curves is

$$A = \int_a^b \left[(x^3 - 3x + 4) - (3x^2 - 2x)\right] dx + \int_b^c \left[(3x^2 - 2x) - (x^3 - 3x + 4)\right] dx$$
$$= \int_a^b (x^3 - 3x^2 - x + 4)\, dx + \int_b^c (-x^3 + 3x^2 + x - 4)\, dx$$
$$= \left[\tfrac{1}{4}x^4 - x^3 - \tfrac{1}{2}x^2 + 4x\right]_a^b + \left[-\tfrac{1}{4}x^4 + x^3 + \tfrac{1}{2}x^2 - 4x\right]_b^c \approx 8.38$$

39. As the figure illustrates, the curves $y = x$ and $y = x^5 - 6x^3 + 4x$ enclose a four-part region symmetric about the origin (since $x^5 - 6x^3 + 4x$ and x are odd functions of x). The curves intersect at values of x where $x^5 - 6x^3 + 4x = x$; that is, where $x(x^4 - 6x^2 + 3) = 0$. That happens at $x = 0$ and where



$$x^2 = \frac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm \sqrt{6};$$ that is, at $x = -\sqrt{3 + \sqrt{6}}$, $-\sqrt{3 - \sqrt{6}}$, 0, $\sqrt{3 - \sqrt{6}}$, and $\sqrt{3 + \sqrt{6}}$.

The exact area is

$$2\int_0^{\sqrt{3+\sqrt{6}}} \left|(x^5 - 6x^3 + 4x) - x\right| dx = 2\int_0^{\sqrt{3+\sqrt{6}}} \left|x^5 - 6x^3 + 3x\right| dx$$
$$= 2\int_0^{\sqrt{3-\sqrt{6}}} (x^5 - 6x^3 + 3x)\, dx + 2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} (-x^5 + 6x^3 - 3x)\, dx$$
$$\overset{\text{CAS}}{=} 12\sqrt{6} - 9$$

41. 1 second $= \frac{1}{3600}$ hour, so 10 s $= \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule.

$\Delta t = \frac{1/360-0}{5} = \frac{1}{1800}$, so

$$\begin{aligned}
\text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\,dt - \int_0^{1/360} v_C\,dt = \int_0^{1/360} (v_K - v_C)\,dt \\
&\approx M_5 = \tfrac{1}{1800}\left[(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5)\right. \\
&\qquad\qquad \left. + (v_K - v_C)(7) + (v_K - v_C)(9)\right] \\
&= \tfrac{1}{1800}[(22-20) + (52-46) + (71-62) + (86-75) + (98-86)] \\
&= \tfrac{1}{1800}(2+6+9+11+12) = \tfrac{1}{1800}(40) = \tfrac{1}{45} \text{ mile, or } 117\tfrac{1}{3} \text{ feet}
\end{aligned}$$

43. Let $h(x)$ denote the height of the wing at x cm from the left end.

$$\begin{aligned}
A \approx M_5 &= \frac{200-0}{5}\,[h(20) + h(60) + h(100) + h(140) + h(180)] \\
&= 40(20.3 + 29.0 + 27.3 + 20.5 + 8.7) = 40(105.8) = 4232 \text{ cm}^2
\end{aligned}$$

45. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t)\,dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_{A} is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_{\text{B}}(t)\,dt = s_{\text{B}}(x)$.

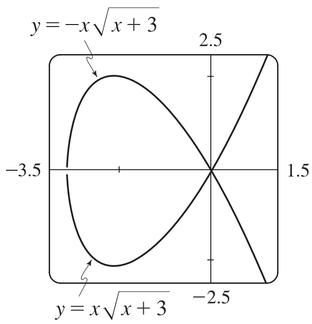
(a) After one minute, the area under curve A is greater than the area under curve B. So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_{\text{A}}(1) - s_{\text{B}}(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B, so car A is still ahead.

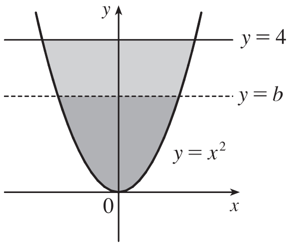
(d) From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

47.



To graph this function, we must first express it as a combination of explicit functions of y; namely, $y = \pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x = -3$ to $x = 0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y = -x\sqrt{x+3}$. So the area is $A = 2\int_{-3}^0 \left(-x\sqrt{x+3}\right) dx$. We substitute $u = x + 3$, so $du = dx$ and the limits change to 0 and 3, and we get

$$\begin{aligned}
A &= -2\int_0^3 [(u-3)\sqrt{u}]\,du = -2\int_0^3 (u^{3/2} - 3u^{1/2})\,du \\
&= -2\left[\tfrac{2}{5}u^{5/2} - 2u^{3/2}\right]_0^3 = -2\left[\tfrac{2}{5}(3^2\sqrt{3}) - 2(3\sqrt{3})\right] = \tfrac{24}{5}\sqrt{3}
\end{aligned}$$

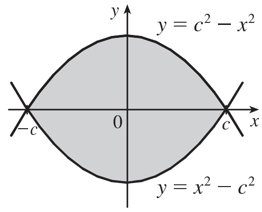
49. 

By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \Rightarrow x = \sqrt{y}$. We are looking for a number b such that

$$\int_0^b \sqrt{y}\,dy = \int_b^4 \sqrt{y}\,dy \Rightarrow \tfrac{2}{3}\left[y^{3/2}\right]_0^b = \tfrac{2}{3}\left[y^{3/2}\right]_b^4 \Rightarrow$$

$$b^{3/2} = 4^{3/2} - b^{3/2} \Rightarrow 2b^{3/2} = 8 \Rightarrow b^{3/2} = 4 \Rightarrow b = 4^{2/3} \approx 2.52.$$

51. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x = -c$ and $x = c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is

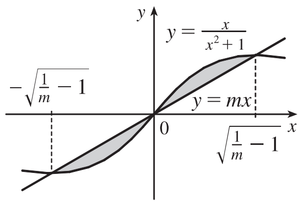


$$A = 4\int_0^c (c^2 - x^2)\,dx = 4\left[c^2x - \tfrac{1}{3}x^3\right]_0^c = 4\left(c^3 - \tfrac{1}{3}c^3\right) = 4\left(\tfrac{2}{3}c^3\right) = \tfrac{8}{3}c^3$$

So $A = 576 \Leftrightarrow \tfrac{8}{3}c^3 = 576 \Leftrightarrow c^3 = 216 \Leftrightarrow c = \sqrt[3]{216} = 6.$

Note that $c = -6$ is another solution, since the graphs are the same.

53. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \Rightarrow$



$x = x(mx^2 + m) \Rightarrow x(mx^2+m) - x = 0 \Rightarrow$

$x(mx^2 + m - 1) = 0 \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow$

$x = 0$ or $x^2 = \dfrac{1-m}{m} \Rightarrow x = 0$ or $x = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y'(0) = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

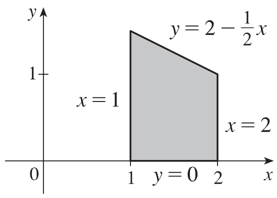
$$2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} = [\ln(1/m - 1 + 1) - m(1/m - 1)] - (\ln 1 - 0)$$

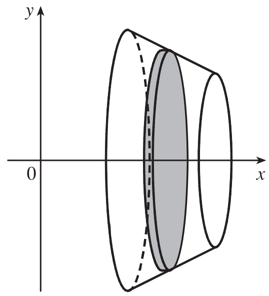
$$= \ln(1/m) - 1 + m = m - \ln m - 1$$

6.2 Volumes

1. A cross-section is a disk with radius $2 - \frac{1}{2}x$, so its area is $A(x) = \pi\left(2 - \frac{1}{2}x\right)^2$.

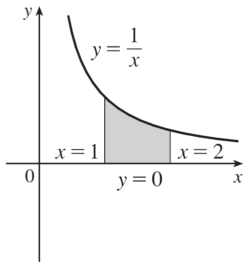
$$\begin{aligned} V &= \int_1^2 A(x)\,dx = \int_1^2 \pi\left(2 - \tfrac{1}{2}x\right)^2\,dx \\ &= \pi \int_1^2 \left(4 - 2x + \tfrac{1}{4}x^2\right) dx \\ &= \pi\left[4x - x^2 + \tfrac{1}{12}x^3\right]_1^2 \\ &= \pi\left[\left(8 - 4 + \tfrac{8}{12}\right) - \left(4 - 1 + \tfrac{1}{12}\right)\right] \\ &= \pi\left(1 + \tfrac{7}{12}\right) = \tfrac{19}{12}\pi \end{aligned}$$

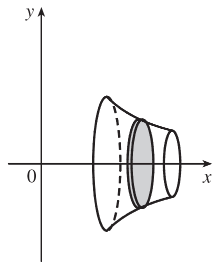




3. A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

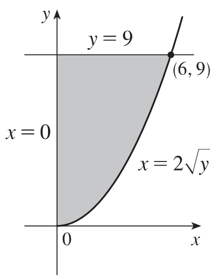
$$\begin{aligned} V &= \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2\,dx \\ &= \pi\int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 \\ &= \pi\left[-\tfrac{1}{2} - (-1)\right] = \tfrac{\pi}{2} \end{aligned}$$

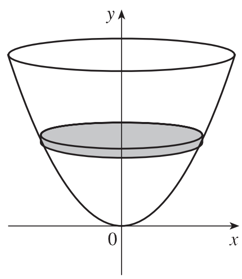




5. A cross-section is a disk with radius $2\sqrt{y}$, so its area is $A(y) = \pi\left(2\sqrt{y}\right)^2$.

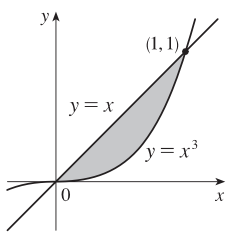
$$\begin{aligned} V &= \int_0^9 A(y)\,dy = \int_0^9 \pi\left(2\sqrt{y}\right)^2 dy = 4\pi\int_0^9 y\,dy \\ &= 4\pi\left[\tfrac{1}{2}y^2\right]_0^9 = 2\pi(81) = 162\pi \end{aligned}$$

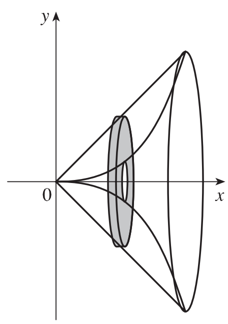




7. A cross-section is a washer (annulus) with inner radius x^3 and outer radius x, so its area is $A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$.

$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx \\ &= \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{1}{3} - \tfrac{1}{7}\right) = \tfrac{4}{21}\pi \end{aligned}$$



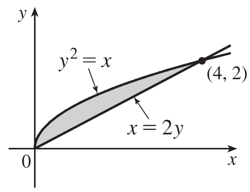


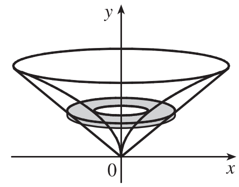
9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is

$A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4)$.

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 (4y^2 - y^4)\,dy$$

$$= \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64}{15}\pi$$





11. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

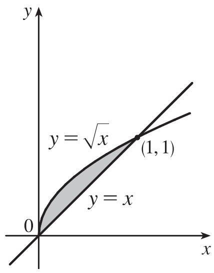
$$A(x) = \pi(1-x)^2 - \pi\left(1 - \sqrt{x}\right)^2$$

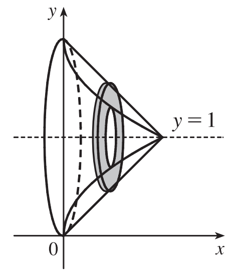
$$= \pi\left[(1 - 2x + x^2) - \left(1 - 2\sqrt{x} + x\right)\right]$$

$$= \pi\left(-3x + x^2 + 2\sqrt{x}\right).$$

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 \left(-3x + x^2 + 2\sqrt{x}\right) dx$$

$$= \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$





13. A cross-section is a washer with inner radius $(1 + \sec x) - 1 = \sec x$ and outer radius $3 - 1 = 2$, so its area is

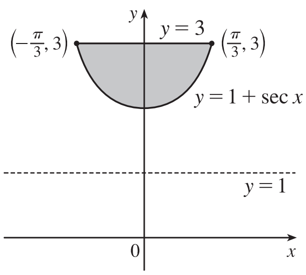
$$A(x) = \pi\left[2^2 - (\sec x)^2\right] = \pi(4 - \sec^2 x).$$

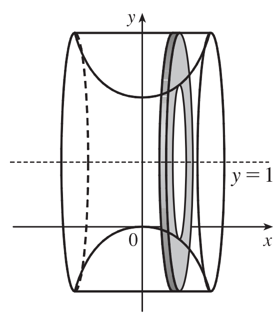
$$V = \int_{-\pi/3}^{\pi/3} A(x)\,dx = \int_{-\pi/3}^{\pi/3} \pi(4 - \sec^2 x)\,dx$$

$$= 2\pi \int_0^{\pi/3} (4 - \sec^2 x)\,dx \qquad \text{[by symmetry]}$$

$$= 2\pi\Big[4x - \tan x\Big]_0^{\pi/3} = 2\pi\left[\left(\tfrac{4\pi}{3} - \sqrt{3}\right) - 0\right]$$

$$= 2\pi\left(\tfrac{4\pi}{3} - \sqrt{3}\right)$$



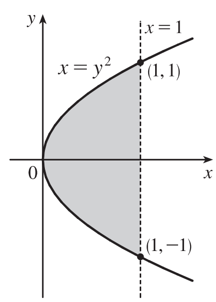


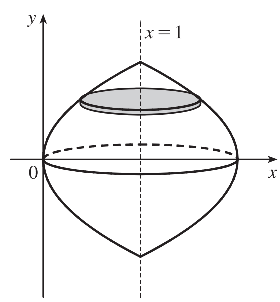
15. $$V = \int_{-1}^{1} \pi(1 - y^2)^2\,dy = 2\int_0^1 \pi(1 - y^2)^2\,dy$$

$$= 2\pi \int_0^1 (1 - 2y^2 + y^4)\,dy$$

$$= 2\pi\left[y - \tfrac{2}{3}y^3 + \tfrac{1}{5}y^5\right]_0^1$$

$$= 2\pi \cdot \tfrac{8}{15} = \tfrac{16}{15}\pi$$



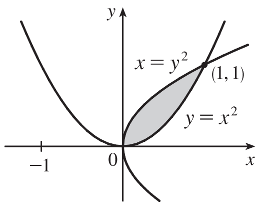


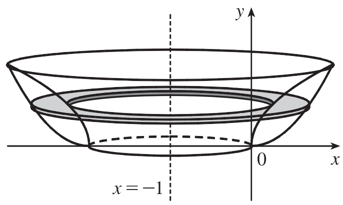
17. $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi \left\{ \left[\sqrt{y} - (-1) \right]^2 - \left[y^2 - (-1) \right]^2 \right\} dy = \pi \int_0^1 \left[\left(\sqrt{y} + 1 \right)^2 - (y^2 + 1)^2 \right] dy$$

$$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1 \right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2 \right) dy$$

$$= \pi \left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 \right]_0^1 = \pi \left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3} \right) = \tfrac{29}{30}\pi$$





19. $\mathcal{R}_1$ about OA (the line $y = 0$): $V = \displaystyle\int_0^1 A(x)\,dx = \int_0^1 \pi (x^3)^2\,dx = \pi \int_0^1 x^6\,dx = \pi \left[\frac{1}{7}x^7 \right]_0^1 = \frac{\pi}{7}$

21. $\mathcal{R}_1$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi \left(1 - \sqrt[3]{y} \right)^2 dy = \pi \int_0^1 (1 - 2y^{1/3} + y^{2/3})\,dy = \pi \left[y - \tfrac{3}{2}y^{4/3} + \tfrac{3}{5}y^{5/3} \right]_0^1$$
$$= \pi \left(1 - \tfrac{3}{2} + \tfrac{3}{5} \right) = \tfrac{\pi}{10}$$

23. $\mathcal{R}_2$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi(1)^2 - \pi \left(\sqrt{x} \right)^2 \right] dx = \pi \int_0^1 (1 - x)\,dx = \pi \left[x - \tfrac{1}{2}x^2 \right]_0^1 = \pi \left(1 - \tfrac{1}{2} \right) = \tfrac{\pi}{2}$$

25. $\mathcal{R}_2$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1)^2 - \pi(1 - y^2)^2 \right] dy = \pi \int_0^1 \left[1 - (1 - 2y^2 + y^4) \right] dy = \pi \int_0^1 (2y^2 - y^4)\,dy$$
$$= \pi \left[\tfrac{2}{3}y^3 - \tfrac{1}{5}y^5 \right]_0^1 = \pi \left(\tfrac{2}{3} - \tfrac{1}{5} \right) = \tfrac{7}{15}\pi$$

27. $\mathcal{R}_3$ about OA (the line $y = 0$):

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\pi \left(\sqrt{x} \right)^2 - \pi (x^3)^2 \right] dx = \pi \int_0^1 (x - x^6)\,dx = \pi \left[\tfrac{1}{2}x^2 - \tfrac{1}{7}x^7 \right]_0^1 = \pi \left(\tfrac{1}{2} - \tfrac{1}{7} \right) = \tfrac{5}{14}\pi.$$

Note: Let $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$. If we rotate $\mathcal{R}$ about any of the segments OA, OC, AB, or BC, we obtain a right circular cylinder of height 1 and radius 1. Its volume is $\pi r^2 h = \pi(1)^2 \cdot 1 = \pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π. Thus, $\frac{\pi}{7} + \frac{\pi}{2} + \frac{5\pi}{14} = \left(\frac{2 + 7 + 5}{14} \right) \pi = \pi$.

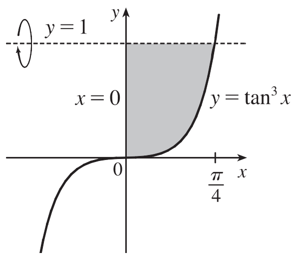
29. $\mathscr{R}_3$ about AB (the line $x = 1$):

$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1-y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy = \pi\int_0^1 \left[(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})\right] dy$$

$$= \pi\int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3})\,dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1 = \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13}{30}\pi$$

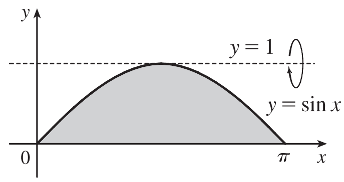
Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have $\frac{\pi}{10} + \frac{7\pi}{15} + \frac{13\pi}{30} = \left(\frac{3+14+13}{30}\right)\pi = \pi$.

31. $V = \pi\displaystyle\int_0^{\pi/4} (1 - \tan^3 x)^2\,dx$



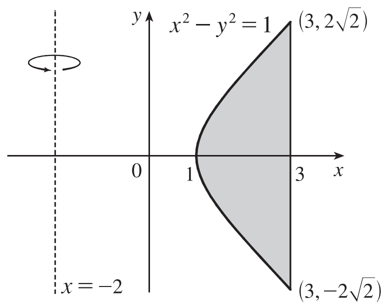
33. $V = \pi\displaystyle\int_0^{\pi} \left[(1-0)^2 - (1 - \sin x)^2\right] dx$

$= \pi\displaystyle\int_0^{\pi} \left[1^2 - (1 - \sin x)^2\right] dx$

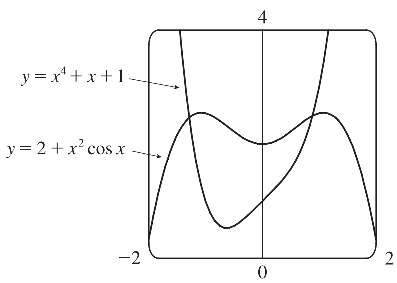


35. $V = \pi\displaystyle\int_{-\sqrt{8}}^{\sqrt{8}} \left\{[3 - (-2)]^2 - \left[\sqrt{y^2+1} - (-2)\right]^2\right\} dy$

$= \pi\displaystyle\int_{-2\sqrt{2}}^{2\sqrt{2}} \left[5^2 - \left(\sqrt{1+y^2} + 2\right)^2\right] dy$



37.

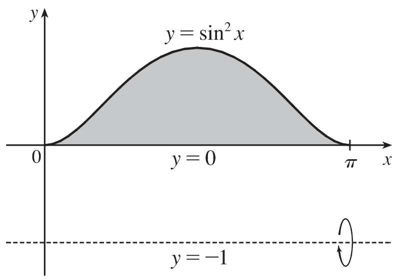


$y = 2 + x^2 \cos x$ and $y = x^4 + x + 1$ intersect at $x = a \approx -1.288$ and $x = b \approx 0.884$.

$V = \pi\displaystyle\int_a^b [(2 + x^2\cos x)^2 - (x^4 + x + 1)^2]dx \approx 23.780$

39. $V = \pi \int_0^{\pi} \left\{ \left[\sin^2 x - (-1)\right]^2 - [0-(-1)]^2 \right\} dx$

$\overset{\text{CAS}}{=} \frac{11}{8}\pi^2$



41. $\pi \int_0^{\pi/2} \cos^2 x \, dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ of the xy-plane about the x-axis.

43. $\pi \int_0^1 (y^4 - y^8)\, dy = \pi \int_0^1 \left[(y^2)^2 - (y^4)^2\right] dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le y \le 1, y^4 \le x \le y^2\right\}$ of the xy-plane about the y-axis.

45. There are 10 subintervals over the 15-cm length, so we'll use $n = 10/2 = 5$ for the Midpoint Rule.

$$V = \int_0^{15} A(x)\, dx \approx M_5 = \tfrac{15-0}{5}[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)]$$
$$= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3$$

47. (a) $V = \int_2^{10} \pi\, [f(x)]^2\, dx \approx \pi \frac{10-2}{4} \left\{ [f(3)]^2 + [f(5)]^2 + [f(7)]^2 + [f(9)]^2 \right\}$

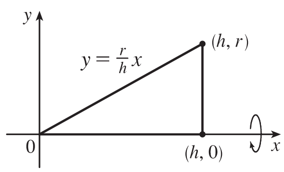
$\approx 2\pi \left[(1.5)^2 + (2.2)^2 + (3.8)^2 + (3.1)^2\right] \approx 196 \text{ units}^3$

(b) $V = \int_0^4 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dy$

$\approx \pi \frac{4-0}{4} \left\{ \left[(9.9)^2 - (2.2)^2\right] + \left[(9.7)^2 - (3.0)^2\right] + \left[(9.3)^2 - (5.6)^2\right] + \left[(8.7)^2 - (6.5)^2\right] \right\}$

$\approx 838 \text{ units}^3$

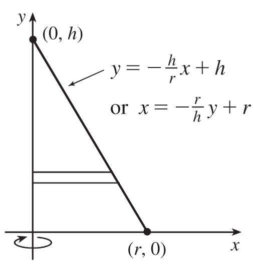
49. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h$$
$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\dfrac{r}{h}y + r$ about the y-axis.

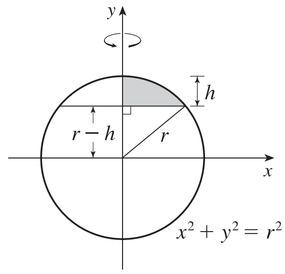
$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \overset{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$
$$= \pi \left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2 h - r^2 h + r^2 h\right) = \tfrac{1}{3}\pi r^2 h$$



* Or use substitution with $u = r - \dfrac{r}{h}y$ and $du = -\dfrac{r}{h}dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

51. $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$



$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\, dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left\{ \left[r^3 - \frac{r^3}{3} \right] - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\}$$

$$= \pi \left\{ \tfrac{2}{3} r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[3r^2 - \left(r^2 - 2rh + h^2\right)\right] \right\}$$

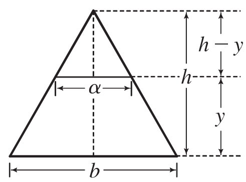
$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3 \right)$$

$$= \tfrac{1}{3}\pi \left(3rh^2 - h^3 \right) = \tfrac{1}{3}\pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3} \right)$$

53. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \dfrac{y}{h}\right)$.

Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \dfrac{y}{h}\right)$. So



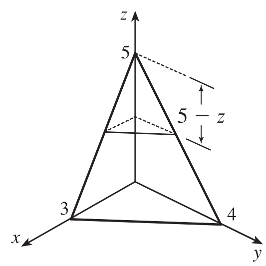
$$V = \int_0^h A(y)\, dy = \int_0^h \left[b\left(1 - \frac{y}{h}\right) \right] \left[2b\left(1 - \frac{y}{h}\right) \right] dy$$

$$= \int_0^h 2b^2 \left(1 - \frac{y}{h}\right)^2 dy = 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2} \right) dy$$

$$= 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = 2b^2 \left[h - h + \tfrac{1}{3}h \right]$$

$= \frac{2}{3} b^2 h \quad [\, = \frac{1}{3} Bh$ where B is the area of the base, as with any pyramid.]

55. A cross-section at height z is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of $(5 - z)/5$. Thus, the triangle at height z has area



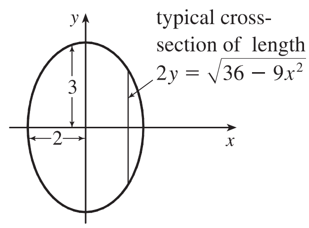
$$A(z) = \frac{1}{2} \cdot 3\left(\frac{5-z}{5}\right) \cdot 4\left(\frac{5-z}{5}\right) = 6\left(1 - \frac{z}{5}\right)^2, \text{ so}$$

$$V = \int_0^5 A(z)\, dz = 6 \int_0^5 \left(1 - \frac{z}{5}\right)^2 dz = 6 \int_1^0 u^2(-5\, du) \qquad \left[\begin{array}{l} u = 1 - z/5, \\ du = -\frac{1}{5}\, dz \end{array} \right]$$

$$= -30\left[\tfrac{1}{3}u^3\right]_1^0 = -30\left(-\tfrac{1}{3}\right) = 10 \text{ cm}^3$$

57. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse,

then $l^2 + l^2 = (2y)^2 \quad \Rightarrow \quad 2l^2 = 4y^2 \quad \Rightarrow \quad l^2 = 2y^2$.

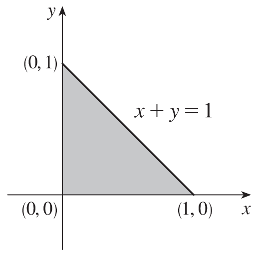


$$V = \textstyle\int_{-2}^{2} A(x)\, dx = 2\int_0^2 A(x)\, dx = 2\int_0^2 \frac{1}{2}(l)(l)\, dx = 2\int_0^2 y^2\, dx$$

$$= 2\textstyle\int_0^2 \frac{1}{4}(36 - 9x^2)\, dx = \frac{9}{2}\int_0^2 (4 - x^2)\, dx$$

$$= \tfrac{9}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2 = \tfrac{9}{2}\left(8 - \tfrac{8}{3}\right) = 24$$

59. The cross-section of the base corresponding to the coordinate x has length $y = 1 - x$. The corresponding square with side s has area $A(x) = s^2 = (1-x)^2 = 1 - 2x + x^2$. Therefore,



$$V = \int_0^1 A(x)\,dx = \int_0^1 (1 - 2x + x^2)\,dx$$

$$= \left[x - x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(1 - 1 + \tfrac{1}{3}\right) - 0 = \tfrac{1}{3}$$

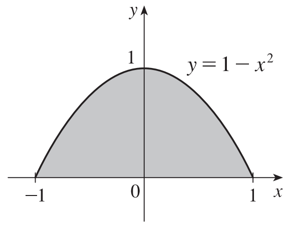
Or: $\int_0^1 (1-x)^2\,dx = \int_1^0 u^2(-du) \quad [u = 1 - x] = \left[\tfrac{1}{3}u^3\right]_0^1 = \tfrac{1}{3}$

61. The cross-section of the base b corresponding to the coordinate x has length $1 - x^2$. The height h also has length $1 - x^2$, so the corresponding isosceles triangle has area $A(x) = \frac{1}{2}bh = \frac{1}{2}(1-x^2)^2$. Therefore,

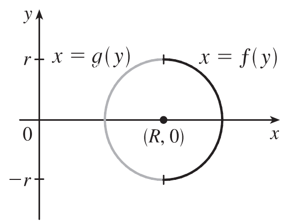
$$V = \int_{-1}^1 \tfrac{1}{2}(1-x^2)^2\,dx$$

$$= 2 \cdot \tfrac{1}{2}\int_0^1 (1 - 2x^2 + x^4)\,dx \qquad \text{[by symmetry]}$$

$$= \left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 = \left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - 0 = \tfrac{8}{15}$$



63. (a) The torus is obtained by rotating the circle $(x-R)^2 + y^2 = r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x = R + \sqrt{r^2 - y^2} = f(y)$ and the left half by $x = R - \sqrt{r^2 - y^2} = g(y)$. So



$$V = \pi \textstyle\int_{-r}^r \left\{[f(y)]^2 - [g(y)]^2\right\} dy$$

$$= 2\pi \textstyle\int_0^r \left[\left(R^2 + 2R\sqrt{r^2-y^2} + r^2 - y^2\right) - \left(R^2 - 2R\sqrt{r^2-y^2} + r^2 - y^2\right)\right] dy$$

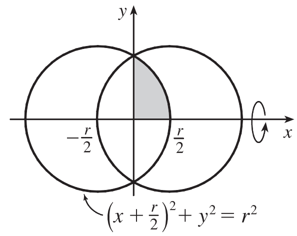
$$= 2\pi \textstyle\int_0^r 4R\sqrt{r^2-y^2}\,dy = 8\pi R \int_0^r \sqrt{r^2-y^2}\,dy$$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so $8\pi R \int_0^r \sqrt{r^2-y^2}\,dy = 8\pi R \cdot \frac{1}{4}\pi r^2 = 2\pi^2 r^2 R$.

65. (a) Volume$(S_1) = \int_0^h A(z)\,dz =$ Volume(S_2) since the cross-sectional area $A(z)$ at height z is the same for both solids.

(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h, that is, $\pi r^2 h$.

67. The volume is obtained by rotating the area common to two circles of radius r, as shown. The volume of the right half is



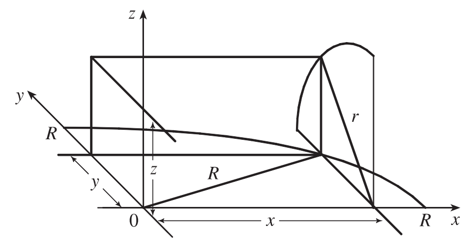
$$V_{\text{right}} = \pi \int_0^{r/2} y^2\,dx = \pi \int_0^{r/2} \left[r^2 - \left(\tfrac{1}{2}r + x\right)^2\right] dx$$

$$= \pi\left[r^2x - \tfrac{1}{3}\left(\tfrac{1}{2}r + x\right)^3\right]_0^{r/2} = \pi\left[\left(\tfrac{1}{2}r^3 - \tfrac{1}{3}r^3\right) - \left(0 - \tfrac{1}{24}r^3\right)\right] = \tfrac{5}{24}\pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

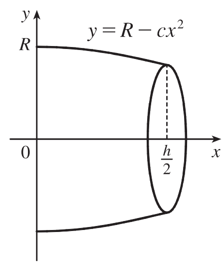
Another solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 51 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi h^2(3r - h) = \frac{2}{3}\pi\left(\frac{1}{2}r\right)^2\left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.

69. Take the x-axis to be the axis of the cylindrical hole of radius r. A quarter of the cross-section through y, perpendicular to the y-axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are $x = \sqrt{R^2 - y^2}$ and $z = \sqrt{r^2 - y^2}$, so $\frac{1}{4}A(y) = xz = \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}$, and



$$V = \int_{-r}^{r} A(y)\,dy = \int_{-r}^{r} 4\sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy = 8\int_0^r \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\,dy$$

71. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c\left(\frac{1}{2}h\right)^2 = R - d = r$.



(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi y^2\,dx = 2\pi \int_0^{h/2} \left(R - cx^2\right)^2 dx = 2\pi\left[R^2x - \tfrac{2}{3}Rcx^3 + \tfrac{1}{5}c^2x^5\right]_0^{h/2}$$

$$= 2\pi\left(\tfrac{1}{2}R^2h - \tfrac{1}{12}Rch^3 + \tfrac{1}{160}c^2h^5\right)$$

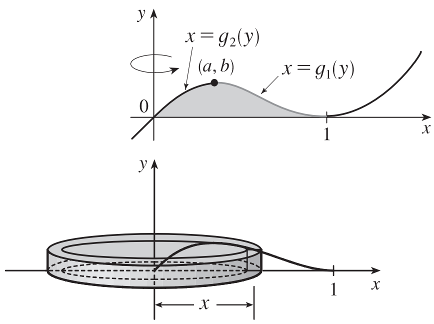
Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h\left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4\right)\right]$.

But $R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = \left(R - \frac{1}{4}ch^2\right)^2 - \frac{1}{40}c^2h^4 = (R - d)^2 - \frac{2}{5}\left(\frac{1}{4}ch^2\right)^2 = r^2 - \frac{2}{5}d^2$.

Substituting this back into V, we see that $V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$, as required.

6.3 Volumes by Cylindrical Shells

1.



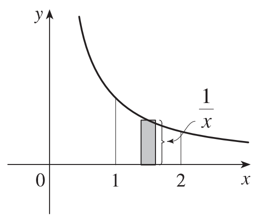
If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

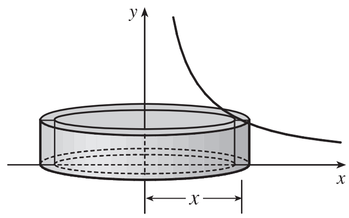
Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x\left[x(x-1)^2\right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\, dx = 2\pi \int_1^2 1\, dx$

$= 2\pi \left[\, x\,\right]_1^2 = 2\pi(2-1) = 2\pi$

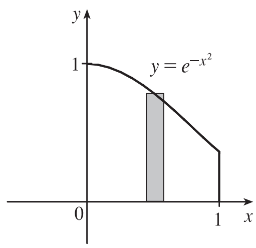


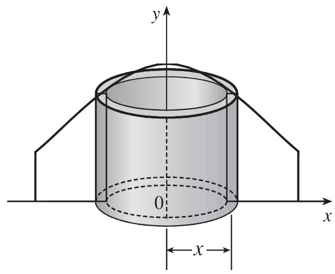


5. $V = \int_0^1 2\pi x e^{-x^2}\, dx$. Let $u = x^2$.

Thus, $du = 2x\, dx$, so

$V = \pi \int_0^1 e^{-u}\, du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

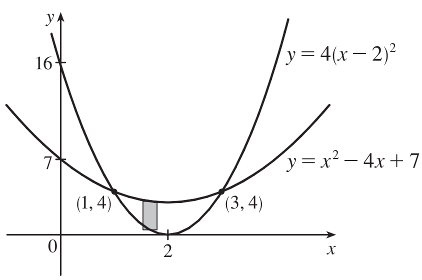


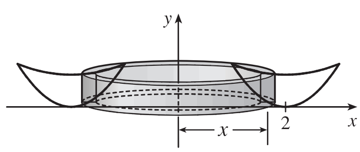


7. The curves intersect when $4(x-2)^2 = x^2 - 4x + 7 \quad\Leftrightarrow\quad 4x^2 - 16x + 16 = x^2 - 4x + 7 \quad\Leftrightarrow$

$3x^2 - 12x + 9 = 0 \quad\Leftrightarrow\quad 3(x^2 - 4x + 3) = 0 \quad\Leftrightarrow\quad 3(x-1)(x-3) = 0$, so $x = 1$ or 3.

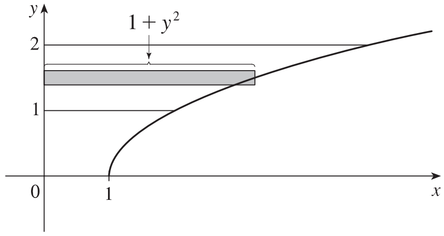
$$\begin{aligned} V &= 2\pi \int_1^3 \left\{x\left[\left(x^2 - 4x + 7\right) - 4(x-2)^2\right]\right\} dx = 2\pi \int_1^3 \left[x(x^2 - 4x + 7 - 4x^2 + 16x - 16)\right] dx \\ &= 2\pi \int_1^3 \left[x(-3x^2 + 12x - 9)\right] dx = 2\pi(-3)\int_1^3 (x^3 - 4x^2 + 3x)\, dx = -6\pi\left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_1^3 \\ &= -6\pi\left[\left(\tfrac{81}{4} - 36 + \tfrac{27}{2}\right) - \left(\tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}\right)\right] = -6\pi\left(20 - 36 + 12 + \tfrac{4}{3}\right) = -6\pi\left(-\tfrac{8}{3}\right) = 16\pi \end{aligned}$$

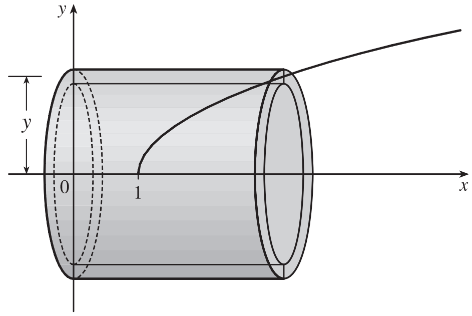




9. $V = \int_1^2 2\pi y(1+y^2)\,dy = 2\pi \int_1^2 (y+y^3)\,dy = 2\pi\left[\frac{1}{2}y^2 + \frac{1}{4}y^4\right]_1^2$

$= 2\pi\left[(2+4) - \left(\frac{1}{2}+\frac{1}{4}\right)\right] = 2\pi\left(\frac{21}{4}\right) = \frac{21}{2}\pi$

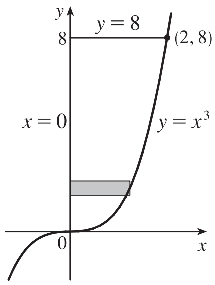


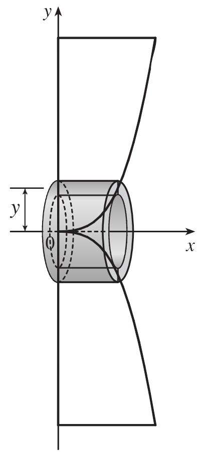


11. $V = 2\pi \displaystyle\int_0^8 \left[y(\sqrt[3]{y} - 0)\right] dy$

$= 2\pi \displaystyle\int_0^8 y^{4/3}\,dy = 2\pi\left[\frac{3}{7}y^{7/3}\right]_0^8$

$= \dfrac{6\pi}{7}(8^{7/3}) = \dfrac{6\pi}{7}(2^7) = \dfrac{768}{7}\pi$





13. The height of the shell is $2 - \left[1 + (y-2)^2\right] = 1 - (y-2)^2 = 1 - \left(y^2 - 4y + 4\right) = -y^2 + 4y - 3$.

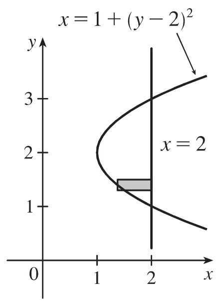
$V = 2\pi \int_1^3 y(-y^2+4y-3)\,dy$

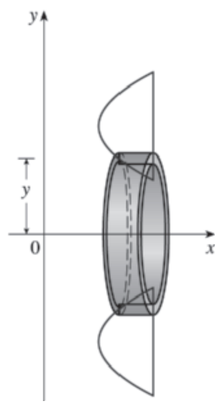
$= 2\pi \int_1^3 (-y^3 + 4y^2 - 3y)\,dy$

$= 2\pi\left[-\frac{1}{4}y^4 + \frac{4}{3}y^3 - \frac{3}{2}y^2\right]_1^3$

$= 2\pi\left[\left(-\frac{81}{4} + 36 - \frac{27}{2}\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right]$

$= 2\pi\left(\frac{8}{3}\right) = \frac{16}{3}\pi$





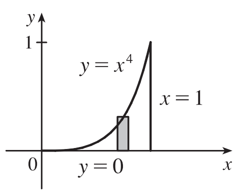
15. The shell has radius $2-x$, circumference $2\pi(2-x)$, and height x^4.

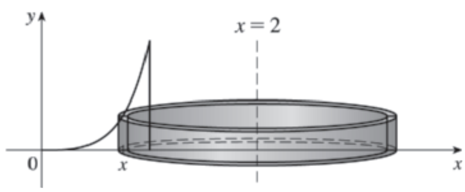
$V = \int_0^1 2\pi(2-x)x^4\,dx$

$= 2\pi \int_0^1 (2x^4 - x^5)\,dx$

$= 2\pi\left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right]_0^1$

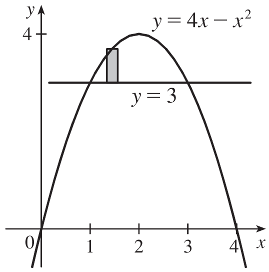
$= 2\pi\left[\left(\frac{2}{5} - \frac{1}{6}\right) - 0\right] = 2\pi\left(\frac{7}{30}\right) = \frac{7}{15}\pi$

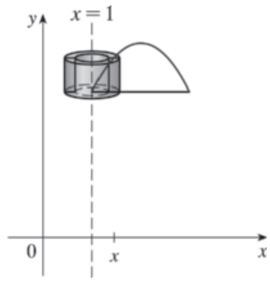




17. The shell has radius $x-1$, circumference $2\pi(x-1)$, and height $(4x-x^2)-3=-x^2+4x-3$.

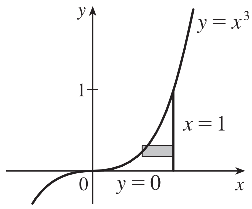
$$\begin{aligned}
V &= \int_1^3 2\pi(x-1)(-x^2+4x-3)\,dx \\
&= 2\pi\int_1^3(-x^3+5x^2-7x+3)\,dx \\
&= 2\pi\left[-\tfrac{1}{4}x^4+\tfrac{5}{3}x^3-\tfrac{7}{2}x^2+3x\right]_1^3 \\
&= 2\pi\left[\left(-\tfrac{81}{4}+45-\tfrac{63}{2}+9\right)-\left(-\tfrac{1}{4}+\tfrac{5}{3}-\tfrac{7}{2}+3\right)\right] \\
&= 2\pi\left(\tfrac{4}{3}\right)=\tfrac{8}{3}\pi
\end{aligned}$$

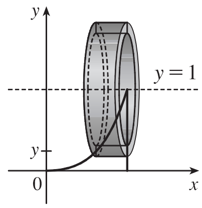




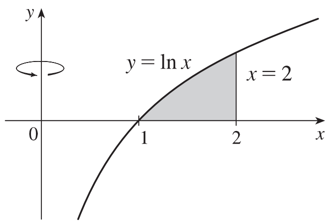
19. The shell has radius $1-y$, circumference $2\pi(1-y)$, and height $1-\sqrt[3]{y}$ $\left[y=x^3 \quad\Leftrightarrow\quad x=\sqrt[3]{y}\right]$.

$$\begin{aligned}
V &= \int_0^1 2\pi(1-y)(1-y^{1/3})\,dy \\
&= 2\pi\int_0^1(1-y-y^{1/3}+y^{4/3})\,dy \\
&= 2\pi\left[y-\tfrac{1}{2}y^2-\tfrac{3}{4}y^{4/3}+\tfrac{3}{7}y^{7/3}\right]_0^1 \\
&= 2\pi\left[\left(1-\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{3}{7}\right)-0\right] \\
&= 2\pi\left(\tfrac{5}{28}\right)=\tfrac{5}{14}\pi
\end{aligned}$$

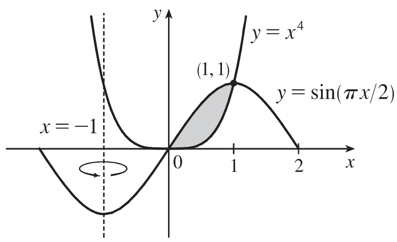




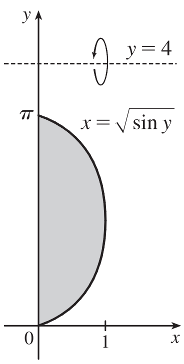
21. $V=\int_1^2 2\pi x\ln x\,dx$



23. $V=\int_0^1 2\pi[x-(-1)]\left(\sin\frac{\pi}{2}x-x^4\right)dx$



25. $V=\int_0^\pi 2\pi(4-y)\sqrt{\sin y}\,dy$

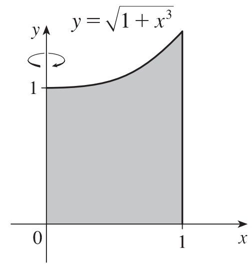


27. $V = \int_0^1 2\pi x\sqrt{1+x^3}\,dx$. Let $f(x) = x\sqrt{1+x^3}$.

Then the Midpoint Rule with $n = 5$ gives

$$\int_0^1 f(x)\,dx \approx \tfrac{1-0}{5}\left[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)\right]$$
$$\approx 0.2(2.9290)$$

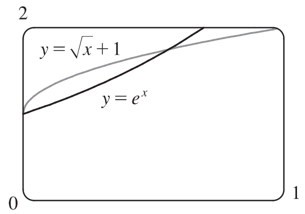
Multiplying by 2π gives $V \approx 3.68$.



29. $\int_0^3 2\pi x^5\,dx = 2\pi\int_0^3 x(x^4)\,dx$. The solid is obtained by rotating the region $0 \le y \le x^4$, $0 \le x \le 3$ about the y-axis using cylindrical shells.

31. $\int_0^1 2\pi(3-y)(1-y^2)\,dy$. The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$, $x = 0$, and $y = 0$ or (ii) $x = y^2$, $x = 1$, and $y = 0$ about the line $y = 3$ using cylindrical shells.

33.

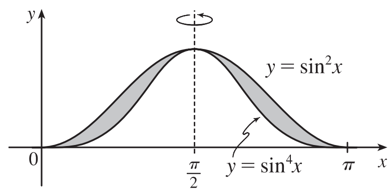


From the graph, the curves intersect at $x = 0$ and $x = a \approx 0.56$, with $\sqrt{x} + 1 > e^x$ on the interval $(0, a)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V = 2\pi\int_0^a x\left[\left(\sqrt{x}+1\right) - e^x\right]dx \approx 0.13.$$

35. $V = 2\pi \displaystyle\int_0^{\pi/2} \left[\left(\tfrac{\pi}{2} - x\right)\left(\sin^2 x - \sin^4 x\right)\right] dx$

$\overset{\text{CAS}}{=} \frac{1}{32}\pi^3$



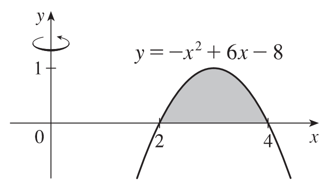
37. Use shells:

$V = \int_2^4 2\pi x(-x^2 + 6x - 8)\,dx = 2\pi\int_2^4(-x^3 + 6x^2 - 8x)\,dx$

$= 2\pi\left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4$

$= 2\pi[(-64 + 128 - 64) - (-4 + 16 - 16)]$

$= 2\pi(4) = 8\pi$



39. Use shells:

$V = \int_1^4 2\pi[x - (-1)][5 - (x + 4/x)]\,dx$

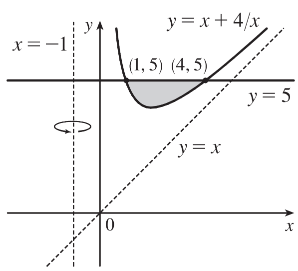
$= 2\pi\int_1^4 (x+1)(5 - x - 4/x)\,dx$

$= 2\pi\int_1^4 (5x - x^2 - 4 + 5 - x - 4/x)\,dx$

$= 2\pi\int_1^4 (-x^2 + 4x + 1 - 4/x)\,dx = 2\pi\left[-\frac{1}{3}x^3 + 2x^2 + x - 4\ln x\right]_1^4$

$= 2\pi\left[\left(-\frac{64}{3} + 32 + 4 - 4\ln 4\right) - \left(-\frac{1}{3} + 2 + 1 - 0\right)\right]$

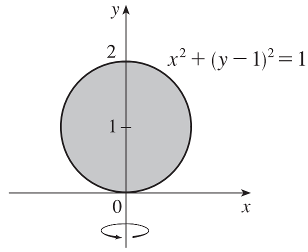
$= 2\pi(12 - 4\ln 4) = 8\pi(3 - \ln 4)$



41. Use disks: $x^2 + (y-1)^2 = 1 \quad \Leftrightarrow \quad x = \pm\sqrt{1-(y-1)^2}$

$$V = \pi \int_0^2 \left[\sqrt{1-(y-1)^2}\right]^2 dy = \pi \int_0^2 (2y - y^2)\, dy$$

$$= \pi\left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \pi\left(4 - \tfrac{8}{3}\right) = \tfrac{4}{3}\pi$$



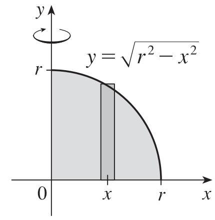
43. Use shells:

$$V = 2\int_0^r 2\pi x\sqrt{r^2 - x^2}\, dx$$

$$= -2\pi \int_0^r (r^2 - x^2)^{1/2}(-2x)\, dx$$

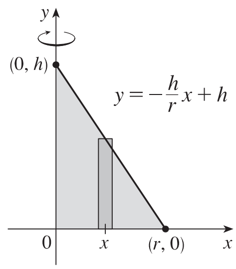
$$= \left[-2\pi \cdot \tfrac{2}{3}(r^2 - x^2)^{3/2}\right]_0^r$$

$$= -\tfrac{4}{3}\pi(0 - r^3) = \tfrac{4}{3}\pi r^3$$



45. $V = 2\pi \displaystyle\int_0^r x\left(-\frac{h}{r}x + h\right) dx = 2\pi h \int_0^r \left(-\frac{x^2}{r} + x\right) dx$

$$= 2\pi h\left[-\frac{x^3}{3r} + \frac{x^2}{2}\right]_0^r = 2\pi h\,\frac{r^2}{6} = \frac{\pi r^2 h}{3}$$



6.4 Work

1. $W = Fd = mgd = (40)(9.8)(1.5) = 588$ J

3. $W = \displaystyle\int_a^b f(x)\, dx = \int_0^9 \frac{10}{(1+x)^2}\, dx = 10\int_1^{10} \frac{1}{u^2}\, du \quad [u = 1+x,\ du = dx] \quad = 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\tfrac{1}{10} + 1\right) = 9$ ft-lb

5. The force function is given by $F(x)$ (in newtons) and the work (in joules) is the area under the curve, given by

$\int_0^8 F(x)\, dx = \int_0^4 F(x)\, dx + \int_4^8 F(x)\, dx = \tfrac{1}{2}(4)(30) + (4)(30) = 180$ J.

7. $10 = f(x) = kx = \tfrac{1}{3}k \quad [4 \text{ inches} = \tfrac{1}{3} \text{ foot}]$, so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches $= \tfrac{1}{2}$ foot, so

$W = \int_0^{1/2} 30x\, dx = \left[15x^2\right]_0^{1/2} = \tfrac{15}{4}$ ft-lb.

9. (a) If $\int_0^{0.12} kx\, dx = 2$ J, then $2 = \left[\tfrac{1}{2}kx^2\right]_0^{0.12} = \tfrac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$ N/m.

Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$\int_{0.05}^{0.10} \frac{2500}{9}x\, dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

(b) $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

11. The distance from 20 cm to 30 cm is 0.1 m, so with $f(x) = kx$, we get $W_1 = \int_0^{0.1} kx\,dx = k\left[\frac{1}{2}x^2\right]_0^{0.1} = \frac{1}{200}k$.

Now $W_2 = \int_{0.1}^{0.2} kx\,dx = k\left[\frac{1}{2}x^2\right]_{0.1}^{0.2} = k\left(\frac{4}{200} - \frac{1}{200}\right) = \frac{3}{200}k$. Thus, $W_2 = 3W_1$.

In Exercises 13–20, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

13. (a) The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50}\tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is $W_1 = \int_0^{25}\frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^{25} = \frac{625}{4}$ ft-lb. The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \int_{25}^{50}\frac{1}{2}\cdot 25\,dx = \frac{25}{2}\left[x\right]_{25}^{50} = \frac{625}{2}$ ft-lb. The total work done in pulling half the rope to the top of the building is $W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4}\cdot 625 = \frac{1875}{4}$ ft-lb.

15. The work needed to lift the cable is $\lim\limits_{n\to\infty}\sum_{i=1}^{n} 2x_i^*\,\Delta x = \int_0^{500} 2x\,dx = \left[x^2\right]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb · 500 ft = 400,000 ft-lb. Thus, the total work required is 250,000 + 400,000 = 650,000 ft-lb.

17. At a height of x meters $(0 \le x \le 12)$, the mass of the rope is $(0.8 \text{ kg/m})(12 - x \text{ m}) = (9.6 - 0.8x)$ kg and the mass of the water is $\left(\frac{36}{12}\text{ kg/m}\right)(12 - x\text{ m}) = (36 - 3x)$ kg. The mass of the bucket is 10 kg, so the total mass is $(9.6 - 0.8x) + (36 - 3x) + 10 = (55.6 - 3.8x)$ kg, and hence, the total force is $9.8(55.6 - 3.8x)$ N. The work needed to lift the bucket Δx m through the ith subinterval of $[0, 12]$ is $9.8(55.6 - 3.8x_i^*)\Delta x$, so the total work is

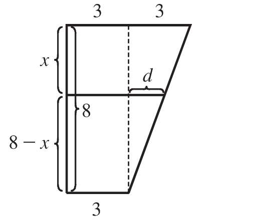
$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9.8(55.6 - 3.8x_i^*)\,\Delta x = \int_0^{12}(9.8)(55.6 - 3.8x)\,dx = 9.8\Big[55.6x - 1.9x^2\Big]_0^{12} = 9.8(393.6) \approx 3857 \text{ J}$$

19. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \le x_i^* \le \frac{1}{2}$) has volume $(2\times 1\times\Delta x)$ m^3, a mass of $2000\,\Delta x$ kg, weighs about $(9.8)(2000\,\Delta x) = 19{,}600\,\Delta x$ N, and thus requires about $19{,}600x_i^*\,\Delta x$ J of work for its removal. So $W = \lim\limits_{n\to\infty}\sum_{i=1}^{n} 19{,}600x_i^*\,\Delta x = \int_0^{1/2} 19{,}600x\,dx = \left[9800x^2\right]_0^{1/2} = 2450$ J.

21. A rectangular "slice" of water Δx m thick and lying x m above the bottom has width x m and volume $8x\,\Delta x$ m^3. It weighs about $(9.8\times 1000)(8x\,\Delta x)$ N, and must be lifted $(5 - x)$ m by the pump, so the work needed is about $(9.8\times 10^3)(5 - x)(8x\,\Delta x)$ J. The total work required is

$$\begin{aligned} W &\approx \int_0^3(9.8\times 10^3)(5 - x)8x\,dx = (9.8\times 10^3)\int_0^3(40x - 8x^2)\,dx = (9.8\times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3 \\ &= (9.8\times 10^3)(180 - 72) = (9.8\times 10^3)(108) = 1058.4\times 10^3 \approx 1.06\times 10^6 \text{ J} \end{aligned}$$

23. Let x measure depth (in feet) below the spout at the top of the tank. A horizontal disk-shaped "slice" of water Δx ft thick and lying at coordinate x has radius $\frac{3}{8}(16-x)$ ft $(\star)$ and volume $\pi r^2\Delta x = \pi\cdot\frac{9}{64}(16-x)^2\,\Delta x$ ft^3. It weighs about $(62.5)\frac{9\pi}{64}(16-x)^2\,\Delta x$ lb and must be lifted x ft by the pump, so the work needed to pump it out is about $(62.5)x\,\frac{9\pi}{64}(16-x)^2\,\Delta x$ ft-lb. The total work required is



$(\star)$ From similar triangles, $\dfrac{d}{8-x} = \dfrac{3}{8}$.

So $r = 3 + d = 3 + \frac{3}{8}(8-x)$

$= \dfrac{3(8)}{8} + \dfrac{3}{8}(8-x)$

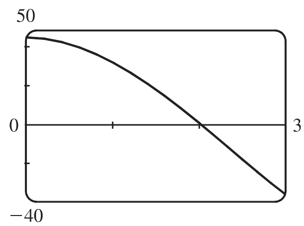
$= \frac{3}{8}(16-x)$

$$W \approx \int_0^8 (62.5)x\,\tfrac{9\pi}{64}(16-x)^2\,dx = (62.5)\tfrac{9\pi}{64}\int_0^8 x(256-32x+x^2)\,dx$$

$$= (62.5)\tfrac{9\pi}{64}\int_0^8 (256x-32x^2+x^3)\,dx = (62.5)\tfrac{9\pi}{64}\left[128x^2-\tfrac{32}{3}x^3+\tfrac{1}{4}x^4\right]_0^8$$

$$= (62.5)\frac{9\pi}{64}\left(\frac{11,264}{3}\right) = 33{,}000\pi \approx 1.04\times 10^5 \text{ ft-lb}$$

25. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 21, except that the work is fixed, and we are trying to find the lower limit of integration:



$4.7\times 10^5 \approx \int_h^3 (9.8\times 10^3)(5-x)8x\,dx = \left(9.8\times 10^3\right)\left[20x^2-\frac{8}{3}x^3\right]_h^3 \quad\Leftrightarrow$

$\frac{4.7}{9.8}\times 10^2 \approx 48 = \left(20\cdot 3^2-\frac{8}{3}\cdot 3^3\right)-\left(20h^2-\frac{8}{3}h^3\right) \quad\Leftrightarrow$

$2h^3-15h^2+45=0$. To find the solution of this equation, we plot $2h^3-15h^2+45$ between $h=0$ and $h=3$. We see that the equation is satisfied for $h\approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.

27. $V=\pi r^2 x$, so V is a function of x and P can also be regarded as a function of x. If $V_1=\pi r^2 x_1$ and $V_2=\pi r^2 x_2$, then

$$W=\int_{x_1}^{x_2} F(x)\,dx=\int_{x_1}^{x_2}\pi r^2 P(V(x))\,dx=\int_{x_1}^{x_2} P(V(x))\,dV(x) \qquad [\text{Let } V(x)=\pi r^2 x,\text{ so } dV(x)=\pi r^2\,dx.]$$

$$=\int_{V_1}^{V_2} P(V)\,dV \quad \text{by the Substitution Rule.}$$

29. $W=\displaystyle\int_a^b F(r)\,dr=\int_a^b G\frac{m_1m_2}{r^2}\,dr=Gm_1m_2\left[\frac{-1}{r}\right]_a^b=Gm_1m_2\left(\frac{1}{a}-\frac{1}{b}\right)$

6.5 Average Value of a Function

1. $f_{\text{ave}}=\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{4-0}\int_0^4(4x-x^2)\,dx=\frac{1}{4}\left[2x^2-\frac{1}{3}x^3\right]_0^4=\frac{1}{4}\left[\left(32-\frac{64}{3}\right)-0\right]=\frac{1}{4}\left(\frac{32}{3}\right)=\frac{8}{3}$

3. $g_{\text{ave}}=\frac{1}{b-a}\int_a^b g(x)\,dx=\frac{1}{8-1}\int_1^8\sqrt[3]{x}\,dx=\frac{1}{7}\left[\frac{3}{4}x^{4/3}\right]_1^8=\frac{3}{28}(16-1)=\frac{45}{28}$

5. $f_{\text{ave}}=\frac{1}{5-0}\int_0^5 te^{-t^2}\,dt=\frac{1}{5}\int_0^{-25}e^u\left(-\frac{1}{2}\,du\right) \qquad \left[u=-t^2,\ du=-2t\,dt,\ t\,dt=-\frac{1}{2}\,du\right]$

$\quad=-\frac{1}{10}\left[e^u\right]_0^{-25}=-\frac{1}{10}\left(e^{-25}-1\right)=\frac{1}{10}\left(1-e^{-25}\right)$

7. $h_{\text{ave}} = \frac{1}{\pi - 0}\int_0^\pi \cos^4 x \sin x\,dx = \frac{1}{\pi}\int_1^{-1} u^4(-du) \quad [u = \cos x, du = -\sin x\,dx]$

$= \frac{1}{\pi}\int_{-1}^{1} u^4\,du = \frac{1}{\pi}\cdot 2\int_0^1 u^4\,du \quad$ [by Theorem 5.5.7] $= \frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5\pi}$

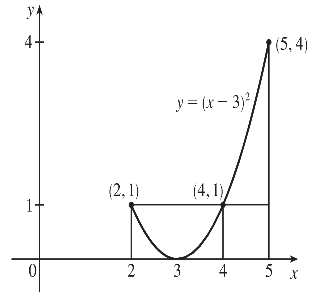
9. (a) $f_{\text{ave}} = \dfrac{1}{5-2}\displaystyle\int_2^5 (x-3)^2\,dx = \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5$

$= \frac{1}{9}\left[2^3 - (-1)^3\right] = \frac{1}{9}(8+1) = 1$

(b) $f(c) = f_{\text{ave}} \quad\Leftrightarrow\quad (c-3)^2 = 1 \quad\Leftrightarrow$

$c - 3 = \pm 1 \quad\Leftrightarrow\quad c = 2$ or 4

(c)



11. (a) $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi (2\sin x - \sin 2x)\,dx$

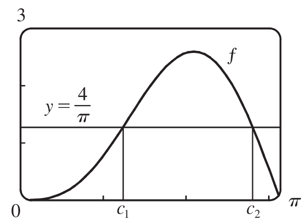
$= \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^\pi$

$= \frac{1}{\pi}\left[\left(2 + \frac{1}{2}\right) - \left(-2 + \frac{1}{2}\right)\right] = \frac{4}{\pi}$

(b) $f(c) = f_{\text{ave}} \quad\Leftrightarrow\quad 2\sin c - \sin 2c = \frac{4}{\pi} \quad\Leftrightarrow$

$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



13. f is continuous on $[1, 3]$, so by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that $\int_1^3 f(x)\,dx = f(c)(3-1) \quad\Rightarrow\quad 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

15. $f_{\text{ave}} = \dfrac{1}{50-20}\displaystyle\int_{20}^{50} f(x)\,dx \approx \frac{1}{30}M_3 = \frac{1}{30}\cdot\frac{50-20}{3}[f(25) + f(35) + f(45)] = \frac{1}{3}(38 + 29 + 48) = \frac{115}{3} = 38\frac{1}{3}$

17. Let $t = 0$ and $t = 12$ correspond to 9 AM and 9 PM, respectively.

$$T_{\text{ave}} = \frac{1}{12-0}\int_0^{12}\left[50 + 14\sin\tfrac{1}{12}\pi t\right]dt = \frac{1}{12}\left[50t - 14\cdot\tfrac{12}{\pi}\cos\tfrac{1}{12}\pi t\right]_0^{12}$$
$$= \frac{1}{12}\left[50\cdot 12 + 14\cdot\tfrac{12}{\pi} + 14\cdot\tfrac{12}{\pi}\right] = \left(50 + \tfrac{28}{\pi}\right)\,{}^\circ\mathrm{F} \approx 59\,{}^\circ\mathrm{F}$$

19. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx = \frac{3}{2}\int_0^8 (x+1)^{-1/2}\,dx = \left[3\sqrt{x+1}\right]_0^8 = 9 - 3 = 6$ kg/m

21. $V_{\text{ave}} = \frac{1}{5}\int_0^5 V(t)\,dt = \frac{1}{5}\int_0^5 \frac{5}{4\pi}\left[1 - \cos\left(\frac{2}{5}\pi t\right)\right]dt = \frac{1}{4\pi}\int_0^5 \left[1 - \cos\left(\frac{2}{5}\pi t\right)\right]dt$

$= \frac{1}{4\pi}\left[t - \frac{5}{2\pi}\sin\left(\frac{2}{5}\pi t\right)\right]_0^5 = \frac{1}{4\pi}[(5-0) - 0] = \frac{5}{4\pi} \approx 0.4$ L

23. Let $F(x) = \int_a^x f(t)\,dt$ for x in $[a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b), so by the Mean Value Theorem there is a number c in (a, b) such that $F(b) - F(a) = F'(c)(b-a)$. But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt - 0 = f(c)(b-a)$.

6 Review

CONCEPT CHECK

1. (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.

(b) Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top. See Figures 11 and 12 in Section 6.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.

(b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi\left(r_{\text{out}}^2\right) - \pi\left(r_{\text{in}}^2\right)$.

4. (a) $V = 2\pi rh\,\Delta r = (\text{circumference})(\text{height})(\text{thickness})$

(b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$, where a and b are the limits on x or y.

(c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. $\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

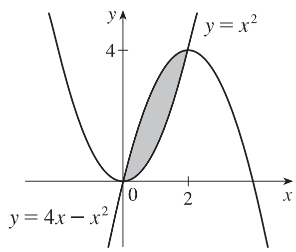
6. (a) The average value of a function f on an interval $[a,b]$ is $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

(b) The Mean Value Theorem for Integrals says that there is a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. For a geometric interpretation of the Mean Value Theorem for Integrals, see Figure 2 in Section 6.5 and the discussion that accompanies it.

EXERCISES

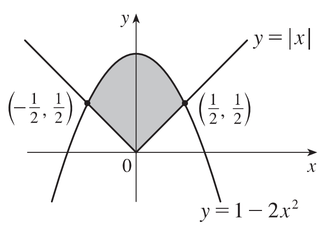
1. The curves intersect when $x^2 = 4x - x^2 \iff 2x^2 - 4x = 0 \iff 2x(x-2) = 0 \iff x = 0$ or 2.

$$A = \int_0^2 \left[(4x - x^2) - x^2\right] dx = \int_0^2 (4x - 2x^2)\,dx$$
$$= \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2 = \left[\left(8 - \tfrac{16}{3}\right) - 0\right] = \tfrac{8}{3}$$



3. If $x \geq 0$, then $|x| = x$, and the graphs intersect when $x = 1 - 2x^2 \iff 2x^2 + x - 1 = 0 \iff (2x-1)(x+1) = 0 \iff x = \frac{1}{2}$ or -1, but $-1 < 0$. By symmetry, we can double the area from $x = 0$ to $x = \frac{1}{2}$.

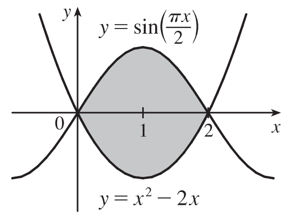
$$A = 2\int_0^{1/2} \left[(1 - 2x^2) - x\right] dx = 2\int_0^{1/2} (-2x^2 - x + 1)\,dx$$
$$= 2\left[-\tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + x\right]_0^{1/2} = 2\left[\left(-\tfrac{1}{12} - \tfrac{1}{8} + \tfrac{1}{2}\right) - 0\right]$$
$$= 2\left(\tfrac{7}{24}\right) = \tfrac{7}{12}$$



5. $A = \int_0^2 \left[\sin\left(\frac{\pi x}{2}\right) - (x^2 - 2x)\right] dx$

$= \left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}x^3 + x^2\right]_0^2$

$= \left(\frac{2}{\pi} - \frac{8}{3} + 4\right) - \left(-\frac{2}{\pi} - 0 + 0\right) = \frac{4}{3} + \frac{4}{\pi}$

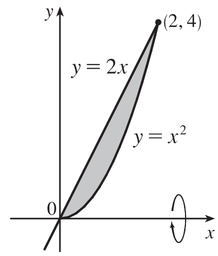


7. Using washers with inner radius x^2 and outer radius $2x$, we have

$V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right] dx = \pi \int_0^2 (4x^2 - x^4)\, dx$

$= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right)$

$= 32\pi \cdot \frac{2}{15} = \frac{64}{15}\pi$

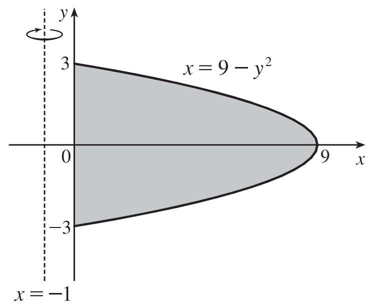


9. $V = \pi \int_{-3}^3 \left\{\left[(9 - y^2) - (-1)\right]^2 - [0 - (-1)]^2\right\} dy$

$= 2\pi \int_0^3 \left[(10 - y^2)^2 - 1\right] dy = 2\pi \int_0^3 (100 - 20y^2 + y^4 - 1)\, dy$

$= 2\pi \int_0^3 (99 - 20y^2 + y^4)\, dy = 2\pi\left[99y - \frac{20}{3}y^3 + \frac{1}{5}y^5\right]_0^3$

$= 2\pi\left(297 - 180 + \frac{243}{5}\right) = \frac{1656}{5}\pi$



11. The graph of $x^2 - y^2 = a^2$ is a hyperbola with right and left branches.

Solving for y gives us $y^2 = x^2 - a^2 \quad \Rightarrow \quad y = \pm\sqrt{x^2 - a^2}$.

We'll use shells and the height of each shell is

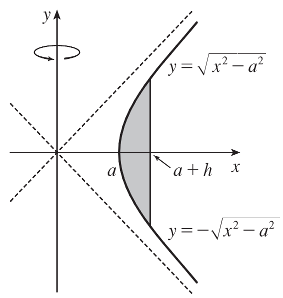
$\sqrt{x^2 - a^2} - \left(-\sqrt{x^2 - a^2}\right) = 2\sqrt{x^2 - a^2}$.

The volume is $V = \int_a^{a+h} 2\pi x \cdot 2\sqrt{x^2 - a^2}\, dx$. To evaluate, let $u = x^2 - a^2$,

so $du = 2x\, dx$ and $x\, dx = \frac{1}{2}\, du$. When $x = a$, $u = 0$, and when $x = a + h$,

$u = (a + h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2$.

Thus, $V = 4\pi \int_0^{2ah+h^2} \sqrt{u}\left(\frac{1}{2}\, du\right) = 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{2ah+h^2} = \frac{4}{3}\pi\left(2ah + h^2\right)^{3/2}$.

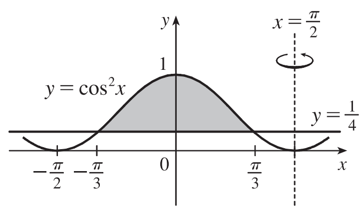


13. A shell has radius $\frac{\pi}{2} - x$, circumference $2\pi\left(\frac{\pi}{2} - x\right)$, and height $\cos^2 x - \frac{1}{4}$.

$y = \cos^2 x$ intersects $y = \frac{1}{4}$ when $\cos^2 x = \frac{1}{4} \quad \Leftrightarrow$

$\cos x = \pm\frac{1}{2} \quad [\,|x| \le \pi/2] \quad \Leftrightarrow \quad x = \pm\frac{\pi}{3}$.

$$V = \int_{-\pi/3}^{\pi/3} 2\pi\left(\frac{\pi}{2} - x\right)\left(\cos^2 x - \frac{1}{4}\right) dx$$



15. (a) A cross-section is a washer with inner radius x^2 and outer radius x.

$$V = \int_0^1 \pi\left[(x)^2 - (x^2)^2\right] dx = \int_0^1 \pi(x^2 - x^4)\, dx = \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left[\tfrac{1}{3} - \tfrac{1}{5}\right] = \tfrac{2}{15}\pi$$

(b) A cross-section is a washer with inner radius y and outer radius $\sqrt{y}$.

$$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - y^2\right] dy = \int_0^1 \pi(y - y^2)\, dy = \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 = \pi\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \tfrac{\pi}{6}$$

(c) A cross-section is a washer with inner radius $2 - x$ and outer radius $2 - x^2$.

$$V = \int_0^1 \pi\left[(2 - x^2)^2 - (2 - x)^2\right] dx = \int_0^1 \pi(x^4 - 5x^2 + 4x)\, dx = \pi\left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 2x^2\right]_0^1 = \pi\left[\tfrac{1}{5} - \tfrac{5}{3} + 2\right] = \tfrac{8}{15}\pi$$

17. (a) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \tan(x^2)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan(x^2)\, dx \approx \tfrac{1}{4}\left[\tan\left(\left(\tfrac{1}{8}\right)^2\right) + \tan\left(\left(\tfrac{3}{8}\right)^2\right) + \tan\left(\left(\tfrac{5}{8}\right)^2\right) + \tan\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{1}{4}(1.53) \approx 0.38$$

(b) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \pi\tan^2(x^2)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x)\, dx \approx \tfrac{1}{4}\pi\left[\tan^2\left(\left(\tfrac{1}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{3}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{5}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{\pi}{4}(1.114) \approx 0.87$$

19. $\int_0^{\pi/2} 2\pi x \cos x\, dx = \int_0^{\pi/2} (2\pi x)\cos x\, dx$

The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the y-axis.

21. $\int_0^{\pi} \pi(2 - \sin x)^2\, dx$

The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le 2 - \sin x\}$ about the x-axis.

23. Take the base to be the disk $x^2 + y^2 \le 9$. Then $V = \int_{-3}^{3} A(x)\, dx$, where $A(x_0)$ is the area of the isosceles right triangle whose hypotenuse lies along the line $x = x_0$ in the xy-plane. The length of the hypotenuse is $2\sqrt{9 - x^2}$ and the length of each leg is $\sqrt{2}\sqrt{9 - x^2}$. $A(x) = \frac{1}{2}\left(\sqrt{2}\sqrt{9 - x^2}\right)^2 = 9 - x^2$, so

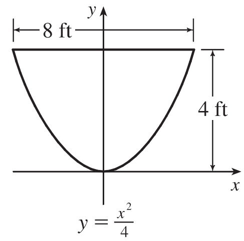
$$V = 2\int_0^3 A(x)\, dx = 2\int_0^3 (9 - x^2)\, dx = 2\left[9x - \tfrac{1}{3}x^3\right]_0^3 = 2(27 - 9) = 36$$

25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x\sin 60^\circ = \frac{\sqrt{3}}{8}x$. Therefore,

$A(x) = \frac{1}{2} \cdot \frac{1}{4}x \cdot \frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $\quad V = \int_0^{20} A(x)\, dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\, dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64 \cdot 3} = \frac{125\sqrt{3}}{3}\ \text{m}^3$.

27. $f(x) = kx \quad\Rightarrow\quad 30\ \text{N} = k(15 - 12)\ \text{cm} \quad\Rightarrow\quad k = 10\ \text{N/cm} = 1000\ \text{N/m}$. $\quad 20\ \text{cm} - 12\ \text{cm} = 0.08\ \text{m} \quad\Rightarrow$

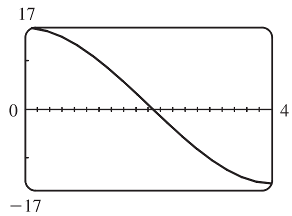
$W = \int_0^{0.08} kx\, dx = 1000\int_0^{0.08} x\, dx = 500\left[x^2\right]_0^{0.08} = 500(0.08)^2 = 3.2\ \text{N·m} = 3.2\ \text{J}$.



29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4,4)$. $4 = a \cdot 4^2 \quad \Rightarrow \quad a = \frac{1}{4} \quad \Rightarrow \quad y = \frac{1}{4}x^2 \quad \Rightarrow \quad x^2 = 4y \quad \Rightarrow$ $x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.

$$W = \int_0^4 \pi\left(2\sqrt{y}\right)^2 62.5(4-y)\,dy = 250\pi \int_0^4 y(4-y)\,dy$$

$$= 250\pi\left[2y^2 - \tfrac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \tfrac{64}{3}\right) = \tfrac{8000\pi}{3} \approx 8378 \text{ ft-lb}$$



(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h) unknown: $W = 4000 \quad \Leftrightarrow$ $250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \quad \Leftrightarrow \quad \frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \quad \Leftrightarrow$ $h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0$. We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0,4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.

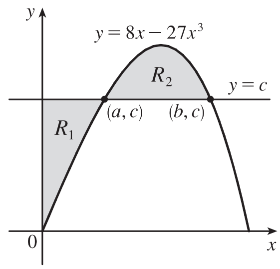
31. $\lim\limits_{h\to 0} f_{\text{ave}} = \lim\limits_{h\to 0} \dfrac{1}{(x+h)-x}\displaystyle\int_x^{x+h} f(t)\,dt = \lim\limits_{h\to 0}\dfrac{F(x+h)-F(x)}{h}$, where $F(x) = \int_a^x f(t)\,dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim\limits_{h\to 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

PROBLEMS PLUS

1. (a) The area under the graph of f from 0 to t is equal to $\int_0^t f(x)\,dx$, so the requirement is that $\int_0^t f(x)\,dx = t^3$ for all t. We differentiate both sides of this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$. This function is positive and continuous, as required.

(b) The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation with respect to b using the Fundamental Theorem of Calculus gives $2b = \pi[f(b)]^2 \quad\Rightarrow\quad f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

3. Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure, $R_1 = R_2 \quad\Rightarrow$



$$\int_0^a \left[c - \left(8x - 27x^3\right)\right] dx = \int_a^b \left[\left(8x - 27x^3\right) - c\right] dx$$

$$\left[cx - 4x^2 + \tfrac{27}{4}x^4\right]_0^a = \left[4x^2 - \tfrac{27}{4}x^4 - cx\right]_a^b$$

$$ac - 4a^2 + \tfrac{27}{4}a^4 = \left(4b^2 - \tfrac{27}{4}b^4 - bc\right) - \left(4a^2 - \tfrac{27}{4}a^4 - ac\right)$$

$$0 = 4b^2 - \tfrac{27}{4}b^4 - bc = 4b^2 - \tfrac{27}{4}b^4 - b\left(8b - 27b^3\right)$$

$$= 4b^2 - \tfrac{27}{4}b^4 - 8b^2 + 27b^4 = \tfrac{81}{4}b^4 - 4b^2$$

$$= b^2\left(\tfrac{81}{4}b^2 - 4\right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \quad\Rightarrow\quad b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.

5. (a) $V = \pi h^2(r - h/3) = \frac{1}{3}\pi h^2(3r - h)$. See the solution to Exercise 6.2.51.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is $V = \frac{1}{3}\pi(1-x)^2\,[3(1) - (1-x)] = \frac{1}{3}\pi(x-1)^2(x+2)$. This volume must be $\frac{1}{3}$ of the total volume of the sphere, which is $\frac{4}{3}\pi(1)^3$. So $\frac{1}{3}\pi(x-1)^2(x+2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \quad\Rightarrow\quad (x^2 - 2x + 1)(x+2) = \frac{4}{3} \quad\Rightarrow\quad x^3 - 3x + 2 = \frac{4}{3} \quad\Rightarrow$ $3x^3 - 9x + 2 = 0$. Using Newton's method with $f(x) = 3x^3 - 9x + 2$, $f'(x) = 9x^2 - 9$, we get $x_{n+1} = x_n - \dfrac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}$. Taking $x_1 = 0$, we get $x_2 \approx 0.2222$, and $x_3 \approx 0.2261 \approx x_4$, so, correct to four decimal places, $x \approx 0.2261$.

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0 \quad\Rightarrow$ $x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \quad\Rightarrow\quad 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with $f(x) = 8x^3 - 12x^2 + 3$, $f'(x) = 24x^2 - 24x$, so $x_{n+1} = x_n - \dfrac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}$. Take $x_1 = 0.5$. Then $x_2 \approx 0.6667$, and $x_3 \approx 0.6736 \approx x_4$. So to four decimal places the depth is 0.6736 m.

(d) (i) From part (a) with $r = 5$ in., the volume of water in the bowl is

$V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi h^2(15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3$. We are given that $\dfrac{dV}{dt} = 0.2\ \text{in}^3/\text{s}$ and we want to find $\dfrac{dh}{dt}$

when $h = 3$. Now $\dfrac{dV}{dt} = 10\pi h\,\dfrac{dh}{dt} - \pi h^2\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10h - h^2)}$. When $h = 3$, we have

$\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10 \cdot 3 - 3^2)} = \dfrac{1}{105\pi} \approx 0.003$ in/s.

(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is

$V = \frac{1}{2} \cdot \frac{4}{3}\pi(5)^3 - \frac{1}{3}\pi(4)^2(15 - 4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3}\pi$. To find the time required to fill the bowl we divide

this volume by the rate: Time $= \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387\ \text{s} \approx 6.5$ min.

7. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA(x)$, where k is some positive constant and $A(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA(x)$ $(\star)$. Also, we know that the total volume of water up to a depth x is $V(x) = \int_0^x A(s)\,ds$, where $A(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A(x)$. Substituting this into equation $\star$, we get $A(x)(dx/dt) = -kA(x) \quad\Rightarrow\quad dx/dt = -k$, a constant.

9. We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = \left(a, 2a^2\right)$, then area A is just $\int_0^a (2x^2 - x^2)\,dx = \int_0^a x^2\,dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of y: $y = 2x^2 \quad\Leftrightarrow\quad x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

$$\int_0^{2a^2}\left[\sqrt{y/2} - C(y)\right]dy = \left[\frac{4}{3}(y/2)^{3/2}\right]_0^{2a^2} - \int_0^{2a^2} C(y)\,dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy$$

where $C(y)$ is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy \quad\Leftrightarrow\quad \int_0^{2a^2} C(y)\,dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem:

$C(2a^2)(4a) = 3a^2 \quad\Rightarrow\quad C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for y: $x = \frac{3}{4}\sqrt{y/2} \quad\Rightarrow$

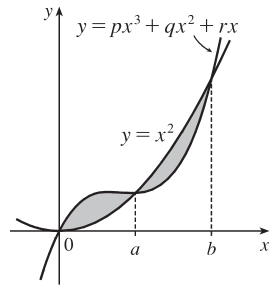
$x^2 = \frac{9}{16}(y/2) \quad\Rightarrow\quad y = \frac{32}{9}x^2$.

11. (a) Stacking disks along the y-axis gives us $V = \int_0^h \pi\,[f(y)]^2\,dy$.

(b) Using the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt} = \pi\,[f(h)]^2\,\dfrac{dh}{dt}$.

(c) $kA\sqrt{h} = \pi[f(h)]^2\,\dfrac{dh}{dt}$. Set $\dfrac{dh}{dt} = C$: $\pi[f(h)]^2\,C = kA\sqrt{h} \;\Rightarrow\; [f(h)]^2 = \dfrac{kA}{\pi C}\sqrt{h} \;\Rightarrow\; f(h) = \sqrt{\dfrac{kA}{\pi C}}\,h^{1/4}$; that is, $f(y) = \sqrt{\dfrac{kA}{\pi C}}\,y^{1/4}$. The advantage of having $\dfrac{dh}{dt} = C$ is that the markings on the container are equally spaced.

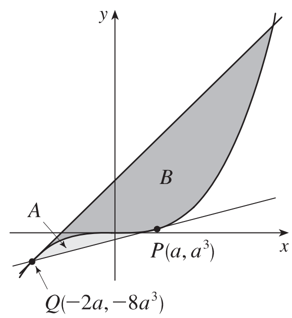
13. The cubic polynomial passes through the origin, so let its equation be $y = px^3 + qx^2 + rx$. The curves intersect when $px^3 + qx^2 + rx = x^2 \;\Leftrightarrow\; px^3 + (q-1)x^2 + rx = 0$. Call the left side $f(x)$. Since $f(a) = f(b) = 0$, another form of f is



$$\begin{aligned} f(x) &= px(x-a)(x-b) = px[x^2 - (a+b)x + ab] \\ &= p[x^3 - (a+b)x^2 + abx] \end{aligned}$$

Since the two areas are equal, we must have $\int_0^a f(x)\,dx = -\int_a^b f(x)\,dx \;\Rightarrow$ $[F(x)]_0^a = [F(x)]_b^a \;\Rightarrow\; F(a) - F(0) = F(a) - F(b) \;\Rightarrow\; F(0) = F(b)$, where F is an antiderivative of f. Now $F(x) = \int f(x)\,dx = \int p[x^3 - (a+b)x^2 + abx]\,dx = p\left[\frac{1}{4}x^4 - \frac{1}{3}(a+b)x^3 + \frac{1}{2}abx^2\right] + C$, so $F(0) = F(b) \;\Rightarrow\; C = p\left[\frac{1}{4}b^4 - \frac{1}{3}(a+b)b^3 + \frac{1}{2}ab^3\right] + C \;\Rightarrow\; 0 = p\left[\frac{1}{4}b^4 - \frac{1}{3}(a+b)b^3 + \frac{1}{2}ab^3\right] \;\Rightarrow$ $0 = 3b - 4(a+b) + 6a$ [multiply by $12/(pb^3)$, $b \neq 0$] $\;\Rightarrow\; 0 = 3b - 4a - 4b + 6a \;\Rightarrow\; b = 2a$. Hence, b is twice the value of a.

15. We assume that P lies in the region of positive x. Since $y = x^3$ is an odd function, this assumption will not affect the result of the calculation. Let $P = \left(a, a^3\right)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so the equation of the tangent is $y - a^3 = 3a^2(x-a) \;\Leftrightarrow\; y = 3a^2x - 2a^3$. We solve this simultaneously with $y = x^3$ to find the other point of intersection: $x^3 = 3a^2x - 2a^3 \;\Leftrightarrow\; (x-a)^2(x+2a) = 0$. So $Q = \left(-2a, -8a^3\right)$ is the other point of intersection. The equation of the tangent at Q is $y - (-8a^3) = 12a^2[x - (-2a)] \;\Leftrightarrow\; y = 12a^2x + 16a^3$. By symmetry,



this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A = \int_{-2a}^{a}\left[x^3 - (3a^2x - 2a^3)\right]dx$ and $B = \int_{-2a}^{4a}\left[(12a^2x + 16a^3) - x^3\right]dx$. We calculate $A = \left[\frac{1}{4}x^4 - \frac{3}{2}a^2x^2 + 2a^3x\right]_{-2a}^{a} = \frac{3}{4}a^4 - (-6a^4) = \frac{27}{4}a^4$, and $B = \left[6a^2x^2 + 16a^3x - \frac{1}{4}x^4\right]_{-2a}^{4a} = 96a^4 - (-12a^4) = 108a^4$. We see that $B = 16A = 2^4A$. This is because our calculation of area B was essentially the same as that of area A, with a replaced by $-2a$, so if we replace a with $-2a$ in our expression for A, we get $\frac{27}{4}(-2a)^4 = 108a^4 = B$.

7 □ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = \ln x$, $dv = x^2\,dx \quad\Rightarrow\quad du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$\int x^2 \ln x\,dx = (\ln x)\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\right)\,dx = \frac{1}{3}x^3\,\ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\,\ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$

$= \frac{1}{3}x^3\,\ln x - \frac{1}{9}x^3 + C \quad \left[\text{or } \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + C\right]$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

3. Let $u = x$, $dv = \cos 5x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,

$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \quad\Rightarrow\quad du = dr$, $v = 2e^{r/2}$. Then $\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C$.

7. Let $u = x^2$, $dv = \sin \pi x\,dx \quad\Rightarrow\quad du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos \pi x$. Then

$I = \int x^2\,\sin \pi x\,dx = -\frac{1}{\pi}x^2\cos \pi x + \frac{2}{\pi}\int x\cos \pi x\,dx$ $(\star)$. Next let $U = x$, $dV = \cos \pi x\,dx \quad\Rightarrow\quad dU = dx$,

$V = \frac{1}{\pi}\sin \pi x$, so $\int x\cos \pi x\,dx = \frac{1}{\pi}x\sin \pi x - \frac{1}{\pi}\int \sin \pi x\,dx = \frac{1}{\pi}x\sin \pi x + \frac{1}{\pi^2}\cos \pi x + C_1$.

Substituting for $\int x\cos \pi x\,dx$ in $(\star)$, we get

$I = -\frac{1}{\pi}x^2\cos \pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin \pi x + \frac{1}{\pi^2}\cos \pi x + C_1\right) = -\frac{1}{\pi}x^2\cos \pi x + \frac{2}{\pi^2}x\sin \pi x + \frac{2}{\pi^3}\cos \pi x + C$, where $C = \frac{2}{\pi}C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\begin{aligned}\int \ln(2x+1)\,dx &= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx \\ &= x\ln(2x+1) - \int\left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C \\ &= \tfrac{1}{2}(2x+1)\,\ln(2x+1) - x + C\end{aligned}$$

11. Let $u = \arctan 4t$, $dv = dt \quad\Rightarrow\quad du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$$\int \arctan 4t\,dt = t\arctan 4t - \int \frac{4t}{1+16t^2}\,dt = t\arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\,dt = t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C.$$

13. Let $u = t$, $dv = \sec^2 2t\,dt \quad\Rightarrow\quad du = dt$, $v = \frac{1}{2}\tan 2t$. Then

$\int t\sec^2 2t\,dt = \frac{1}{2}t\tan 2t - \frac{1}{2}\int \tan 2t\,dt = \frac{1}{2}t\,\tan 2t - \frac{1}{4}\ln|\sec 2t| + C.$

15. First let $u = (\ln x)^2$, $dv = dx \quad\Rightarrow\quad du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \quad\Rightarrow$

$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

17. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad\Rightarrow\quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \quad\Rightarrow\quad dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad\Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

19. Let $u = t$, $dv = \sin 3t\,dt \quad\Rightarrow\quad du = dt$, $v = -\frac{1}{3}\cos 3t$. Then

$\int_0^\pi t\,\sin 3t\,dt = \left[-\frac{1}{3}t\,\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}.$

21. Let $u = t$, $dv = \cosh t\,dt \quad\Rightarrow\quad du = dt$, $v = \sinh t$. Then

$$\begin{aligned}\int_0^1 t\cosh t\,dt &= \left[t\sinh t\right]_0^1 - \int_0^1 \sinh t\,dt = (\sinh 1 - \sinh 0) - \left[\cosh t\right]_0^1 = \sinh 1 - (\cosh 1 - \cosh 0)\\ &= \sinh 1 - \cosh 1 + 1.\end{aligned}$$

We can use the definitions of sinh and cosh to write the answer in terms of e:

$\sinh 1 - \cosh 1 + 1 = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^1 + e^{-1}) + 1 = -e^{-1} + 1 = 1 - 1/e.$

23. Let $u = \ln x$, $dv = x^{-2}\,dx \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

25. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \quad\Rightarrow\quad du = dy$, $v = -\frac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

27. Let $u = \cos^{-1}x$, $dv = dx \quad\Rightarrow\quad du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2} \cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\tfrac{1}{2}dt\right],$$ where $t = 1 - x^2 \quad\Rightarrow$

$dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^1 t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right).$

29. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \Rightarrow du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx = \sin x \ln(\sin x) - \sin x + C$.

Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln \sin x - 1) + C$.

31. Let $u = (\ln x)^2$, $dv = x^4\,dx \Rightarrow du = 2\,\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \Rightarrow dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2\,dx = \tfrac{32}{5}(\ln 2)^2 - 2\left(\tfrac{32}{25}\ln 2 - \tfrac{31}{125}\right) = \tfrac{32}{5}(\ln 2)^2 - \tfrac{64}{25}\ln 2 + \tfrac{62}{125}$.

33. Let $y = \sqrt{x}$, so that $dy = \frac{1}{2}x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2y}\,dx$. Thus, $\int \cos\sqrt{x}\,dx = \int \cos y\,(2y\,dy) = 2\int y\cos y\,dy$. Now use parts with $u = y$, $dv = \cos y\,dy$, $du = dy$, $v = \sin y$ to get $\int y\cos y\,dy = y\sin y - \int \sin y\,dy = y\sin y + \cos y + C_1$, so $\int \cos\sqrt{x}\,dx = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

35. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos(\theta^2)\cdot\tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

37. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

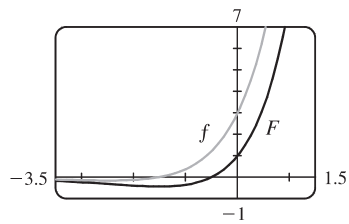
$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

In Exercises 39–42, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

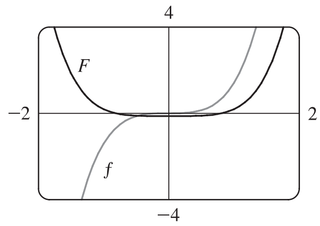
39. Let $u = 2x + 3$, $dv = e^x\,dx \Rightarrow du = 2\,dx$, $v = e^x$. Then

$$\begin{aligned}\int (2x+3)e^x\,dx &= (2x+3)e^x - 2\int e^x\,dx = (2x+3)e^x - 2e^x + C \\ &= (2x+1)\,e^x + C\end{aligned}$$



We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.

41. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \quad \Rightarrow \quad du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.



Then

$$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C$$

Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.

43. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

45. (a) From Example 6, $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\cos x\,\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x\,\sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

47. Let $u = (\ln x)^n$, $dv = dx \quad \Rightarrow \quad du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int (\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$.

49. $\int \tan^n x\,dx = \int \tan^{n-2} x\,\tan^2 x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx = \int \tan^{n-2} x\,\sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= I - \int \tan^{n-2} x\,dx$.

Let $u = \tan^{n-2} x$, $dv = \sec^2 x\,dx \quad \Rightarrow \quad du = (n-2)\,\tan^{n-3} x\,\sec^2 x\,dx$, $v = \tan x$. Then, by Equation 2,

$$I = \tan^{n-1} x - (n-2)\int \tan^{n-2} x\,\sec^2 x\,dx$$
$$1I = \tan^{n-1} x - (n-2)I$$
$$(n-1)I = \tan^{n-1} x$$
$$I = \frac{\tan^{n-1} x}{n-1}$$

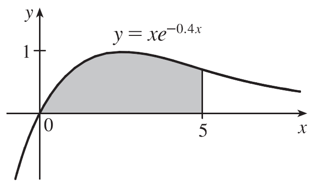
Returning to the original integral, $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$.

51. By repeated applications of the reduction formula in Exercise 47,

$$\begin{aligned}\int(\ln x)^3\,dx &= x\,(\ln x)^3 - 3\int(\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int(\ln x)^1\,dx\right]\\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x)^1 - 1\int(\ln x)^0\,dx\right]\\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int 1\,dx = x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C\end{aligned}$$

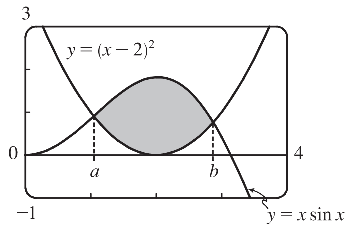
53. Area $= \int_0^5 xe^{-0.4x}dx$. Let $u = x$, $dv = e^{-0.4x}\,dx \quad\Rightarrow$ $du = dx$, $v = -2.5e^{-0.4x}$. Then

$$\begin{aligned}\text{area} &= \left[-2.5xe^{-0.4x}\right]_0^5 + 2.5\int_0^5 e^{-0.4x}\,dx\\ &= -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5\\ &= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \text{ or } \tfrac{25}{4} - \tfrac{75}{4}e^{-2}\end{aligned}$$



55. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

$$\begin{aligned}\text{area} &= \int_a^b \left[x\sin x - (x-2)^2\right]dx\\ &= \left[-x\cos x + \sin x - \tfrac{1}{3}(x-2)^3\right]_a^b \qquad \text{[by Example 1]}\\ &\approx 2.81358 - 0.63075 = 2.18283\end{aligned}$$



57. $V = \int_0^1 2\pi x\cos(\pi x/2)\,dx$. Let $u = x$, $dv = \cos(\pi x/2)\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$V = 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi\cdot\frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx = 2\pi\left(\frac{2}{\pi} - 0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 = 4 + \frac{8}{\pi}(0-1) = 4 - \frac{8}{\pi}.$$

59. Volume $= \int_{-1}^0 2\pi(1-x)e^{-x}\,dx$. Let $u = 1 - x$, $dv = e^{-x}\,dx \quad\Rightarrow\quad du = -\,dx$, $v = -e^{-x}$.

$$V = 2\pi\left[(1-x)(-e^{-x})\right]_{-1}^0 - 2\pi\int_{-1}^0 e^{-x}\,dx = 2\pi\left[(x-1)(e^{-x}) + e^{-x}\right]_{-1}^0 = 2\pi\left[xe^{-x}\right]_{-1}^0 = 2\pi(0+e) = 2\pi e$$

61. The average value of $f(x) = x^2\ln x$ on the interval $[1,3]$ is $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 x^2\ln x\,dx = \tfrac{1}{2}I$.

Let $u = \ln x$, $dv = x^2\,dx \quad\Rightarrow\quad du = (1/x)\,dx$, $v = \frac{1}{3}x^3$.

So $I = \left[\frac{1}{3}x^3\ln x\right]_1^3 - \int_1^3 \frac{1}{3}x^2\,dx = (9\ln 3 - 0) - \left[\frac{1}{9}x^3\right]_1^3 = 9\ln 3 - \left(3 - \frac{1}{9}\right) = 9\ln 3 - \frac{26}{9}$.

Thus, $f_{\text{ave}} = \frac{1}{2}I = \frac{1}{2}\left(9\ln 3 - \frac{26}{9}\right) = \frac{9}{2}\ln 3 - \frac{13}{9}$.

63. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \quad\Rightarrow\quad du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \quad\Rightarrow\quad dU = dw$, $V = -e^{-w}$. Then

$$\begin{aligned}s(t) &= -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)\\ &= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}\end{aligned}$$

65. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \quad\Rightarrow\quad du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7-2) = 12 - 5 - 5 = 2.$

We used the fact that f'' is continuous to guarantee that I exists.

67. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$,

which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x)\,dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2\,f'(x)\,dx = \left[x^2\,f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2\,f(b) - a^2\,f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \quad \Rightarrow$

$V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x\,f(x)\,dx\right] = \int_a^b 2\pi x\,f(x)\,dx$.

7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $\int \sin^3 x\,\cos^2 x\,dx = \int \sin^2 x\,\cos^2 x\,\sin x\,dx = \int (1 - \cos^2 x)\cos^2 x\,\sin x\,dx \overset{c}{=} \int (1 - u^2)u^2(-du)$

$$= \int (u^2 - 1)u^2\,du = \int (u^4 - u^2)\,du = \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C$$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x\,\cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,\cos^2 x\,\cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1 - \sin^2 x)\cos x\,dx \overset{s}{=} \int_1^{\sqrt{2}/2} u^5(1 - u^2)\,du$

$$= \int_1^{\sqrt{2}/2} (u^5 - u^7)\,du = \left[\tfrac{1}{6}u^6 - \tfrac{1}{8}u^8\right]_1^{\sqrt{2}/2} = \left(\tfrac{1/8}{6} - \tfrac{1/16}{8}\right) - \left(\tfrac{1}{6} - \tfrac{1}{8}\right) = -\tfrac{11}{384}$$

5. Let $y = \pi x$, so $dy = \pi\,dx$ and

$$\begin{aligned}
\int \sin^2(\pi x)\,\cos^5(\pi x)\,dx &= \tfrac{1}{\pi}\int \sin^2 y\,\cos^5 y\,dy = \tfrac{1}{\pi}\int \sin^2 y\,\cos^4 y\,\cos y\,dy \\
&= \tfrac{1}{\pi}\int \sin^2 y\,(1 - \sin^2 y)^2\,\cos y\,dy \overset{s}{=} \tfrac{1}{\pi}\int u^2(1 - u^2)^2\,du = \tfrac{1}{\pi}\int (u^2 - 2u^4 + u^6)\,du \\
&= \tfrac{1}{\pi}\left(\tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7\right) + C = \tfrac{1}{3\pi}\sin^3 y - \tfrac{2}{5\pi}\sin^5 y + \tfrac{1}{7\pi}\sin^7 y + C \\
&= \tfrac{1}{3\pi}\sin^3(\pi x) - \tfrac{2}{5\pi}\sin^5(\pi x) + \tfrac{1}{7\pi}\sin^7(\pi x) + C
\end{aligned}$$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$ [half-angle identity]

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{2} + 0\right) - (0 + 0)\right] = \tfrac{\pi}{4}$$

9. $\int_0^\pi \sin^4(3t)\,dt = \int_0^\pi \left[\sin^2(3t)\right]^2\,dt = \int_0^\pi \left[\tfrac{1}{2}(1 - \cos 6t)\right]^2\,dt = \tfrac{1}{4}\int_0^\pi (1 - 2\cos 6t + \cos^2 6t)\,dt$

$$\begin{aligned}
&= \tfrac{1}{4}\int_0^\pi \left[1 - 2\cos 6t + \tfrac{1}{2}(1 + \cos 12t)\right]dt = \tfrac{1}{4}\int_0^\pi \left(\tfrac{3}{2} - 2\cos 6t + \tfrac{1}{2}\cos 12t\right)dt \\
&= \tfrac{1}{4}\left[\tfrac{3}{2}t - \tfrac{1}{3}\sin 6t + \tfrac{1}{24}\sin 12t\right]_0^\pi = \tfrac{1}{4}\left[\left(\tfrac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] = \tfrac{3\pi}{8}
\end{aligned}$$

11. $\int (1 + \cos\theta)^2\,d\theta = \int (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \theta + 2\sin\theta + \tfrac{1}{2}\int (1 + \cos 2\theta)\,d\theta$

$$= \theta + 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta + C$$

13. $\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2} \tfrac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2} \tfrac{1}{4}(2\sin x\,\cos x)^2 dx = \tfrac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$

$$= \tfrac{1}{4}\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4x)\,dx = \tfrac{1}{8}\int_0^{\pi/2} (1 - \cos 4x)\,dx = \tfrac{1}{8}\left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

15. $\displaystyle\int \frac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha = \int \frac{\cos^4\alpha}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha = \int \frac{\left(1-\sin^2\alpha\right)^2}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha \overset{s}{=} \int \frac{(1-u^2)^2}{\sqrt{u}}\,du$

$$= \int \frac{1-2u^2+u^4}{u^{1/2}}\,du = \int \left(u^{-1/2} - 2u^{3/2} + u^{7/2}\right)du = 2u^{1/2} - \tfrac{4}{5}u^{5/2} + \tfrac{2}{9}u^{9/2} + C$$

$$= \tfrac{2}{45}u^{1/2}(45 - 18u^2 + 5u^4) + C = \tfrac{2}{45}\sqrt{\sin\alpha}\,(45 - 18\sin^2\alpha + 5\sin^4\alpha) + C$$

17. $\displaystyle\int \cos^2 x\,\tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}\,dx \overset{c}{=} \int \frac{(1-u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right]du$

$$= -\ln|u| + \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

19. $\displaystyle\int \frac{\cos x + \sin 2x}{\sin x}\,dx = \int \frac{\cos x + 2\sin x\,\cos x}{\sin x}\,dx = \int \frac{\cos x}{\sin x}\,dx + \int 2\cos x\,dx \overset{s}{=} \int \frac{1}{u}\,du + 2\sin x$

$$= \ln|u| + 2\sin x + C = \ln|\sin x| + 2\sin x + C$$

Or: Use the formula $\int \cot x\,dx = \ln|\sin x| + C$.

21. Let $u = \tan x$, $du = \sec^2 x\,dx$. Then $\int \sec^2 x\,\tan x\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tan^2 x + C$.

Or: Let $v = \sec x$, $dv = \sec x\,\tan x\,dx$. Then $\int \sec^2 x\,\tan x\,dx = \int v\,dv = \tfrac{1}{2}v^2 + C = \tfrac{1}{2}\sec^2 x + C$.

23. $\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$

25. $\int \sec^6 t\,dt = \int \sec^4 t\cdot\sec^2 t\,dt = \int(\tan^2 t + 1)^2\sec^2 t\,dt = \int(u^2+1)^2\,du \qquad [u = \tan t,\ du = \sec^2 t\,dt]$

$$= \int(u^4 + 2u^2 + 1)\,du = \tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u + C = \tfrac{1}{5}\tan^5 t + \tfrac{2}{3}\tan^3 t + \tan t + C$$

27. $\int_0^{\pi/3}\tan^5 x\,\sec^4 x\,dx = \int_0^{\pi/3}\tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx = \int_0^{\sqrt{3}} u^5(u^2+1)\,du \qquad [u = \tan x,\ du = \sec^2 x\,dx]$

$$= \int_0^{\sqrt{3}}(u^7 + u^5)\,du = \left[\tfrac{1}{8}u^8 + \tfrac{1}{6}u^6\right]_0^{\sqrt{3}} = \tfrac{81}{8} + \tfrac{27}{6} = \tfrac{81}{8} + \tfrac{9}{2} = \tfrac{81}{8} + \tfrac{36}{8} = \tfrac{117}{8}$$

Alternate solution:

$\int_0^{\pi/3}\tan^5 x\,\sec^4 x\,dx = \int_0^{\pi/3}\tan^4 x\,\sec^3 x\,\sec x\,\tan x\,dx = \int_0^{\pi/3}(\sec^2 x - 1)^2\,\sec^3 x\,\sec x\,\tan x\,dx$

$$= \int_1^2(u^2-1)^2u^3\,du \quad [u = \sec x,\ du = \sec x\,\tan x\,dx] \quad = \int_1^2(u^4 - 2u^2 + 1)u^3\,du$$

$$= \int_1^2(u^7 - 2u^5 + u^3)\,du = \left[\tfrac{1}{8}u^8 - \tfrac{1}{3}u^6 + \tfrac{1}{4}u^4\right]_1^2 = \left(32 - \tfrac{64}{3} + 4\right) - \left(\tfrac{1}{8} - \tfrac{1}{3} + \tfrac{1}{4}\right) = \tfrac{117}{8}$$

29. $\int \tan^3 x\,\sec x\,dx = \int \tan^2 x\,\sec x\,\tan x\,dx = \int(\sec^2 x - 1)\,\sec x\,\tan x\,dx$

$$= \int(u^2-1)\,du \quad [u = \sec x,\ du = \sec x\,\tan x\,dx] \quad = \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\sec^3 x - \sec x + C$$

31. $\int \tan^5 x\,dx = \int(\sec^2 x - 1)^2\,\tan x\,dx = \int \sec^4 x\,\tan x\,dx - 2\int \sec^2 x\,\tan x\,dx + \int \tan x\,dx$

$$= \int \sec^3 x\,\sec x\,\tan x\,dx - 2\int \tan x\,\sec^2 x\,dx + \int \tan x\,dx$$

$$= \tfrac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \tfrac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\,]$$

33. $\displaystyle\int \frac{\tan^3\theta}{\cos^4\theta}\,d\theta = \int \tan^3\theta\,\sec^4\theta\,d\theta = \int \tan^3\theta\cdot(\tan^2\theta + 1)\cdot\sec^2\theta\,d\theta$

$$= \int u^3(u^2+1)\,du \qquad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$$

$$= \int(u^5 + u^3)\,du = \tfrac{1}{6}u^6 + \tfrac{1}{4}u^4 + C = \tfrac{1}{6}\tan^6\theta + \tfrac{1}{4}\tan^4\theta + C$$

35. Let $u = x, dv = \sec x \tan x\,dx \Rightarrow du = dx, v = \sec x$. Then

$\int x \sec x \tan x\,dx = x \sec x - \int \sec x\,dx = x \sec x - \ln|\sec x + \tan x| + C$.

37. $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\,dx = \left[-\cot x - x\right]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

39. $\int \cot^3 \alpha \csc^3 \alpha\,d\alpha = \int \cot^2 \alpha \csc^2 \alpha \cdot \csc \alpha \cot \alpha\,d\alpha = \int (\csc^2 \alpha - 1) \csc^2 \alpha \cdot \csc \alpha \cot \alpha\,d\alpha$

$= \int (u^2 - 1)u^2 \cdot (-du) \qquad [u = \csc \alpha,\ du = -\csc \alpha \cot \alpha\,d\alpha]$

$= \int (u^2 - u^4)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3 \alpha - \frac{1}{5}\csc^5 \alpha + C$

41. $I = \displaystyle\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \Rightarrow$

$du = (-\csc x \cot x + \csc^2 x)\,dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

43. $\int \sin 8x \cos 5x\,dx \overset{2a}{=} \int \frac{1}{2}[\sin(8x - 5x) + \sin(8x + 5x)]\,dx = \frac{1}{2}\int \sin 3x\,dx + \frac{1}{2}\int \sin 13x\,dx$

$= -\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$

45. $\int \sin 5\theta \sin \theta\,d\theta \overset{2b}{=} \int \frac{1}{2}[\cos(5\theta - \theta) - \cos(5\theta + \theta)]\,d\theta = \frac{1}{2}\int \cos 4\theta\,d\theta - \frac{1}{2}\int \cos 6\theta\,d\theta = \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$

47. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx = \int \left(\cos^2 x - \sin^2 x\right)dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

49. Let $u = \tan(t^2) \Rightarrow du = 2t\sec^2(t^2)\,dt$. Then $\int t\sec^2(t^2)\tan^4(t^2)\,dt = \int u^4\left(\frac{1}{2}\,du\right) = \frac{1}{10}u^5 + C = \frac{1}{10}\tan^5(t^2) + C$.

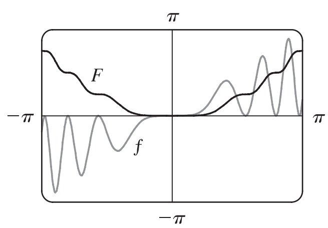
In Exercises 51–54, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

51. Let $u = x^2$, so that $du = 2x\,dx$. Then

$\int x\sin^2(x^2)\,dx = \int \sin^2 u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int \frac{1}{2}(1 - \cos 2u)\,du$

$= \frac{1}{4}\left(u - \frac{1}{2}\sin 2u\right) + C = \frac{1}{4}u - \frac{1}{4}\left(\frac{1}{2} \cdot 2\sin u \cos u\right) + C$

$= \frac{1}{4}x^2 - \frac{1}{4}\sin(x^2)\cos(x^2) + C$

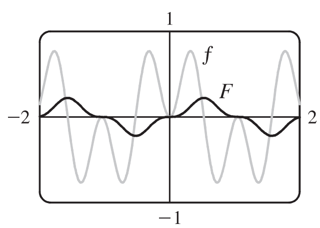


We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

53. $\int \sin 3x \sin 6x\,dx = \int \frac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)]\,dx$

$= \frac{1}{2}\int (\cos 3x - \cos 9x)\,dx$

$= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$



Notice that $f(x) = 0$ whenever F has a horizontal tangent.

55. $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,(1 - \sin^2 x)\cos x\,dx$

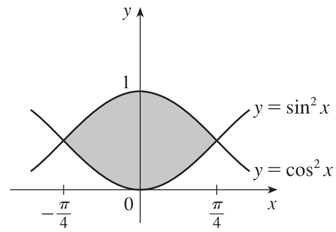
$= \frac{1}{2\pi}\int_0^0 u^2(1 - u^2)\,du \qquad [\text{where } u = \sin x]$

$= 0$

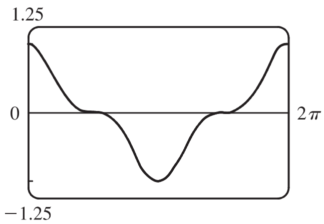
57. $A = \int_{-\pi/4}^{\pi/4} (\cos^2 x - \sin^2 x)\,dx = \int_{-\pi/4}^{\pi/4} \cos 2x\,dx$

$= 2\int_0^{\pi/4} \cos 2x\,dx = 2\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \left[\sin 2x\right]_0^{\pi/4}$

$= 1 - 0 = 1$



59.

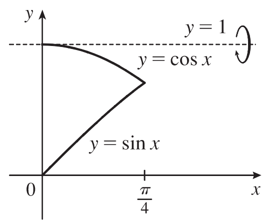


It seems from the graph that $\int_0^{2\pi} \cos^3 x\,dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

61. Using disks, $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi\int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

63. Using washers,

$$\begin{aligned} V &= \int_0^{\pi/4} \pi\left[(1 - \sin x)^2 - (1 - \cos x)^2\right] dx \\ &= \pi\int_0^{\pi/4} \left[(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)\right] dx \\ &= \pi\int_0^{\pi/4} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\,dx \\ &= \pi\int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x)\,dx = \pi\left[2\sin x + 2\cos x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= \pi\left[\left(\sqrt{2} + \sqrt{2} - \tfrac{1}{2}\right) - (0 + 2 - 0)\right] = \pi\left(2\sqrt{2} - \tfrac{5}{2}\right) \end{aligned}$$



65. $s = f(t) = \int_0^t \sin\omega u\,\cos^2\omega u\,du$. Let $y = \cos\omega u \quad\Rightarrow\quad dy = -\omega\sin\omega u\,du$. Then

$s = -\frac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos\omega t} = \frac{1}{3\omega}(1 - \cos^3\omega t)$.

67. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\,dx = \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

69. $\int_{-\pi}^{\pi} \cos mx\,\cos nx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\,dx$.

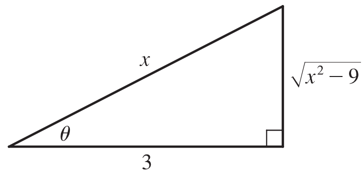
If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} + \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m+n)x]\,dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\dfrac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$.

7.3 Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and

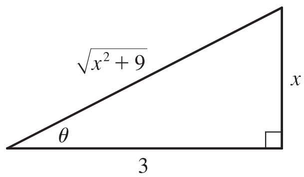


$$\begin{aligned}\sqrt{x^2-9} &= \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}\\ &= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.\end{aligned}$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta\cdot 3\tan\theta}\,3\sec\theta\,\tan\theta\,d\theta = \tfrac{1}{9}\int \cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and



$$\begin{aligned}\sqrt{x^2+9} &= \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta + 1)} = \sqrt{9\sec^2\theta}\\ &= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

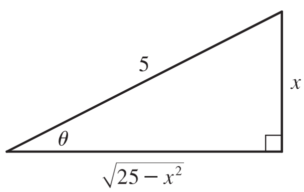
$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= \int \frac{3^3\tan^3\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\,\sec\theta d\theta = 3^3\int \tan^2\theta\,\tan\theta\,\sec\theta\,d\theta\\ &= 3^3\int(\sec^2\theta - 1)\,\tan\theta\,\sec\theta\,d\theta = 3^3\int(u^2-1)\,du \qquad [u = \sec\theta,\, du = \sec\theta\,\tan\theta\,d\theta]\\ &= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{(x^2+9)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C\\ &= \tfrac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C \quad\text{or}\quad \tfrac{1}{3}(x^2-18)\sqrt{x^2+9} + C\end{aligned}$$

5. Let $t = \sec\theta$, so $dt = \sec\theta\,\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

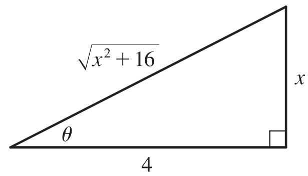
$$\begin{aligned}\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt &= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta\\ &= \int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}\\ &= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}\end{aligned}$$

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then

$$\begin{aligned}\int \frac{1}{x^2\sqrt{25-x^2}}\,dx &= \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}\,5\cos\theta\,d\theta = \frac{1}{25}\int \csc^2\theta\,d\theta\\ &= -\frac{1}{25}\cot\theta + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C\end{aligned}$$



9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\, d\theta$ and



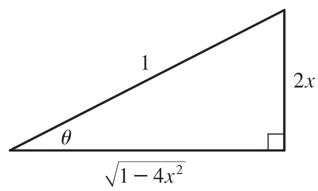
$$\begin{aligned}\sqrt{x^2+16} &= \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta+1)}\\ &= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|\\ &= 4\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1$$

$$= \ln\left|\sqrt{x^2+16} + x\right| - \ln|4| + C_1 = \ln\left(\sqrt{x^2+16} + x\right) + C, \text{ where } C = C_1 - \ln 4.$$

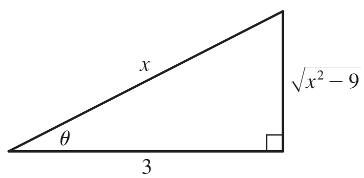
(Since $\sqrt{x^2+16} + x > 0$, we don't need the absolute value.)

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\, d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\begin{aligned}\textstyle\int \sqrt{1-4x^2}\, dx &= \textstyle\int \cos\theta\left(\frac{1}{2}\cos\theta\right) d\theta = \frac{1}{4}\int (1+\cos 2\theta)\, d\theta\\ &= \textstyle\frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{4}(\theta + \sin\theta\,\cos\theta) + C\\ &= \textstyle\frac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\,\right] + C\end{aligned}$$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\,\tan\theta\, d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\, dx = \int \frac{3\tan\theta}{27\sec^3\theta}\, 3\sec\theta\,\tan\theta\, d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta} d\theta$$

$$= \textstyle\frac{1}{3}\int \sin^2\theta\, d\theta = \frac{1}{3}\int \frac{1}{2}(1-\cos 2\theta)\, d\theta = \frac{1}{6}\theta - \frac{1}{12}\sin 2\theta + C = \frac{1}{6}\theta - \frac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

15. Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$ and $x = a \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

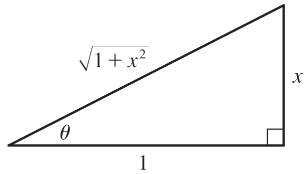
$$\textstyle\int_0^a x^2\sqrt{a^2-x^2}\, dx = \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\, a\cos\theta\, d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\, d\theta = a^4\int_0^{\pi/2}\left[\frac{1}{2}(2\sin\theta\,\cos\theta)\right]^2 d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\, d\theta = \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{8}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{\pi}{16}a^4$$

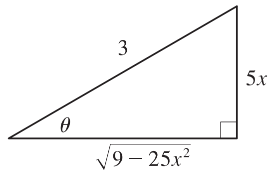
17. Let $u = x^2 - 7$, so $du = 2x\, dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



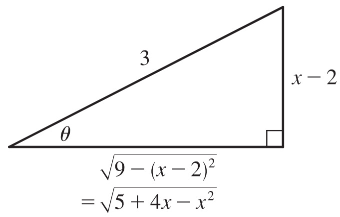
$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$
$$= \int(\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$
$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.41]}$$
$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then



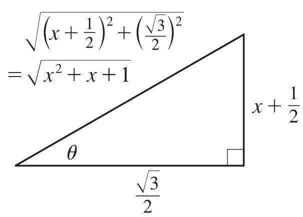
$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2} \frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$
$$= \tfrac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$
$$= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi$$

23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let $x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\,d\theta$. Then



$$\int\sqrt{5+4x-x^2}\,dx = \int\sqrt{9-(x-2)^2}\,dx = \int\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta$$
$$= \int\sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta$$
$$= \tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$
$$= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C$$

25. $x^2 + x + 1 = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$. Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sec\theta$. Then



[continued]

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$

$$= \int\left(\tfrac{\sqrt{3}}{2}\tan\theta - \tfrac{1}{2}\right)\sec\theta\,d\theta = \int \tfrac{\sqrt{3}}{2}\tan\theta\,\sec\theta\,d\theta - \int \tfrac{1}{2}\sec\theta\,d\theta$$

$$= \tfrac{\sqrt{3}}{2}\sec\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C_1$$

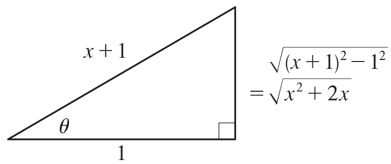
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right| + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1} + \left(x+\tfrac{1}{2}\right)\right]\right| + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C_1$$

$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C, \quad \text{where } C = C_1 - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}}$$

27. $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x+1)^2 - 1$. Let $x + 1 = 1\sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x} = \tan\theta$. Then



$$\int \sqrt{x^2+2x}\,dx = \int \tan\theta\,(\sec\theta\,\tan\theta\,d\theta) = \int \tan^2\theta\,\sec\theta\,d\theta$$

$$= \int(\sec^2\theta - 1)\,\sec\theta\,d\theta = \int \sec^3\theta\,d\theta - \int\sec\theta\,d\theta$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| + C$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C = \tfrac{1}{2}(x+1)\sqrt{x^2+2x} - \tfrac{1}{2}\ln\left|x + 1 + \sqrt{x^2+2x}\right| + C$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int\sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\,d\theta,\\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\,\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7\frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha\frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha\tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha(\sec^2\theta - 1)\,d\theta = \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\,\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$.

Let $x = r\cos u \;\Rightarrow\; dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

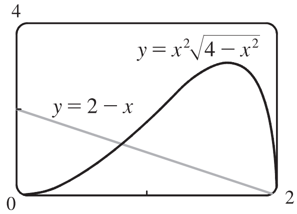
$$\begin{aligned}\int\sqrt{r^2-x^2}\,dx &= \int r\sin u\,(-r\sin u)\,du = -r^2\int\sin^2 u\,du = -\tfrac{1}{2}r^2(u-\sin u\,\cos u)+C\\ &= -\tfrac{1}{2}r^2\cos^{-1}(x/r)+\tfrac{1}{2}x\sqrt{r^2-x^2}+C\end{aligned}$$

so

$$\begin{aligned}\text{area of region } PQR &= \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r)+x\sqrt{r^2-x^2}\,\right]_{r\cos\theta}^{r}\\ &= \tfrac{1}{2}\left[0-(-r^2\theta+r\cos\theta\,r\sin\theta)\right] = \tfrac{1}{2}r^2\theta-\tfrac{1}{2}r^2\sin\theta\,\cos\theta\end{aligned}$$

and thus, (area of sector POR) $=$ (area of $\triangle POQ$) $+$ (area of region PQR) $= \frac{1}{2}r^2\theta$.

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2-x$ intersect at about $x = a \approx 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2-x$ on $(a,2)$. So the area bounded by the curve and the line is



$A \approx \int_a^2\left[x^2\sqrt{4-x^2}-(2-x)\right]dx = \int_a^2 x^2\sqrt{4-x^2}\,dx - \left[2x-\frac{1}{2}x^2\right]_a^2$.

To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Then

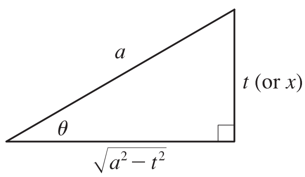
$dx = 2\cos\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = a \;\Rightarrow\; \theta = \alpha = \sin^{-1}(a/2) \approx 0.416$. So

$$\begin{aligned}\int_a^2 x^2\sqrt{4-x^2}\,dx &\approx \int_\alpha^{\pi/2}4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_\alpha^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_\alpha^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta\\ &= 2\left[\theta-\tfrac{1}{4}\sin 4\theta\right]_\alpha^{\pi/2} = 2\left[\left(\tfrac{\pi}{2}-0\right)-\left(\alpha-\tfrac{1}{4}(0.996)\right)\right]\approx 2.81\end{aligned}$$

Thus, $A \approx 2.81 - \left[\left(2\cdot 2-\frac{1}{2}\cdot 2^2\right)-\left(2a-\frac{1}{2}a^2\right)\right] \approx 2.10$.

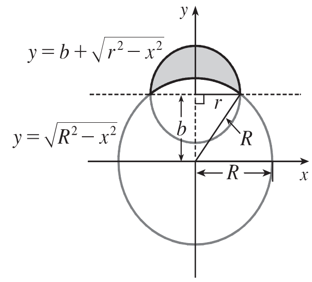
39. (a) Let $t = a\sin\theta$, $dt = a\cos\theta\,d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$ and $t = x \;\Rightarrow\; \theta = \sin^{-1}(x/a)$. Then

$$\begin{aligned}\int_0^x\sqrt{a^2-t^2}\,dt &= \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\,d\theta)\\ &= a^2\int_0^{\sin^{-1}(x/a)}\cos^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)}(1+\cos 2\theta)\,d\theta\\ &= \frac{a^2}{2}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\left[\theta+\sin\theta\,\cos\theta\right]_0^{\sin^{-1}(x/a)}\\ &= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right)-0\right]\\ &= \tfrac{1}{2}a^2\sin^{-1}(x/a)+\tfrac{1}{2}x\sqrt{a^2-x^2}\end{aligned}$$



(b) The integral $\int_0^x\sqrt{a^2-t^2}\,dt$ represents the area under the curve $y = \sqrt{a^2-t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2+y^2 = a^2$. The triangular region has base x and height $\sqrt{a^2-x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2-x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$.

41. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is



$$\begin{aligned} A &= \int_{-r}^{r} \left[\left(b + \sqrt{r^2 - x^2}\right) - \sqrt{R^2 - x^2}\,\right] dx \\ &= 2\int_0^r \left(b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}\,\right) dx \\ &= 2\int_0^r b\,dx + 2\int_0^r \sqrt{r^2 - x^2}\,dx - 2\int_0^r \sqrt{R^2 - x^2}\,dx \end{aligned}$$

The first integral is just $2br = 2r\sqrt{R^2 - r^2}$. The second integral represents the area of a quarter-circle of radius r, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\begin{aligned} \int \sqrt{a^2 - x^2}\,dx &= \int a^2 \cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \left(\tfrac{1}{2}a^2\right)\int (1 + \cos 2\theta)\,d\theta \\ &= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\,\cos\theta) + C \\ &= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C \end{aligned}$$

Thus, the desired area is

$$\begin{aligned} A &= 2r\sqrt{R^2 - r^2} + 2\left(\tfrac{1}{4}\pi r^2\right) - \left[R^2 \arcsin(x/R) + x\sqrt{R^2 - x^2}\,\right]_0^r \\ &= 2r\sqrt{R^2 - r^2} + \tfrac{1}{2}\pi r^2 - \left[R^2\arcsin(r/R) + r\sqrt{R^2 - r^2}\,\right] = r\sqrt{R^2 - r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R) \end{aligned}$$

43. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \quad\Rightarrow\quad x = \pm\sqrt{r^2 - (y-R)^2}$, so $g(y) = 2\sqrt{r^2 - (y-R)^2}$ and

$$\begin{aligned} V &= \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u + R)\sqrt{r^2 - u^2}\,du \qquad [\text{where } u = y - R] \\ &= 4\pi\int_{-r}^{r} u\sqrt{r^2 - u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2 - u^2}\,du \qquad \begin{bmatrix}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{bmatrix} \\ &= 4\pi\left[-\tfrac{1}{3}(r^2 - u^2)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0 - 0) + 4\pi R r^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta \\ &= 2\pi R r^2\int_{-\pi/2}^{\pi/2}(1 + \cos 2\theta)\,d\theta = 2\pi R r^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2 \end{aligned}$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2 - y^2}\,dy$ as in Exercise 6.2.63(a), but evaluate the integral using $y = r\sin\theta$.

7.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

3. (a) $\dfrac{x^4+1}{x^5+4x^3} = \dfrac{x^4+1}{x^3\,(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+4}$

(b) $\dfrac{1}{(x^2-9)^2} = \dfrac{1}{[(x+3)(x-3)]^2} = \dfrac{1}{(x+3)^2(x-3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2}$

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ [or use long division] $= 1 + \dfrac{1}{(x^2-1)(x^2+1)}$

$$= 1 + \frac{1}{(x-1)(x+1)(x^2+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+4} + \dfrac{Et+F}{(t^2+4)^2}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}dx = \int\left(1 + \frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2) + B(x+5)\,(*)$, or equivalently, $x-9 = (A+B)x - 2A + 5B$. Equating coefficients of x on each side of the equation gives us $1 = A + B$ (**1**) and equating constants gives us $-9 = -2A + 5B$ (**2**). Adding two times (**1**) to (**2**) gives us $-7 = 7B \;\Leftrightarrow\; B = -1$ and hence, $A = 2$. [Alternatively, to find the coefficients A and B, we may use substitution as follows: substitute 2 for x in $(*)$ to get $-7 = 7B \;\Leftrightarrow\; B = -1$, then substitute -5 for x in $(*)$ to get $-14 = -7A \;\Leftrightarrow\; A = 2$.] Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int\left(\frac{2}{x+5} + \frac{-1}{x-2}\right)dx = 2\ln|x+5| - \ln|x-2| + C.$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1) + B(x+1)$.

Substituting 1 for x gives $1 = 2B \;\Leftrightarrow\; B = \frac{1}{2}$. Substituting -1 for x gives $1 = -2A \;\Leftrightarrow\; A = -\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3\left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1|\right]_2^3$$

$$= \left(-\tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2\right) - \left(-\tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2 + \ln 3 - \ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

15. $\dfrac{x^3-2x^2-4}{x^3-2x^2}=1+\dfrac{-4}{x^2(x-2)}$. Write $\dfrac{-4}{x^2(x-2)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-2}$. Multiplying both sides by $x^2(x-2)$ gives $-4=Ax(x-2)+B(x-2)+Cx^2$. Substituting 0 for x gives $-4=-2B \iff B=2$. Substituting 2 for x gives $-4=4C \iff C=-1$. Equating coefficients of x^2, we get $0=A+C$, so $A=1$. Thus,

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx=\int_3^4\left(1+\frac{1}{x}+\frac{2}{x^2}-\frac{1}{x-2}\right)dx=\left[x+\ln|x|-\frac{2}{x}-\ln|x-2|\right]_3^4$$
$$=\left[\left(4+\ln 4-\tfrac{1}{2}-\ln 2\right)-\left(3+\ln 3-\tfrac{2}{3}-0\right)\right]=\tfrac{7}{6}+\ln\tfrac{2}{3}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)}=\dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12=A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy=\int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy=\left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$=2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2$$
$$=2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{9}{5}(3\ln 2-\ln 3)=\tfrac{9}{5}\ln\tfrac{8}{3}$$

19. $\dfrac{1}{(x+5)^2(x-1)}=\dfrac{A}{x+5}+\dfrac{B}{(x+5)^2}+\dfrac{C}{x-1} \quad\Rightarrow\quad 1=A(x+5)(x-1)+B(x-1)+C(x+5)^2$.

Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1=A(3)(-3)+B(-3)+C(3^2)=-9A-3B+9C=-9A+\frac{1}{2}+\frac{1}{4}=-9A+\frac{3}{4}$, so $9A=-\frac{1}{4}$ and $A=-\frac{1}{36}$. Now

$$\int\frac{1}{(x+5)^2(x-1)}\,dx=\int\left[\frac{-1/36}{x+5}-\frac{1/6}{(x+5)^2}+\frac{1/36}{x-1}\right]dx=-\frac{1}{36}\ln|x+5|+\frac{1}{6(x+5)}+\frac{1}{36}\ln|x-1|+C.$$

21.

$$\begin{array}{r|l} & \quad\quad\quad\quad\; x \\ x^2+4 & x^3+0x^2+0x+4 \\ & x^3 \quad\quad\;\; +4x \\ \hline & \quad\quad\;\; -4x+4 \end{array}$$

By long division, $\dfrac{x^3+4}{x^2+4}=x+\dfrac{-4x+4}{x^2+4}$. Thus,

$$\int\frac{x^3+4}{x^2+4}\,dx=\int\left(x+\frac{-4x+4}{x^2+4}\right)dx=\int\left(x-\frac{4x}{x^2+4}+\frac{4}{x^2+2^2}\right)dx$$
$$=\frac{1}{2}x^2-4\cdot\frac{1}{2}\ln\left|x^2+4\right|+4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C=\frac{1}{2}x^2-2\ln(x^2+4)+2\tan^{-1}\left(\frac{x}{2}\right)+C$$

23. $\dfrac{5x^2+3x-2}{x^3+2x^2}=\dfrac{5x^2+3x-2}{x^2(x+2)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2=Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C \quad\Rightarrow\quad A=2$. So

$$\int\frac{5x^2+3x-2}{x^3+2x^2}\,dx=\int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x+2}\right)dx=2\ln|x|+\frac{1}{x}+3\ln|x+2|+C.$$

25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)\left(x^2+9\right)$ to get

$10 = A\left(x^2+9\right) + (Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10 = 10A \quad \Leftrightarrow \quad A = 1$. Substituting 0 for x gives $10 = 9A - C \quad \Rightarrow \quad C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0 = A + B \quad \Rightarrow$ $B = -1$. Thus,

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right) dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right) dx \\ &= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) + C\end{aligned}$$

In the second term we used the substitution $u = x^2 + 9$ and in the last term we used Formula 10.

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $\left(x^2+1\right)\left(x^2+2\right)$ to get

$x^3+x^2+2x+1 = (Ax+B)\left(x^2+2\right) + (Cx+D)\left(x^2+1\right) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = \left(Ax^3+Bx^2+2Ax+2B\right) + \left(Cx^3+Dx^2+Cx+D\right) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$\begin{aligned} A + C &= 1 \quad \textbf{(1)} \qquad & B + D &= 1 \quad \textbf{(2)} \\ 2A + C &= 2 \quad \textbf{(3)} \qquad & 2B + D &= 1 \quad \textbf{(4)} \end{aligned}$$

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$, so $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 0$, so $D = 1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+2}\right) dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u = x^2+1$ so $du = 2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$. For $\displaystyle\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+\left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln\left(x^2+1\right) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

29.
$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}\,dx &= \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4} \\ &= \frac{1}{2}\ln\left|x^2+2x+5\right| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x+1 = 2u, \\ \text{and } dx = 2\,du\end{array}\right] \\ &= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C\end{aligned}$$

31. $\dfrac{1}{x^3-1} = \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1} \quad \Rightarrow \quad 1 = A\left(x^2+x+1\right) + (Bx+C)(x-1)$.

Take $x = 1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$,

so $B = -\frac{1}{3}$, $C = -\frac{2}{3}$ $\Rightarrow$

$$\begin{aligned}\int \frac{1}{x^3-1}\,dx &= \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx \\ &= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}\,dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4} \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln\!\left(x^2+x+1\right) - \tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\!\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\!\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K\end{aligned}$$

33. Let $u = x^4 + 4x^2 + 3$, so that $du = (4x^3 + 8x)\,dx = 4(x^3 + 2x)\,dx$, $x = 0 \Rightarrow u = 3$, and $x = 1 \Rightarrow u = 8$.

Then $\displaystyle\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx = \int_3^8 \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\Big[\ln|u|\Big]_3^8 = \frac{1}{4}(\ln 8 - \ln 3) = \frac{1}{4}\ln\frac{8}{3}.$

35. $\displaystyle\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2} \Rightarrow 1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$. Setting $x = 0$

gives $1 = 16A$, so $A = \frac{1}{16}$. Now compare coefficients.

$$\begin{aligned}1 &= \tfrac{1}{16}(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2+4) + Dx^2 + Ex \\ 1 &= \tfrac{1}{16}x^4 + \tfrac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex \\ 1 &= \left(\tfrac{1}{16} + B\right)x^4 + Cx^3 + \left(\tfrac{1}{2} + 4B + D\right)x^2 + (4C+E)x + 1\end{aligned}$$

So $B + \frac{1}{16} = 0 \Rightarrow B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \Rightarrow D = -\frac{1}{4}$, and $4C + E = 0 \Rightarrow E = 0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2+4)^2} &= \int\left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2}\right)dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln\left|x^2+4\right| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C \\ &= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C\end{aligned}$$

37. $\displaystyle\frac{x^2-3x+7}{(x^2-4x+6)^2} = \frac{Ax+B}{x^2-4x+6} + \frac{Cx+D}{(x^2-4x+6)^2} \Rightarrow x^2 - 3x + 7 = (Ax+B)(x^2-4x+6) + Cx + D \Rightarrow$

$x^2 - 3x + 7 = Ax^3 + (-4A + B)x^2 + (6A - 4B + C)x + (6B + D)$. So $A = 0$, $-4A + B = 1 \Rightarrow B = 1$,

$6A - 4B + C = -3 \Rightarrow C = 1$, $6B + D = 7 \Rightarrow D = 1$. Thus,

$$\begin{aligned}I &= \int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx = \int\left(\frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2}\right)dx \\ &= \int \frac{1}{(x-2)^2+2}\,dx + \int \frac{x-2}{(x^2-4x+6)^2}\,dx + \int \frac{3}{(x^2-4x+6)^2}\,dx \\ &= I_1 + I_2 + I_3.\end{aligned}$$

[continued]

$$I_1 = \int \frac{1}{(x-2)^2 + \left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + C_1$$

$$I_2 = \frac{1}{2}\int \frac{2x-4}{(x^2-4x+6)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\left(-\frac{1}{u}\right) + C_2 = -\frac{1}{2(x^2-4x+6)} + C_2$$

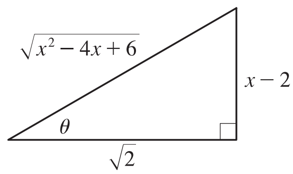
$$I_3 = 3\int \frac{1}{\left[(x-2)^2 + \left(\sqrt{2}\right)^2\right]^2}\,dx = 3\int \frac{1}{[2(\tan^2\theta+1)]^2}\,\sqrt{2}\sec^2\theta\,d\theta \qquad \begin{bmatrix} x-2=\sqrt{2}\tan\theta, \\ dx=\sqrt{2}\sec^2\theta\,d\theta \end{bmatrix}$$

$$= \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{3\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_3 = \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_3$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\cdot\frac{x-2}{\sqrt{x^2-4x+6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2-4x+6}} + C_3$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C_3$$



So $I = I_1 + I_2 + I_3 \qquad [C = C_1 + C_2 + C_3]$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-1}{2(x^2-4x+6)} + \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C$$

$$= \left(\frac{4\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}\right)\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)-2}{4(x^2-4x+6)} + C = \frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$$

39. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u\,du \Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u\,du}{(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

41. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1 + \frac{4}{u^2-4}\right)du \qquad \text{[by long division]}$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)} \quad (\star)$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(\star)$ is

$$2 + 8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right)du = 2 + 8\Big[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\Big]_3^4 = 2 + \Big[2\ln|u-2| - 2\ln|u+2|\Big]_3^4$$

$$= 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 = 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5}$$

$$= 2 + 2\ln\tfrac{5}{3} \text{ or } 2 + \ln\left(\tfrac{5}{3}\right)^2 = 2 + \ln\tfrac{25}{9}$$

43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4 - u)\,du$$
$$= \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$
$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]}$$
$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u} \quad \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} = \int \frac{u^2\,(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du$$
$$= 2\ln|u+2| - \ln|u+1| + C = \ln\frac{(e^x+2)^2}{e^x+1} + C$$

49. Let $u = \tan t$, so that $du = \sec^2 t\,dt$. Then $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt = \int \frac{1}{u^2 + 3u + 2}\,du = \int \frac{1}{(u+1)(u+2)}\,du.$

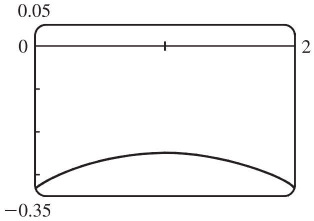
Now $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2} \quad \Rightarrow \quad 1 = A(u+2) + B(u+1).$

Setting $u = -2$ gives $1 = -B$, so $B = -1$. Setting $u = -1$ gives $1 = A$.

Thus, $\displaystyle\int \frac{1}{(u+1)(u+2)}\,du = \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \ln|u+1| - \ln|u+2| + C = \ln|\tan t + 1| - \ln|\tan t + 2| + C.$

51. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2 - x + 2}\,dx$, $v = x$, and (by integration by parts)

$$\int \ln(x^2 - x + 2)\,dx = x\ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2}\,dx = x\ln(x^2 - x + 2) - \int \left(2 + \frac{x-4}{x^2 - x + 2}\right) dx$$
$$= x\ln(x^2 - x + 2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2 - x + 2}\,dx + \frac{7}{2}\int \frac{dx}{(x - \frac{1}{2})^2 + \frac{7}{4}}$$
$$= x\ln(x^2 - x + 2) - 2x - \frac{1}{2}\ln(x^2 - x + 2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x - \frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right]$$
$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1} u + C$$
$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C$$

53. 

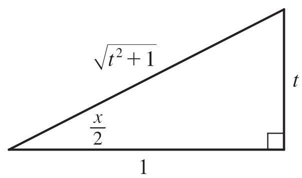
From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2\cdot 0.3)=-0.6$. Now $\dfrac{1}{x^2-2x-3}=\dfrac{1}{(x-3)(x+1)}=\dfrac{A}{x-3}+\dfrac{B}{x+1}\quad\Leftrightarrow$ $1=(A+B)x+A-3B$, so $A=-B$ and $A-3B=1\quad\Leftrightarrow\quad A=\frac{1}{4}$ and $B=-\frac{1}{4}$, so the integral becomes

$$\int_0^2\frac{dx}{x^2-2x-3}=\frac{1}{4}\int_0^2\frac{dx}{x-3}-\frac{1}{4}\int_0^2\frac{dx}{x+1}=\frac{1}{4}\Big[\ln|x-3|-\ln|x+1|\Big]_0^2=\frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2$$
$$=\tfrac{1}{4}\left(\ln\tfrac{1}{3}-\ln 3\right)=-\tfrac{1}{2}\ln 3\approx-0.55$$

55. $$\int\frac{dx}{x^2-2x}=\int\frac{dx}{(x-1)^2-1}=\int\frac{du}{u^2-1}\qquad[\text{put } u=x-1]$$
$$=\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C\quad[\text{by Equation 6}]\quad=\frac{1}{2}\ln\left|\frac{x-2}{x}\right|+C$$

57. (a) If $t=\tan\left(\dfrac{x}{2}\right)$, then $\dfrac{x}{2}=\tan^{-1}t$. The figure gives

$$\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}}\text{ and }\sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}.$$



(b) $$\cos x=\cos\left(2\cdot\frac{x}{2}\right)=2\cos^2\left(\frac{x}{2}\right)-1$$
$$=2\left(\frac{1}{\sqrt{1+t^2}}\right)^2-1=\frac{2}{1+t^2}-1=\frac{1-t^2}{1+t^2}$$

(c) $$\frac{x}{2}=\arctan t\quad\Rightarrow\quad x=2\arctan t\quad\Rightarrow\quad dx=\frac{2}{1+t^2}\,dt$$

59. Let $t=\tan(x/2)$. Then, using the expressions in Exercise 57, we have

$$\int\frac{1}{3\sin x-4\cos x}\,dx=\int\frac{1}{3\left(\dfrac{2t}{1+t^2}\right)-4\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2}=2\int\frac{dt}{3(2t)-4(1-t^2)}=\int\frac{dt}{2t^2+3t-2}$$
$$=\int\frac{dt}{(2t-1)(t+2)}=\int\left[\frac{2}{5}\frac{1}{2t-1}-\frac{1}{5}\frac{1}{t+2}\right]dt\qquad[\text{using partial fractions}]$$
$$=\tfrac{1}{5}\Big[\ln|2t-1|-\ln|t+2|\Big]+C=\frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right|+C=\frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right|+C$$

61. Let $t=\tan(x/2)$. Then, by Exercise 57,

$$\int_0^{\pi/2}\frac{\sin 2x}{2+\cos x}\,dx=\int_0^{\pi/2}\frac{2\sin x\,\cos x}{2+\cos x}\,dx=\int_0^1\frac{2\cdot\dfrac{2t}{1+t^2}\cdot\dfrac{1-t^2}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt=\int_0^1\frac{\dfrac{8t(1-t^2)}{(1+t^2)^2}}{2(1+t^2)+(1-t^2)}\,dt$$
$$=\int_0^1 8t\cdot\frac{1-t^2}{(t^2+3)(t^2+1)^2}\,dt=I$$

If we now let $u = t^2$, then $\dfrac{1-t^2}{(t^2+3)(t^2+1)^2} = \dfrac{1-u}{(u+3)(u+1)^2} = \dfrac{A}{u+3} + \dfrac{B}{u+1} + \dfrac{C}{(u+1)^2} \quad\Rightarrow$

$1 - u = A(u+1)^2 + B(u+3)(u+1) + C(u+3)$. Set $u = -1$ to get $2 = 2C$, so $C = 1$. Set $u = -3$ to get $4 = 4A$, so $A = 1$. Set $u = 0$ to get $1 = 1 + 3B + 3$, so $B = -1$. So

$$\begin{aligned} I &= \int_0^1 \left[\frac{8t}{t^2+3} - \frac{8t}{t^2+1} + \frac{8t}{(t^2+1)^2}\right] dt = \left[4\ln(t^2+3) - 4\ln(t^2+1) - \frac{4}{t^2+1}\right]_0^1 \\ &= (4\ln 4 - 4\ln 2 - 2) - (4\ln 3 - 0 - 4) = 8\ln 2 - 4\ln 2 - 4\ln 3 + 2 = 4\ln\tfrac{2}{3} + 2 \end{aligned}$$

63. By long division, $\dfrac{x^2+1}{3x-x^2} = -1 + \dfrac{3x+1}{3x-x^2}$. Now

$\dfrac{3x+1}{3x-x^2} = \dfrac{3x+1}{x(3-x)} = \dfrac{A}{x} + \dfrac{B}{3-x} \quad\Rightarrow\quad 3x+1 = A(3-x) + Bx$. Set $x = 3$ to get $10 = 3B$, so $B = \frac{10}{3}$. Set $x = 0$ to get $1 = 3A$, so $A = \frac{1}{3}$. Thus, the area is

$$\begin{aligned} \int_1^2 \frac{x^2+1}{3x-x^2}\,dx &= \int_1^2 \left(-1 + \frac{\frac{1}{3}}{x} + \frac{\frac{10}{3}}{3-x}\right)dx = \left[-x + \tfrac{1}{3}\ln|x| - \tfrac{10}{3}\ln|3-x|\right]_1^2 \\ &= \left(-2 + \tfrac{1}{3}\ln 2 - 0\right) - \left(-1 + 0 - \tfrac{10}{3}\ln 2\right) = -1 + \tfrac{11}{3}\ln 2 \end{aligned}$$

65. $\dfrac{P+S}{P\,[(r-1)P-S]} = \dfrac{A}{P} + \dfrac{B}{(r-1)P-S} \quad\Rightarrow\quad P + S = A\,[(r-1)P - S] + BP = [(r-1)A + B]\,P - AS \quad\Rightarrow$

$(r-1)A + B = 1,\ -A = 1 \quad\Rightarrow\quad A = -1,\ B = r$. Now

$$t = \int \frac{P+S}{P\,[(r-1)P-S]}\,dP = \int \left[\frac{-1}{P} + \frac{r}{(r-1)P-S}\right] dP = -\int \frac{dP}{P} + \frac{r}{r-1}\int \frac{r-1}{(r-1)P-S}\,dP$$

so $t = -\ln P + \dfrac{r}{r-1}\ln|(r-1)P - S| + C$. Here $r = 0.10$ and $S = 900$, so

$t = -\ln P + \frac{0.1}{-0.9}\ln|-0.9P - 900| + C = -\ln P - \frac{1}{9}\ln(|-1|\,|0.9P + 900|) = -\ln P - \frac{1}{9}\ln(0.9P + 900) + C$.

When $t = 0$, $P = 10{,}000$, so $0 = -\ln 10{,}000 - \frac{1}{9}\ln(9900) + C$. Thus, $C = \ln 10{,}000 + \frac{1}{9}\ln 9900$ [≈ 10.2326], so our equation becomes

$$\begin{aligned} t &= \ln 10{,}000 - \ln P + \tfrac{1}{9}\ln 9900 - \tfrac{1}{9}\ln(0.9P + 900) = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{9900}{0.9P + 900} \\ &= \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{1100}{0.1P + 100} = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{11{,}000}{P + 1000} \end{aligned}$$

67. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $\displaystyle\int f(x)\,dx = \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$

$$+\frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx + C$$

$$= \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$

$$+\tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right)+37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right]+C$$

$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$

$$+\tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right]+C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879}-\frac{334\ln(2x+1)}{323}-\frac{3146\ln(3x-7)}{80{,}155}$$
$$+\frac{11{,}049\ln(x^2+x+5)}{260{,}015}+\frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

69. There are only finitely many values of x where $Q(x)=0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x)=G(x)/Q(x)$, so $F(x)=G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a)=0$, then $Q(x)\neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a}F(x) && \text{[by continuity of } F\text{]}\\ &= \lim_{x\to a}G(x) && \text{[whenever } Q(x)\neq 0\text{]}\\ &= G(a) && \text{[by continuity of } G\text{]}\end{aligned}$$

7.5 Strategy for Integration

1. Let $u=\sin x$, so that $du=\cos x\,dx$. Then $\int\cos x(1+\sin^2 x)\,dx=\int(1+u^2)\,du=u+\frac{1}{3}u^3+C=\sin x+\frac{1}{3}\sin^3 x+C$.

3. $\displaystyle\int\frac{\sin x+\sec x}{\tan x}\,dx=\int\left(\frac{\sin x}{\tan x}+\frac{\sec x}{\tan x}\right)dx=\int(\cos x+\csc x)\,dx=\sin x+\ln|\csc x-\cot x|+C$

5. $\displaystyle\int_0^2\frac{2t}{(t-3)^2}\,dt=\int_{-3}^{-1}\frac{2(u+3)}{u^2}\,du\quad\begin{bmatrix}u=t-3,\\ du=dt\end{bmatrix}\quad=\int_{-3}^{-1}\left(\frac{2}{u}+\frac{6}{u^2}\right)du=\left[2\ln|u|-\frac{6}{u}\right]_{-3}^{-1}$

$$=(2\ln 1+6)-(2\ln 3+2)=4-2\ln 3\ \text{ or }\ 4-\ln 9$$

7. Let $u=\arctan y$. Then $du=\dfrac{dy}{1+y^2}$ $\Rightarrow$ $\displaystyle\int_{-1}^{1}\frac{e^{\arctan y}}{1+y^2}\,dy=\int_{-\pi/4}^{\pi/4}e^u\,du=\left[e^u\right]_{-\pi/4}^{\pi/4}=e^{\pi/4}-e^{-\pi/4}$.

9. $\int_1^3 r^4 \ln r\,dr \quad \left[\begin{array}{ll} u = \ln r, & dv = r^4\,dr, \\ du = \dfrac{dr}{r} & v = \frac{1}{5}r^5 \end{array}\right] = \left[\frac{1}{5}r^5 \ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$= \frac{243}{5}\ln 3 - \left(\frac{243}{25} - \frac{1}{25}\right) = \frac{243}{5}\ln 3 - \frac{242}{25}$

11. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \qquad [u = x-2,\ du = dx]$

$= \frac{1}{2}\ln(u^2+1) + \tan^{-1} u + C = \frac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$

13. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta \sin^2\theta \sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$= -\int u^5(1-u^2)\,du \quad \left[\begin{array}{l} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{array}\right]$

$= \int (u^7 - u^5)\,du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8\theta - \frac{1}{6}\cos^6\theta + C$

Another solution:

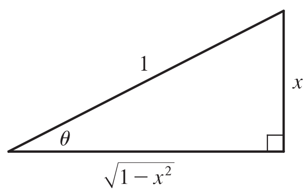
$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2 \cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

$= \int u^3(1-u^2)^2\,du \quad \left[\begin{array}{l} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{array}\right] \quad = \int u^3(1-2u^2+u^4)\,du$

$= \int (u^3 - 2u^5 + u^7)\,du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^6\theta + \frac{1}{8}\sin^8\theta + C$

15. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$



17. $\int x\sin^2 x\,dx \quad \left[\begin{array}{ll} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{array}\right]$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \int \left(\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right) dx$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C \quad$ [where $s = \sin x$, $ds = \cos x\,dx$].

A slightly different method is to write $\int x\sin^2 x\,dx = \int x \cdot \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x} e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with

$u = \arctan t$, $dv = 2t\,dt \quad \Rightarrow \quad du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int \left(1 - \frac{1}{1+t^2}\right) dt = t^2 \arctan t - t + \arctan t + C$$

$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \quad \Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8 \cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2 = \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}.$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \quad \Rightarrow \quad 6x+22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

27. Let $u = 1 + e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \quad \Rightarrow \quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C.$

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x} + 1$. This gives the answer in the form $-\ln(e^{-x}+1) + C$.

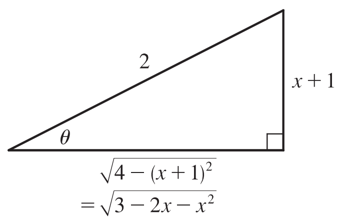
29.
$$\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right) dw = \Big[3w - 7\ln|w+2|\Big]_0^5 = 15 - 7\ln 7 + 7\ln 2$$
$$= 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\tfrac{2}{7}$$

31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

33. $3 - 2x - x^2 = -(x^2+2x+1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$$\begin{aligned}
\int \sqrt{3-2x-x^2}\,dx &= \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,d\theta \\
&= 4\int \cos^2\theta\,d\theta = 2\int (1+\cos 2\theta)\,d\theta \\
&= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2} + C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C
\end{aligned}$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8 \sin x\,dx = 0 \quad$ [by (5.5.7)(b)].

37. $\int_0^{\pi/4} \cos^2\theta\,\tan^2\theta\,d\theta = \int_0^{\pi/4} \sin^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0-0) = \frac{\pi}{8} - \frac{1}{4}$

39. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta = \int \frac{1}{u^2 - u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\displaystyle\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \quad\Rightarrow\quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1 - \cos\theta| + C$].

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \quad\Rightarrow\quad du = d\theta$ and $v = \tan\theta - \theta$. So

$$\begin{aligned}\int \theta\tan^2\theta\,d\theta &= \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C\\ &= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C\end{aligned}$$

43. Let $u = 1 + e^x$, so that $du = e^x\,dx$. Then $\int e^x\sqrt{1+e^x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then

$\int e^x\sqrt{1+e^x}\,dx = \int u \cdot 2u\,du = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

45. Let $t = x^3$. Then $dt = 3x^2\,dx \quad\Rightarrow\quad I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C$.

47. Let $u = x - 1$, so that $du = dx$. Then

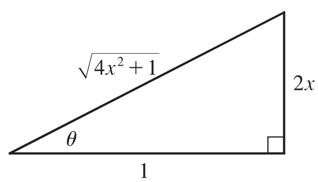
$$\begin{aligned}\int x^3(x-1)^{-4}\,dx &= \int(u+1)^3u^{-4}\,du = \int(u^3 + 3u^2 + 3u + 1)u^{-4}\,du = \int(u^{-1} + 3u^{-2} + 3u^{-3} + u^{-4})\,du\\ &= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C\end{aligned}$$

49. Let $u = \sqrt{4x+1} \quad\Rightarrow\quad u^2 = 4x + 1 \quad\Rightarrow\quad 2u\,du = 4\,dx \quad\Rightarrow\quad dx = \frac{1}{2}u\,du$. So

$$\begin{aligned}\int \frac{1}{x\sqrt{4x+1}}\,dx &= \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]}\\ &= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C\end{aligned}$$

51. Let $2x = \tan\theta \quad\Rightarrow\quad x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta\\ &= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C]\\ &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]\end{aligned}$$



53. $$\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx \qquad \left[\begin{matrix} u = x^2, & dv = \sinh(mx)\,dx,\\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx)\end{matrix}\right]$$

$$\begin{aligned}&= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \frac{1}{m}\int\sinh(mx)\,dx\right) \qquad \left[\begin{matrix} U = x, & dV = \cosh(mx)\,dx,\\ dU = dx & V = \frac{1}{m}\sinh(mx)\end{matrix}\right]\\ &= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C\end{aligned}$$

55. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then $\displaystyle\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u\,du}{u^2 + u^2 \cdot u} = \int \frac{2}{u(1+u)}\,du = I$.

Now $\dfrac{2}{u(1+u)} = \dfrac{A}{u} + \dfrac{B}{1+u} \quad\Rightarrow\quad 2 = A(1+u) + Bu$. Set $u = -1$ to get $2 = -B$, so $B = -2$. Set $u = 0$ to get $2 = A$.

Thus, $I = \displaystyle\int \left(\frac{2}{u} - \frac{2}{1+u}\right) du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\sqrt{x} - 2\ln\left(1 + \sqrt{x}\right) + C$.

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad\Rightarrow$

$$\int x\,\sqrt[3]{x+c}\,dx = \int (u^3 - c)u \cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C = \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

59. Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$\begin{aligned}\int \cos x\,\cos^3(\sin x)\,dx &= \int \cos^3 u\,du = \int \cos^2 u\,\cos u\,du = \int (1 - \sin^2 u)\cos u\,du \\ &= \int (\cos u - \sin^2 u\,\cos u)\,du = \sin u - \tfrac{1}{3}\sin^3 u + C = \sin(\sin x) - \tfrac{1}{3}\sin^3(\sin x) + C\end{aligned}$$

61. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\begin{aligned}\int \sqrt{x}\,e^{\sqrt{x}}\,dx &= \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \left[\begin{matrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{matrix}\right] \\ &= 2y^2e^y - \int 4ye^y\,dy \qquad \left[\begin{matrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{matrix}\right] \\ &= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C \\ &= 2(y^2 - 2y + 2)e^y + C = 2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C\end{aligned}$$

63. Let $u = \cos^2 x$, so that $du = 2\cos x\,(-\sin x)\,dx$. Then

$$\int \frac{\sin 2x}{1 + \cos^4 x}\,dx = \int \frac{2\sin x\,\cos x}{1 + (\cos^2 x)^2}\,dx = \int \frac{1}{1+u^2}\,(-du) = -\tan^{-1} u + C = -\tan^{-1}(\cos^2 x) + C.$$

65.
$$\begin{aligned}\int \frac{dx}{\sqrt{x+1} + \sqrt{x}} &= \int \left(\frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x\sqrt{x}}}{\sqrt{x+1} - \sqrt{x}}\right) dx = \int \left(\sqrt{x+1} - \sqrt{x}\right) dx \\ &= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C\end{aligned}$$

67. Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$, $x = \sqrt{3} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$, and $x = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. Then

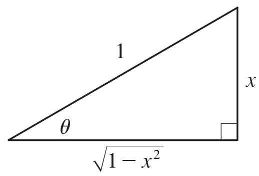
$$\begin{aligned}\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx &= \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\,\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\,(\tan^2\theta + 1)}{\tan^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \left(\frac{\sec\theta\,\tan^2\theta}{\tan^2\theta} + \frac{\sec\theta}{\tan^2\theta}\right) d\theta \\ &= \int_{\pi/4}^{\pi/3} (\sec\theta + \csc\theta\,\cot\theta)\,d\theta = \Big[\ln|\sec\theta + \tan\theta| - \csc\theta\Big]_{\pi/4}^{\pi/3} \\ &= \left(\ln\left|2 + \sqrt{3}\right| - \tfrac{2}{\sqrt{3}}\right) - \left(\ln\left|\sqrt{2} + 1\right| - \sqrt{2}\right) = \sqrt{2} - \tfrac{2}{\sqrt{3}} + \ln\left(2 + \sqrt{3}\right) - \ln\left(1 + \sqrt{2}\right)\end{aligned}$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\,\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right) du = u - \ln|1+u| + C = e^x - \ln(1+e^x) + C.$$

71. Let $\theta = \arcsin x$, so that $d\theta = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $x = \sin\theta$. Then

$$\int \frac{x + \arcsin x}{\sqrt{1-x^2}}\,dx = \int (\sin\theta + \theta)\,d\theta = -\cos\theta + \tfrac{1}{2}\theta^2 + C$$
$$= -\sqrt{1-x^2} + \tfrac{1}{2}(\arcsin x)^2 + C$$



73. $\dfrac{1}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} \quad \Rightarrow \quad 1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C).$

So $0 = A + B = C - 2B$, $1 = 4A - 2C$. Setting $x = 2$ gives $A = \frac{1}{8} \quad \Rightarrow \quad B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right) dx = \frac{1}{8}\int \frac{dx}{x-2} - \frac{1}{16}\int \frac{2x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$
$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

75. Let $y = \sqrt{1+e^x}$, so that $y^2 = 1 + e^x$, $2y\,dy = e^x\,dx$, $e^x = y^2 - 1$, and $x = \ln(y^2 - 1)$. Then

$$\int \frac{xe^x}{\sqrt{1+e^x}}\,dx = \int \frac{\ln(y^2-1)}{y}(2y\,dy) = 2\int [\ln(y+1) + \ln(y-1)]\,dy$$
$$= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C \qquad \text{[by Example 7.1.2]}$$
$$= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C$$
$$= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C$$
$$= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x}\,\ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C$$
$$= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \quad \Rightarrow \quad \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\,\cos x\,dx = \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int (1-\cos^2 x)\sin x\,dx$$
$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\,dy \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$$
$$= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2 e^{x^2}\,dx + \int e^{x^2}\,dx = \int x\Big(2xe^{x^2}\Big)\,dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \quad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \quad = xe^{x^2} + C$$

7.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution $u = \sqrt{2}\,x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$.

Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\int \frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx \overset{33}{=} \sqrt{2}\left[-\frac{1}{x}\sqrt{\tfrac{7}{2}-x^2} - \sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right] + C$$

$$= -\frac{1}{x}\sqrt{7-2x^2} - \sqrt{2}\,\sin^{-1}\!\left(\sqrt{\tfrac{2}{7}}\,x\right) + C$$

3. Let $u = \pi x \quad \Rightarrow \quad du = \pi\,dx$, so

$$\int \sec^3(\pi x)\,dx = \tfrac{1}{\pi}\int \sec^3 u\,du \overset{71}{=} \tfrac{1}{\pi}\left(\tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u|\right) + C$$

$$= \tfrac{1}{2\pi}\sec \pi x\tan \pi x + \tfrac{1}{2\pi}\ln|\sec \pi x + \tan \pi x| + C$$

5. $$\int_0^1 2x\cos^{-1}x\,dx \overset{91}{=} 2\left[\frac{2x^2-1}{4}\cos^{-1}x - \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\left[\left(\tfrac{1}{4}\cdot 0 - 0\right) - \left(-\tfrac{1}{4}\cdot\tfrac{\pi}{2} - 0\right)\right] = 2\left(\tfrac{\pi}{8}\right) = \tfrac{\pi}{4}$$

7. Let $u = \pi x$, so that $du = \pi\,dx$. Then

$$\int \tan^3(\pi x)\,dx = \int \tan^3 u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\int \tan^3 u\,du \overset{69}{=} \tfrac{1}{\pi}\left[\tfrac{1}{2}\tan^2 u + \ln|\cos u|\right] + C$$

$$= \tfrac{1}{2\pi}\tan^2(\pi x) + \tfrac{1}{\pi}\ln|\cos(\pi x)| + C$$

9. Let $u = 2x$ and $a = 3$. Then $du = 2\,dx$ and

$$\int \frac{dx}{x^2\sqrt{4x^2+9}} = \int \frac{\frac{1}{2}\,du}{\frac{u^2}{4}\sqrt{u^2+a^2}} = 2\int \frac{du}{u^2\sqrt{a^2+u^2}} \overset{28}{=} -2\,\frac{\sqrt{a^2+u^2}}{a^2 u} + C$$

$$= -2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x} + C = -\frac{\sqrt{4x^2+9}}{9x} + C$$

11. $$\int_{-1}^{0} t^2 e^{-t}\,dt \overset{97}{=} \left[\frac{1}{-1}t^2 e^{-t}\right]_{-1}^{0} - \frac{2}{-1}\int_{-1}^{0} te^{-t}\,dt = e + 2\int_{-1}^{0} te^{-t}\,dt \overset{96}{=} e + 2\left[\frac{1}{(-1)^2}\,(-t-1)\,e^{-t}\right]_{-1}^{0}$$

$$= e + 2\left[-e^0 + 0\right] = e - 2$$

13. $$\int \frac{\tan^3(1/z)}{z^2}\,dz \quad \begin{bmatrix} u = 1/z, \\ du = -dz/z^2 \end{bmatrix} \quad = -\int \tan^3 u\,du \overset{69}{=} -\tfrac{1}{2}\tan^2 u - \ln|\cos u| + C$$

$$= -\tfrac{1}{2}\tan^2\left(\tfrac{1}{z}\right) - \ln\left|\cos\left(\tfrac{1}{z}\right)\right| + C$$

15. Let $u = e^x$, so that $du = e^x\,dx$ and $e^{2x} = u^2$. Then

$$\int e^{2x}\arctan(e^x)\,dx = \int u^2\arctan u\left(\frac{du}{u}\right) = \int u\arctan u\,du$$

$$\overset{92}{=} \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C = \frac{1}{2}(e^{2x}+1)\arctan(e^x) - \frac{1}{2}e^x + C$$

17. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y-1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$. Then $z = a^2 - u^2$, $du = 2\,dy$, and

$$\begin{aligned}
\int y\sqrt{6+4y-4y^2}\,dy &= \int y\sqrt{z}\,dy = \int \tfrac{1}{2}(u+1)\sqrt{a^2-u^2}\,\tfrac{1}{2}\,du = \tfrac{1}{4}\int u\sqrt{a^2-u^2}\,du + \tfrac{1}{4}\int\sqrt{a^2-u^2}\,du \\
&= \tfrac{1}{4}\int\sqrt{a^2-u^2}\,du - \tfrac{1}{8}\int(-2u)\sqrt{a^2-u^2}\,du \\
&\overset{30}{=} \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{1}{8}\int\sqrt{w}\,dw \qquad \begin{bmatrix} w = a^2-u^2, \\ dw = -2u\,du \end{bmatrix} \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1) - \frac{1}{12}(6+4y-4y^2)\right] + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} + C \\
&\qquad = \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8}\right)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C \\
&\qquad = \frac{1}{24}(8y^2-2y-15)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C
\end{aligned}$$

19. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\begin{aligned}
\int \sin^2 x\,\cos x\,\ln(\sin x)\,dx &= \int u^2\ln u\,du \overset{101}{=} \frac{u^{2+1}}{(2+1)^2}[(2+1)\ln u - 1] + C = \tfrac{1}{9}u^3(3\ln u - 1) + C \\
&= \tfrac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C
\end{aligned}$$

21. Let $u = e^x$ and $a = \sqrt{3}$. Then $du = e^x\,dx$ and

$$\int\frac{e^x}{3-e^{2x}}\,dx = \int\frac{du}{a^2-u^2} \overset{19}{=} \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C = \frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C.$$

23. $\int\sec^5 x\,dx \overset{77}{=} \frac{1}{4}\tan x\,\sec^3 x + \frac{3}{4}\int\sec^3 x\,dx \overset{77}{=} \frac{1}{4}\tan x\,\sec^3 x + \frac{3}{4}\left(\frac{1}{2}\tan x\,\sec x + \frac{1}{2}\int\sec x\,dx\right)$

$$\overset{14}{=} \tfrac{1}{4}\tan x\,\sec^3 x + \tfrac{3}{8}\tan x\,\sec x + \tfrac{3}{8}\ln|\sec x + \tan x| + C$$

25. Let $u = \ln x$ and $a = 2$. Then $du = dx/x$ and

$$\begin{aligned}
\int\frac{\sqrt{4+(\ln x)^2}}{x}\,dx &= \int\sqrt{a^2+u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + C \\
&= \tfrac{1}{2}(\ln x)\sqrt{4+(\ln x)^2} + 2\ln\left[\ln x + \sqrt{4+(\ln x)^2}\right] + C
\end{aligned}$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{41}{=} \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C.$$

29. $$\int \frac{x^4\,dx}{\sqrt{x^{10}-2}} = \int \frac{x^4\,dx}{\sqrt{(x^5)^2-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-2}} \qquad \begin{bmatrix} u = x^5, \\ du = 5x^4\,dx \end{bmatrix}$$

$$\overset{43}{=} \tfrac{1}{5}\ln\left|u + \sqrt{u^2-2}\,\right| + C = \tfrac{1}{5}\ln\left|x^5 + \sqrt{x^{10}-2}\,\right| + C$$

31. Using cylindrical shells, we get

$$V = 2\pi \int_0^2 x \cdot x\sqrt{4-x^2}\,dx = 2\pi \int_0^2 x^2\sqrt{4-x^2}\,dx \overset{31}{=} 2\pi\left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2} + \frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\pi[(0 + 2\sin^{-1}1) - (0 + 2\sin^{-1}0] = 2\pi\left(2\cdot\frac{\pi}{2}\right) = 2\pi^2$$

33. (a) $$\frac{d}{du}\left[\frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C\right] = \frac{1}{b^3}\left[b + \frac{ba^2}{(a+bu)^2} - \frac{2ab}{(a+bu)}\right]$$

$$= \frac{1}{b^3}\left[\frac{b(a+bu)^2 + ba^2 - (a+bu)2ab}{(a+bu)^2}\right] = \frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right] = \frac{u^2}{(a+bu)^2}$$

(b) Let $t = a + bu \quad\Rightarrow\quad dt = b\,du$. Note that $u = \dfrac{t-a}{b}$ and $du = \dfrac{1}{b}\,dt$.

$$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\int \frac{(t-a)^2}{t^2}\,dt = \frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}\,dt = \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right)dt$$

$$= \frac{1}{b^3}\left(t - 2a\ln|t| - \frac{a^2}{t}\right) + C = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

35. Maple and Mathematica both give $\int \sec^4 x\,dx = \frac{2}{3}\tan x + \frac{1}{3}\tan x\,\sec^2 x$, while Derive gives the second term as $\dfrac{\sin x}{3\cos^3 x} = \dfrac{1}{3}\dfrac{\sin x}{\cos x}\dfrac{1}{\cos^2 x} = \dfrac{1}{3}\tan x\,\sec^2 x$. Using Formula 77, we get $\int \sec^4 x\,dx = \frac{1}{3}\tan x\,\sec^2 x + \frac{2}{3}\int \sec^2 x\,dx = \frac{1}{3}\tan x\,\sec^2 x + \frac{2}{3}\tan x + C$.

37. Derive gives $\int x^2\sqrt{x^2+4}\,dx = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4} + x\right)$. Maple gives $\frac{1}{4}x(x^2+4)^{3/2} - \frac{1}{2}x\sqrt{x^2+4} - 2\,\text{arcsinh}\left(\frac{1}{2}x\right)$. Applying the command `convert(%,ln);` yields

$$\tfrac{1}{4}x(x^2+4)^{3/2} - \tfrac{1}{2}x\sqrt{x^2+4} - 2\ln\left(\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{x^2+4}\right) = \tfrac{1}{4}x(x^2+4)^{1/2}\left[(x^2+4) - 2\right] - 2\ln\left[\left(x + \sqrt{x^2+4}\right)/2\right]$$

$$= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4} + x\right) + 2\ln 2$$

Mathematica gives $\frac{1}{4}x(2+x^2)\sqrt{3+x^2} - 2\,\text{arcsinh}(x/2)$. Applying the `TrigToExp` and `Simplify` commands gives $\frac{1}{4}\left[x(2+x^2)\sqrt{4+x^2} - 8\log\left(\frac{1}{2}\left(x + \sqrt{4+x^2}\,\right)\right)\right] = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(x + \sqrt{4+x^2}\right) + 2\ln 2$, so all are equivalent (without constant).

Now use Formula 22 to get

$$\begin{aligned}\int x^2\sqrt{2^2+x^2}\,dx &= \frac{x}{8}(2^2+2x^2)\sqrt{2^2+x^2}-\frac{2^4}{8}\ln\bigl(x+\sqrt{2^2+x^2}\,\bigr)+C\\ &= \frac{x}{8}(2)(2+x^2)\sqrt{4+x^2}-2\ln\bigl(x+\sqrt{4+x^2}\,\bigr)+C\\ &= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4}-2\ln\bigl(\sqrt{x^2+4}+x\bigr)+C\end{aligned}$$

39. Maple gives $\int x\sqrt{1+2x}\,dx=\frac{1}{10}(1+2x)^{5/2}-\frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2+\frac{1}{15}x-\frac{1}{15}\right)$, and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\int x\sqrt{1+2x}\,dx=\frac{2}{15(2)^2}(3\cdot 2x-2\cdot 1)(1+2x)^{3/2}+C=\tfrac{1}{30}(6x-2)(1+2x)^{3/2}+C=\tfrac{1}{15}(3x-1)(1+2x)^{3/2}.$$

41. Maple gives $\int \tan^5 x\,dx=\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\frac{1}{2}\ln(1+\tan^2 x)$, Mathematica gives $\int \tan^5 x\,dx=\frac{1}{4}[-1-2\cos(2x)]\sec^4 x-\ln(\cos x)$, and Derive gives $\int \tan^5 x\,dx=\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x-\ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x<0$, which is not the case. Using Formula 75, $\int \tan^5 x\,dx=\frac{1}{5-1}\tan^{5-1}x-\int \tan^{5-2}x\,dx=\frac{1}{4}\tan^4 x-\int\tan^3 x\,dx$. Using Formula 69, $\int \tan^3 x\,dx=\frac{1}{2}\tan^2 x+\ln|\cos x|+C$, so $\int\tan^5 x\,dx=\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x-\ln|\cos x|+C$.

43. (a) $F(x)=\displaystyle\int f(x)\,dx=\int\frac{1}{x\sqrt{1-x^2}}\,dx\overset{35}{=}-\frac{1}{1}\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C=-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C.$

f has domain $\{x\mid x\neq 0,\,1-x^2>0\}=\{x\mid x\neq 0,\ |x|<1\}=(-1,0)\cup(0,1)$. F has the same domain.

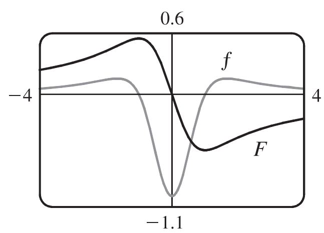
(b) Derive gives $F(x)=\ln\bigl(\sqrt{1-x^2}-1\bigr)-\ln x$ and Mathematica gives $F(x)=\ln x-\ln\bigl(1+\sqrt{1-x^2}\,\bigr)$. Both are correct if you take absolute values of the logarithm arguments, and both would then have the same domain. Maple gives $F(x)=-\operatorname{arctanh}\bigl(1/\sqrt{1-x^2}\,\bigr)$. This function has domain

$\bigl\{x\bigm||x|<1,-1<1/\sqrt{1-x^2}<1\bigr\}=\bigl\{x\bigm||x|<1,\,1/\sqrt{1-x^2}<1\bigr\}=\bigl\{x\bigm||x|<1,\sqrt{1-x^2}>1\bigr\}=\emptyset,$

the empty set! If we apply the command `convert(%,ln);` to Maple's answer, we get

$-\dfrac{1}{2}\ln\left(\dfrac{1}{\sqrt{1-x^2}}+1\right)+\dfrac{1}{2}\ln\left(1-\dfrac{1}{\sqrt{1-x^2}}\right)$, which has the same domain, $\emptyset$.

45. Maple gives the antiderivative



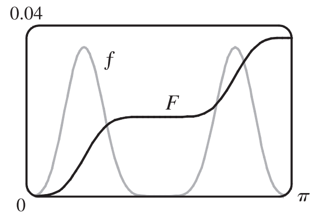
$$F(x)=\int\frac{x^2-1}{x^4+x^2+1}\,dx=-\tfrac{1}{2}\ln(x^2+x+1)+\tfrac{1}{2}\ln(x^2-x+1).$$

We can see that at 0, this antiderivative is 0. From the graphs, it appears that F has a maximum at $x=-1$ and a minimum at $x=1$ [since $F'(x)=f(x)$ changes sign at these x-values], and that F has inflection points at $x\approx-1.7$, $x=0$, and $x\approx 1.7$ [since $f(x)$ has extrema at these x-values].

47. Since $f(x) = \sin^4 x \cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives

$\int f(x)\,dx = -\frac{1}{10}\sin^3 x \cos^7 x - \frac{3}{80}\sin x \cos^7 x + \frac{1}{160}\cos^5 x \sin x + \frac{1}{128}\cos^3 x \sin x + \frac{3}{256}\cos x \sin x + \frac{3}{256}x$

and this expression is 0 at $x = 0$.



F has a minimum at $x = 0$ and a maximum at $x = \pi$.

F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

7.7 Approximate Integration

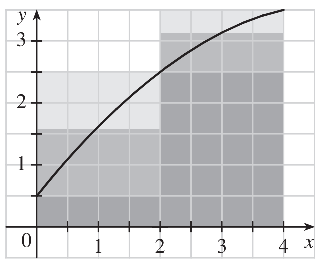
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0) + f(2)] = 2(0.5 + 2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2) + f(4)] = 2(2.5 + 3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1) + f(3)] \approx 2(1.6 + 3.2) = 9.6$$

(b)



L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

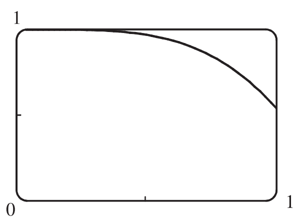
This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

5. $f(x) = x^2 \sin x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$

(a) $M_8 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right)\right] \approx 5.932957$

(b) $S_8 = \frac{\pi}{8 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$

≈ 5.869247

Actual: $\int_0^\pi x^2 \sin x\, dx \overset{84}{=} \left[-x^2 \cos x\right]_0^\pi + 2\int_0^\pi x \cos x\, dx \overset{83}{=} \left[-\pi^2(-1) - 0\right] + 2\left[\cos x + x \sin x\right]_0^\pi$

$= \pi^2 + 2[(-1+0) - (1+0)] = \pi^2 - 4 \approx 5.869604$

Errors: $E_M = \text{actual} - M_8 = \int_0^\pi x^2 \sin x\, dx - M_8 \approx -0.063353$

$E_S = \text{actual} - S_8 = \int_0^\pi x^2 \sin x\, dx - S_8 \approx 0.000357$

7. $f(x) = \sqrt[4]{1+x^2}$, $\Delta x = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.413790$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 2.411453$

(c) $S_8 = \frac{1}{4 \cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.412232$

9. $f(x) = \dfrac{\ln x}{1+x}$, $\Delta x = \dfrac{2-1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 2f(1.9) + f(2)] \approx 0.146879$

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.147391$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7)$
$+ 2f(1.8) + 4f(1.9) + f(2)]$

≈ 0.147219

11. $f(t) = \sin(e^{t/2})$, $\Delta t = \dfrac{\frac{1}{2} - 0}{8} = \dfrac{1}{16}$

(a) $T_8 = \frac{1}{16 \cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

(c) $S_8 = \frac{1}{16 \cdot 3}\left[f(0) + 4f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 4f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451976$

13. $f(t) = e^{\sqrt{t}} \sin t$, $\Delta t = \dfrac{4-0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.513618$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 4.748256$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.675111$

15. $f(x) = \dfrac{\cos x}{x}$, $\Delta x = \dfrac{5-1}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.526123$

17. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2\cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2\cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.074915$

19. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2\cdot 1 + 4\cdot 1\cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6\cdot 1^3/(12\cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \;\Leftrightarrow\; \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \;\Leftrightarrow\; \dfrac{1}{2n^2} \le \dfrac{1}{10^4} \;\Leftrightarrow\; 2n^2 \ge 10^4 \;\Leftrightarrow\; n^2 \ge 5000 \;\Leftrightarrow\; n \ge 71$. Take $n = 71$ for T_n. For E_M, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \;\Leftrightarrow\; 4n^2 \ge 10^4 \;\Leftrightarrow\; n^2 \ge 2500 \;\Leftrightarrow\; n \ge 50$. Take $n = 50$ for M_n.

21. $f(x) = \sin x$, $\Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10}$

(a) $T_{10} = \frac{\pi}{10\cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.983524$

$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right)\right] \approx 2.008248$

$S_{10} = \frac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 2.000110$

Since $I = \int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = 1 - (-1) = 2$, $E_T = I - T_{10} \approx 0.016476$, $E_M = I - M_{10} \approx -0.008248$, and $E_S = I - S_{10} \approx -0.000110$.

(b) $f(x) = \sin x \;\Rightarrow\; \left|f^{(n)}(x)\right| \le 1$, so take $K = 1$ for all error estimates.

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{1(\pi-0)^3}{12(10)^2} = \frac{\pi^3}{1200} \approx 0.025839. \qquad |E_M| \le \frac{|E_T|}{2} = \frac{\pi^3}{2400} \approx 0.012919.$$

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{1(\pi-0)^5}{180(10)^4} = \frac{\pi^5}{1{,}800{,}000} \approx 0.000170.$$

The actual error is about 64% of the error estimate in all three cases.

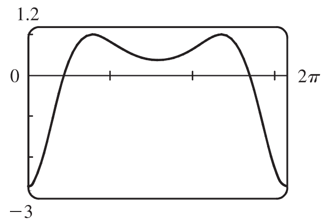
(c) $|E_T| \le 0.00001 \;\Leftrightarrow\; \dfrac{\pi^3}{12n^2} \le \dfrac{1}{10^5} \;\Leftrightarrow\; n^2 \ge \dfrac{10^5\pi^3}{12} \;\Rightarrow\; n \ge 508.3$. Take $n = 509$ for T_n.

$|E_M| \le 0.00001 \;\Leftrightarrow\; \dfrac{\pi^3}{24n^2} \le \dfrac{1}{10^5} \;\Leftrightarrow\; n^2 \ge \dfrac{10^5\pi^3}{24} \;\Rightarrow\; n \ge 359.4$. Take $n = 360$ for M_n.

$|E_S| \le 0.00001 \;\Leftrightarrow\; \dfrac{\pi^5}{180n^4} \le \dfrac{1}{10^5} \;\Leftrightarrow\; n^4 \ge \dfrac{10^5\pi^5}{180} \;\Rightarrow\; n \ge 20.3$.

Take $n = 22$ for S_n (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$.

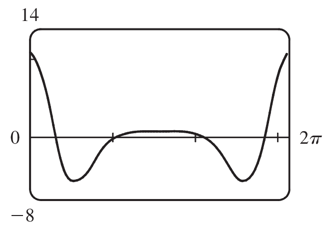
With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph $f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x\,\cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$.

With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$

$n^4 \ge 5{,}915{,}362 \quad\Leftrightarrow\quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$.

($K = 10.9$ leads to the same value of n.)

25. $I = \int_0^1 xe^x\,dx = \left[(x-1)e^x\right]_0^1$ [parts or Formula 96] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

$n = 5$:
$$L_5 = \tfrac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$$
$$R_5 = \tfrac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$$
$$T_5 = \tfrac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$$
$$M_5 = \tfrac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$$

$$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$$
$$E_R \approx 1 - 1.286599 = -0.286599$$
$$E_T \approx 1 - 1.014771 = -0.014771$$
$$E_M \approx 1 - 0.992621 = 0.007379$$

$n = 10$: $L_{10} = \frac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$

$R_{10} = \frac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$

$T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.003696$

$M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$

$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$

$E_R \approx 1 - 1.139610 = -0.139610$

$E_T \approx 1 - 1.003696 = -0.003696$

$E_M \approx 1 - 0.998152 = 0.001848$

$n = 20$: $L_{20} = \frac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$

$R_{20} = \frac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$

$T_{20} = \frac{1}{20 \cdot 2}\{f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)\} \approx 1.000924$

$M_{20} = \frac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$

$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$

$E_R \approx 1 - 1.068881 = -0.068881$

$E_T \approx 1 - 1.000924 = -0.000924$

$E_M \approx 1 - 0.999538 = 0.000462$

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $I = \int_0^2 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5} - 0 = 6.4$, $f(x) = x^4$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$n = 6$: $T_6 = \frac{2}{6 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f\left(\frac{3}{3}\right) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right)\right] + f(2)\right\} \approx 6.695473$

$M_6 = \frac{2}{6}\left[f\left(\frac{1}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{5}{6}\right) + f\left(\frac{7}{6}\right) + f\left(\frac{9}{6}\right) + f\left(\frac{11}{6}\right)\right] \approx 6.252572$

$S_6 = \frac{2}{6 \cdot 3}\left[f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 4f\left(\frac{3}{3}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{5}{3}\right) + f(2)\right] \approx 6.403292$

$E_T = I - T_6 \approx 6.4 - 6.695473 = -0.295473$

$E_M \approx 6.4 - 6.252572 = 0.147428$

$E_S \approx 6.4 - 6.403292 = -0.003292$

$$n = 12: \quad T_{12} = \tfrac{2}{12 \cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{2}{6}\right) + f\left(\tfrac{3}{6}\right) + \cdots + f\left(\tfrac{11}{6}\right)\right] + f(2)\right\} \approx 6.474023$$

$$M_6 = \tfrac{2}{12}\left[f\left(\tfrac{1}{12}\right) + f\left(\tfrac{3}{12}\right) + f\left(\tfrac{5}{12}\right) + \cdots + f\left(\tfrac{23}{12}\right)\right] \approx 6.363008$$

$$S_6 = \tfrac{2}{12 \cdot 3}\left[f(0) + 4f\left(\tfrac{1}{6}\right) + 2f\left(\tfrac{2}{6}\right) + 4f\left(\tfrac{3}{6}\right) + 2f\left(\tfrac{4}{6}\right) + \cdots + 4f\left(\tfrac{11}{6}\right) + f(2)\right] \approx 6.400206$$

$$E_T = I - T_{12} \approx 6.4 - 6.474023 = -0.074023$$

$$E_M \approx 6.4 - 6.363008 = 0.036992$$

$$E_S \approx 6.4 - 6.400206 = -0.000206$$

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (b - a)/n = (6 - 0)/6 = 1$

(a) $T_6 = \frac{\Delta x}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[3 + 2(5) + 2(4) + 2(2) + 2(2.8) + 2(4) + 1]$

$= \frac{1}{2}(39.6) = 19.8$

(b) $M_6 = \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)]$

$\approx 1[4.5 + 4.7 + 2.6 + 2.2 + 3.4 + 3.2]$

$= 20.6$

(c) $S_6 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[3 + 4(5) + 2(4) + 4(2) + 2(2.8) + 4(4) + 1]$

$= \frac{1}{3}(61.6) = 20.5\overline{3}$

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2 - 0)/4 = 0.8$.

$$\int_0^{3.2} f(x)\,dx \approx M_4 = 0.8[f(0.4) + f(1.2) + f(2.0) + f(2.8)] = 0.8[6.5 + 6.4 + 7.6 + 8.8]$$
$$= 0.8(29.3) = 23.44$$

(b) $-4 \le f''(x) \le 1 \quad \Rightarrow \quad |f''(x)| \le 4$, so use $K = 4$, $a = 0$, $b = 3.2$, and $n = 4$ in Theorem 3.

So $|E_M| \le \dfrac{4(3.2 - 0)^3}{24(4)^2} = \dfrac{128}{375} = 0.341\overline{3}$.

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)]\\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3}\text{ ft/s}\end{aligned}$$

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = \frac{6-0}{12} = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) + 2P(3)\\ &\qquad + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)]\\ &= \tfrac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604)\\ &\qquad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052]\\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3}\text{ megawatt-hours}\end{aligned}$$

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi \int_2^{10} x f(x)\,dx = 2\pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$\begin{aligned}I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)]\\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)]\\ &= \tfrac{1}{3}(395.2)\end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

39. Using disks, $V = \int_1^5 \pi(e^{-1/x})^2\,dx = \pi \int_1^5 e^{-2/x}\,dx = \pi I_1$. Now use Simpson's Rule with $f(x) = e^{-2/x}$ to approximate I_1. $I_1 \approx S_8 = \frac{5-1}{3(8)}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)] \approx \frac{1}{6}(11.4566)$

Thus, $V \approx \pi \cdot \frac{1}{6}(11.4566) \approx 6.0$ cubic units.

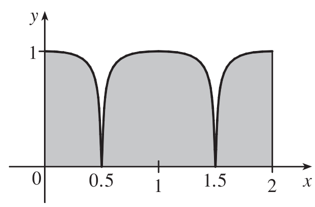
41. $I(\theta) = \dfrac{N^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi N d \sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8 \times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7}$, so

$M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

43. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2\cdot 1 + 1] = 2$.

The Midpoint Rule gives $M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0+0] = 0$, so the Trapezoidal Rule is more accurate.



45. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

47. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and

$M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$T_{2n} = \tfrac{1}{2}\left(\tfrac{1}{2}\Delta x\right)[f(x_0) + 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]$$

so

$$\begin{aligned}\tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] + \tfrac{1}{4}\Delta x[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\ &= T_{2n}\end{aligned}$$

7.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is

$A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty} \left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim\limits_{t\to\infty} \int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\, du = 3\,dx] \quad = -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$$

so $I = \lim\limits_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim\limits_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}.$ Convergent

7. $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim\limits_{t\to-\infty} \int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim\limits_{t\to-\infty}\left[-2\sqrt{2-w}\,\right]_t^{-1}$ $[u = 2-w,\, du = -dw]$

$= \lim\limits_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\,\right] = \infty.$ Divergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim\limits_{t\to\infty} \int_4^t e^{-y/2}\,dy = \lim\limits_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim\limits_{t\to\infty}(-2e^{-t/2} + 2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}.$

Convergent

11. $\int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 \frac{x\,dx}{1+x^2} + \int_0^\infty \frac{x\,dx}{1+x^2}$ and

$\int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim\limits_{t\to-\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_t^0 = \lim\limits_{t\to-\infty}\left[0 - \frac{1}{2}\ln(1+t^2)\right] = -\infty.$ Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}.$

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$ Convergent

15. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim\limits_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim\limits_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim\limits_{t\to\infty}(-\cos t + 1)$. This limit does not exist, so the integral is divergent. Divergent

17. $\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim\limits_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \frac{1}{2}\lim\limits_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \frac{1}{2}\lim\limits_{t\to\infty}\left[\ln(t^2+2t) - \ln 3\right] = \infty.$

Divergent

19. $\int_0^\infty se^{-5s}\,ds = \lim\limits_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim\limits_{t\to\infty}\left[-\frac{1}{5}se^{-5s} - \frac{1}{25}e^{-5s}\right]$ $\left[\begin{array}{l}\text{by integration by}\\ \text{parts with } u = s\end{array}\right]$

$= \lim\limits_{t\to\infty}\left(-\frac{1}{5}te^{-5t} - \frac{1}{25}e^{-5t} + \frac{1}{25}\right) = 0 - 0 + \frac{1}{25}$ [by l'Hospital's Rule]

$= \frac{1}{25}.$ Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t \quad \left[\begin{array}{l}\text{by substitution with}\\ u=\ln x,\, du=dx/x\end{array}\right] \quad = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

23. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\int_0^\infty \frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\displaystyle\int \frac{x^2\,dx}{9+x^6} \quad \left[\begin{array}{l}u=x^3\\ du=3x^2\,dx\end{array}\right] \quad = \int \frac{\frac{1}{3}\,du}{9+u^2} \quad \left[\begin{array}{l}u=3v\\ du=3\,dv\end{array}\right] \quad = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$$= \frac{1}{9}\tan^{-1}v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C,$$

so $\displaystyle 2\int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t = 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}.$

Convergent

25. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_1^{\ln t} u^{-3}\,du \quad \left[\begin{array}{l}u=\ln x,\\ du=dx/x\end{array}\right] \quad = \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}.$$ Convergent

27. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+}\int_t^1 3x^{-5}\,dx = \lim_{t\to 0^+}\left[-\frac{3}{4x^4}\right]_t^1 = -\frac{3}{4}\lim_{t\to 0^+}\left(1 - \frac{1}{t^4}\right) = \infty.$ Divergent

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14}(x+2)^{-1/4}\,dx = \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[16^{3/4} - (t+2)^{3/4}\right]$

$= \frac{4}{3}(8-0) = \frac{32}{3}.$ Convergent

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty.$ Divergent

33. There is an infinite discontinuity at $x=1$. $\int_0^{33}(x-1)^{-1/5}\,dx = \int_0^1 (x-1)^{-1/5}\,dx + \int_1^{33}(x-1)^{-1/5}\,dx$. Here

$\int_0^1 (x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/5}\,dx = \lim_{t\to 1^-}\left[\frac{5}{4}(x-1)^{4/5}\right]_0^t = \lim_{t\to 1^-}\left[\frac{5}{4}(t-1)^{4/5} - \frac{5}{4}\right] = -\frac{5}{4}$ and

$\int_1^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\int_t^{33}(x-1)^{-1/5}\,dx = \lim_{t\to 1^+}\left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33} = \lim_{t\to 1^+}\left[\frac{5}{4}\cdot 16 - \frac{5}{4}(t-1)^{4/5}\right] = 20.$

Thus, $\int_0^{33}(x-1)^{-1/5}\,dx = -\frac{5}{4} + 20 = \frac{75}{4}.$ Convergent

35. $\displaystyle I = \int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2 = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}.$

Now $\displaystyle\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \quad\Rightarrow\quad 1 = A(x-5) + B(x-1).$

Set $x=5$ to get $1=4B$, so $B=\frac{1}{4}$. Set $x=1$ to get $1=-4A$, so $A=-\frac{1}{4}$. Thus

$$I_1 = \lim_{t\to 1^-} \int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5} \right) dx = \lim_{t\to 1^-} \left[-\frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x-5| \right]_0^t$$

$$= \lim_{t\to 1^-} \left[\left(-\tfrac{1}{4} \ln|t-1| + \tfrac{1}{4} \ln|t-5| \right) - \left(-\tfrac{1}{4} \ln|-1| + \tfrac{1}{4} \ln|-5| \right) \right]$$

$$= \infty, \quad \text{since } \lim_{t\to 1^-} \left(-\tfrac{1}{4} \ln|t-1| \right) = \infty.$$

Since I_1 is divergent, I is divergent.

37. $$\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^t \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{1/t} ue^u\,(-du) \qquad \left[\begin{array}{l} u = 1/x, \\ du = -dx/x^2 \end{array} \right]$$

$$= \lim_{t\to 0^-} \left[(u-1)e^u \right]_{1/t}^{-1} \quad \left[\begin{array}{c} \text{use parts} \\ \text{or Formula 96} \end{array} \right] \quad = \lim_{t\to 0^-} \left[-2e^{-1} - \left(\frac{1}{t} - 1 \right) e^{1/t} \right]$$

$$= -\frac{2}{e} - \lim_{s\to -\infty} (s-1)e^s \quad [s = 1/t] \quad = -\frac{2}{e} - \lim_{s\to -\infty} \frac{s-1}{e^{-s}} \stackrel{\text{H}}{=} -\frac{2}{e} - \lim_{s\to -\infty} \frac{1}{-e^{-s}}$$

$$= -\frac{2}{e} - 0 = -\frac{2}{e}. \qquad \text{Convergent}$$

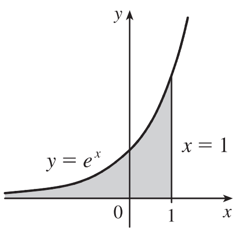
39. $I = \int_0^2 z^2 \ln z\,dz = \lim_{t\to 0^+} \int_t^2 z^2 \ln z\,dz = \lim_{t\to 0^+} \left[\dfrac{z^3}{3^2} (3\ln z - 1) \right]_t^2 \qquad \left[\begin{array}{c} \text{integrate by parts} \\ \text{or use Formula 101} \end{array} \right]$

$= \lim_{t\to 0^+} \left[\tfrac{8}{9}(3\ln 2 - 1) - \tfrac{1}{9} t^3 (3\ln t - 1) \right] = \tfrac{8}{3} \ln 2 - \tfrac{8}{9} - \tfrac{1}{9} \lim_{t\to 0^+} \left[t^3 (3\ln t - 1) \right] = \tfrac{8}{3} \ln 2 - \tfrac{8}{9} - \tfrac{1}{9} L.$

Now $L = \lim_{t\to 0^+} \left[t^3 (3\ln t - 1) \right] = \lim_{t\to 0^+} \dfrac{3\ln t - 1}{t^{-3}} \stackrel{\text{H}}{=} \lim_{t\to 0^+} \dfrac{3/t}{-3/t^4} = \lim_{t\to 0^+} (-t^3) = 0.$

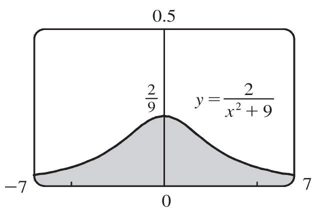
Thus, $L = 0$ and $I = \frac{8}{3} \ln 2 - \frac{8}{9}$. Convergent

41.



Area $= \int_{-\infty}^1 e^x\,dx = \lim_{t\to -\infty} \left[e^x \right]_t^1 = e - \lim_{t\to -\infty} e^t = e$

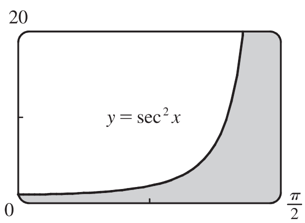
43.



$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\,dx = 2 \cdot 2 \int_0^{\infty} \frac{1}{x^2+9}\,dx = 4 \lim_{t\to\infty} \int_0^t \frac{1}{x^2+9}\,dx$$

$$= 4 \lim_{t\to\infty} \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^t = \frac{4}{3} \lim_{t\to\infty} \left[\tan^{-1} \frac{t}{3} - 0 \right] = \frac{4}{3} \cdot \frac{\pi}{2} = \frac{2\pi}{3}$$

45.



Area $= \int_0^{\pi/2} \sec^2 x\,dx = \lim_{t\to (\pi/2)^-} \int_0^t \sec^2 x\,dx = \lim_{t\to (\pi/2)^-} \left[\tan x \right]_0^t$

$= \lim_{t\to (\pi/2)^-} (\tan t - 0) = \infty$

Infinite area

47. (a)

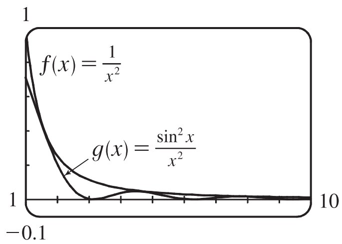
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \quad \Rightarrow \quad 0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent [Equation 2 with $p = 2 > 1$], $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

49. For $x > 0$, $\dfrac{x}{x^3+1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3+1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3+1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx$ is also convergent.

51. For $x > 1$, $f(x) = \dfrac{x+1}{\sqrt{x^4 - x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \le 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

53. For $0 < x \le 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$. Now

$$I = \int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+} \Big[-2x^{-1/2}\Big]_t^1 = \lim_{t\to 0^+}\left(-2 + \frac{2}{\sqrt{t}}\right) = \infty,$$ so I is divergent, and by comparison, $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ is divergent.

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} \quad \begin{bmatrix} u=\sqrt{x},\, x=u^2, \\ dx = 2u\,du \end{bmatrix} \quad = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C, \text{ so}$$

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$

57. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \infty.$ Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+}\frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}.$

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.

59. First suppose $p = -1$. Then

$\displaystyle\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$\displaystyle\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p < -1$, then $p + 1 < 0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\begin{aligned}\textstyle\int_0^1 x^p \ln x\,dx &= \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}} \\ &= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}\end{aligned}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

61. (a) $I = \int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$, and $\int_{0}^{\infty} x\,dx = \lim\limits_{t\to\infty} \int_0^t x\,dx = \lim\limits_{t\to\infty} \left[\frac{1}{2}x^2\right]_0^t = \lim\limits_{t\to\infty} \left[\frac{1}{2}t^2 - 0\right] = \infty$,

so I is divergent.

(b) $\int_{-t}^{t} x\,dx = \left[\frac{1}{2}x^2\right]_{-t}^{t} = \frac{1}{2}t^2 - \frac{1}{2}t^2 = 0$, so $\lim\limits_{t\to\infty} \int_{-t}^{t} x\,dx = 0$. Therefore, $\int_{-\infty}^{\infty} x\,dx \neq \lim\limits_{t\to\infty} \int_{-t}^{t} x\,dx$.

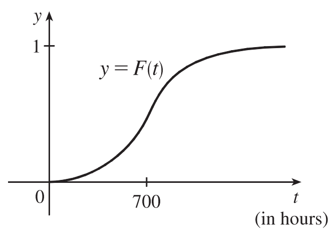
63. Volume $= \displaystyle\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \pi \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \pi < \infty.$

65. Work $= \displaystyle\int_R^\infty F\,dr = \lim_{t\to\infty} \int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R} - \frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work,

so $\frac{1}{2}mv_0^2 = \dfrac{GmM}{R} \quad\Rightarrow\quad v_0 = \sqrt{\dfrac{2GM}{R}}$.

67. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

69. $I = \displaystyle\int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_a^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a.$

$I < 0.001 \quad\Rightarrow\quad \frac{\pi}{2} - \tan^{-1} a < 0.001 \quad\Rightarrow\quad \tan^{-1} a > \frac{\pi}{2} - 0.001 \quad\Rightarrow\quad a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000.$

71. (a) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty} \left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\dfrac{1}{s}$ only if $s > 0$.

Therefore $F(s) = \dfrac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty} \int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty} \left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$

$$= \lim_{n\to\infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$$

This converges only if $1 - s < 0 \quad\Rightarrow\quad s > 1$, in which case $F(s) = \dfrac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim\limits_{n\to\infty} \int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \quad\Rightarrow\quad du = dt$,

$v = -\dfrac{e^{-st}}{s}$. Then $F(s) = \lim\limits_{n\to\infty} \left[-\dfrac{t}{s}e^{-st} - \dfrac{1}{s^2}e^{-st}\right]_0^n = \lim\limits_{n\to\infty} \left(\dfrac{-n}{se^{sn}} - \dfrac{1}{s^2e^{sn}} + 0 + \dfrac{1}{s^2}\right) = \dfrac{1}{s^2}$ only if $s > 0$.

Therefore, $F(s) = \dfrac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

73. $G(s) = \int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \quad\Rightarrow\quad du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty} \left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ and $\lim\limits_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem,

$\lim\limits_{t\to\infty} f(t)e^{-st} = 0$ for $s > a \quad\Rightarrow\quad G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

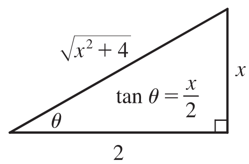
75. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2}xe^{-x^2}\right]_0^t + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty} \left[-\frac{t}{2e^{t^2}}\right] + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

77. For the first part of the integral, let $x = 2\tan\theta \quad\Rightarrow\quad dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$



From the figure, $\tan\theta = \dfrac{x}{2}$, and $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$. So

$$I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx = \lim_{t\to\infty} \left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t$$

$$= \lim_{t\to\infty} \left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right]$$

$$= \lim_{t\to\infty} \left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right] = \ln\left(\lim_{t\to\infty} \frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}$$

Now $L = \lim\limits_{t\to\infty} \dfrac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim\limits_{t\to\infty} \dfrac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \dfrac{2}{C\lim\limits_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges.
If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$.
If $C > 1$, $L = 0$ and I diverges to $-\infty$.

79. No, $I = \int_0^\infty f(x)\,dx$ must be *divergent*. Since $\lim\limits_{x\to\infty} f(x) = 1$, there must exist an N such that if $x \ge N$, then $f(x) \ge \frac{1}{2}$.

Thus, $I = I_1 + I_2 = \int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$, where I_1 is an ordinary definite integral that has a finite value, and I_2 is improper and diverges by comparison with the divergent integral $\int_N^\infty \frac{1}{2}\,dx$.

7 Review

CONCEPT CHECK

1. See Formula 7.1.1 or 7.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.

2. See the Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$ on page 462.

3. If $\sqrt{a^2 - x^2}$ occurs, try $x = a\sin\theta$; if $\sqrt{a^2 + x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2 - a^2}$ occurs, try $x = a\sec\theta$. See the Table of Trigonometric Substitutions on page 467.

4. See Equation 2 and Expressions 7, 9, and 11 in Section 7.4.

5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 7.7. We would expect the best estimate to be given by Simpson's Rule.

6. See Definitions 1(a), (b), and (c) in Section 7.8.

7. See Definitions 3(b), (a), and (c) in Section 7.8.

8. See the Comparison Theorem after Example 8 in Section 7.8.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator, $\dfrac{x(x^2+4)}{x^2-4} = x + \dfrac{8x}{x^2-4} = x + \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

3. False. It can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

$$\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-} \left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to 1^-} \tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

7. False. See Exercise 61 in Section 7.8.

9. (a) True. See the end of Section 7.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1,\infty)$ with $\lim\limits_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x)+g(x)]\,dx = 0$ [convergent].

EXERCISES

1. $\displaystyle\int_0^5 \frac{x}{x+10}\,dx = \int_0^5 \left(1 - \frac{10}{x+10}\right) dx = \big[x - 10\ln(x+10)\big]_0^5 = 5 - 10\ln 15 + 10\ln 10$

$\quad = 5 + 10\ln\frac{10}{15} = 5 + 10\ln\frac{2}{3}$

3. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta = \Big[\ln(1+\sin\theta)\Big]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$

5. $\int_0^{\pi/2}\sin^3\theta\,\cos^2\theta\,d\theta = \int_0^{\pi/2}(1-\cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta = \int_1^0(1-u^2)u^2\,(-du)$ $\quad\left[\begin{array}{l} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{array}\right]$

$\quad = \int_0^1(u^2-u^4)\,du = \left[\frac13 u^3 - \frac15 u^5\right]_0^1 = \left(\frac13 - \frac15\right) - 0 = \frac{2}{15}$

7. Let $u = \ln t$, $du = dt/t$. Then $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt = \int \sin u\,du = -\cos u + C = -\cos(\ln t) + C$.

9. $\displaystyle\int_1^4 x^{3/2}\ln x\,dx$ $\quad\left[\begin{array}{ll} u = \ln x, & dv = x^{3/2}\,dx, \\ du = dx/x & v = \frac25 x^{5/2} \end{array}\right]$ $\displaystyle = \frac25\left[x^{5/2}\ln x\right]_1^4 - \frac25\int_1^4 x^{3/2}\,dx = \frac25(32\ln 4 - \ln 1) - \frac25\left[\frac25 x^{5/2}\right]_1^4$

$\quad = \frac25(64\ln 2) - \frac{4}{25}(32-1) = \frac{128}{5}\ln 2 - \frac{124}{25}$ $\quad\left[\text{or } \frac{64}{5}\ln 4 - \frac{124}{25}\right]$

11. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3}\frac{\tan\theta}{\sec\theta}\sec\theta\,\tan\theta\,d\theta = \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta - 1)\,d\theta = \big[\tan\theta - \theta\big]_0^{\pi/3} = \sqrt3 - \tfrac{\pi}{3}.$$

13. Let $t = \sqrt[3]{x}$. Then $t^3 = x$ and $3t^2\,dt = dx$, so $\int e^{\sqrt[3]{x}}\,dx = \int e^t\cdot 3t^2\,dt = 3I$. To evaluate I, let $u = t^2$, $dv = e^t\,dt \;\Rightarrow\; du = 2t\,dt$, $v = e^t$, so $I = \int t^2e^t\,dt = t^2e^t - \int 2te^t\,dt$. Now let $U = t$, $dV = e^t\,dt \;\Rightarrow\; dU = dt$, $V = e^t$. Thus, $I = t^2e^t - 2\left[te^t - \int e^t\,dt\right] = t^2e^t - 2te^t + 2e^t + C_1$, and hence

$3I = 3e^t(t^2 - 2t + 2) + C = 3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$.

15. $\displaystyle\frac{x-1}{x^2+2x} = \frac{x-1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \;\Rightarrow\; x - 1 = A(x+2) + Bx$. Set $x = -2$ to get $-3 = -2B$, so $B = \frac32$. Set $x = 0$ to get $-1 = 2A$, so $A = -\frac12$. Thus, $\displaystyle\int\frac{x-1}{x^2+2x}\,dx = \int\left(\frac{-\frac12}{x} + \frac{\frac32}{x+2}\right)dx = -\frac12\ln|x| + \frac32\ln|x+2| + C$.

17. Integrate by parts with $u = x$, $dv = \sec x\,\tan x\,dx \;\Rightarrow\; du = dx$, $v = \sec x$:

$\int x\sec x\,\tan x\,dx = x\sec x - \int \sec x\,dx \overset{14}{=} x\sec x - \ln|\sec x + \tan x| + C$.

19. $\displaystyle\int\frac{x+1}{9x^2+6x+5}\,dx = \int\frac{x+1}{(9x^2+6x+1)+4}\,dx = \int\frac{x+1}{(3x+1)^2+4}\,dx$ $\quad\left[\begin{array}{l} u = 3x+1, \\ du = 3\,dx \end{array}\right]$

$$= \int\frac{\left[\frac13(u-1)\right]+1}{u^2+4}\left(\frac13\,du\right) = \frac13\cdot\frac13\int\frac{(u-1)+3}{u^2+4}\,du$$

$$= \frac19\int\frac{u}{u^2+4}\,du + \frac19\int\frac{2}{u^2+2^2}\,du = \frac19\cdot\frac12\ln(u^2+4) + \frac29\cdot\frac12\tan^{-1}\left(\frac12 u\right) + C$$

$$= \tfrac{1}{18}\ln(9x^2+6x+5) + \tfrac19\tan^{-1}\left[\tfrac12(3x+1)\right] + C$$

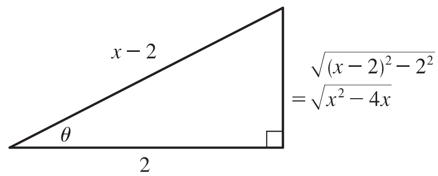
21. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x}} = \int \frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int \frac{dx}{\sqrt{(x-2)^2-2^2}}$

$$= \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\tan\theta} \qquad \begin{bmatrix} x-2=2\sec\theta, \\ dx=2\sec\theta\,\tan\theta\,d\theta \end{bmatrix}$$

$$= \textstyle\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C_1$$

$$= \ln\left|\frac{x-2}{2}+\frac{\sqrt{x^2-4x}}{2}\right|+C_1$$

$$= \ln\left|x-2+\sqrt{x^2-4x}\right|+C, \text{ where } C=C_1-\ln 2$$

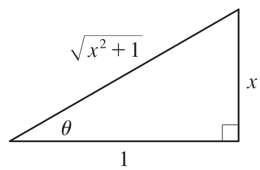


23. Let $x=\tan\theta$, so that $dx=\sec^2\theta\,d\theta$. Then

$$\int \frac{dx}{x\sqrt{x^2+1}} = \int \frac{\sec^2\theta\,d\theta}{\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \textstyle\int \csc\theta\,d\theta = \ln|\csc\theta-\cot\theta|+C$$

$$= \ln\left|\frac{\sqrt{x^2+1}}{x}-\frac{1}{x}\right|+C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right|+C$$



25. $\dfrac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2} \quad\Rightarrow\quad 3x^3-x^2+6x-4=(Ax+B)\big(x^2+2\big)+(Cx+D)\big(x^2+1\big).$

Equating the coefficients gives $A+C=3$, $B+D=-1$, $2A+C=6$, and $2B+D=-4 \quad\Rightarrow$

$A=3$, $C=0$, $B=-3$, and $D=2$. Now

$$\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} = \frac{3}{2}\ln\big(x^2+1\big)-3\tan^{-1}x+\sqrt{2}\tan^{-1}\!\left(\frac{x}{\sqrt{2}}\right)+C.$$

27. $\int_0^{\pi/2}\cos^3x\,\sin 2x\,dx = \int_0^{\pi/2}\cos^3x\,(2\sin x\,\cos x)\,dx = \int_0^{\pi/2}2\cos^4x\,\sin x\,dx = \left[-\frac{2}{5}\cos^5x\right]_0^{\pi/2} = \frac{2}{5}$

29. The product of an odd function and an even function is an odd function, so $f(x)=x^5\sec x$ is an odd function.

By Theorem 5.5.7(b), $\int_{-1}^{1}x^5\sec x\,dx=0$.

31. Let $u=\sqrt{e^x-1}$. Then $u^2=e^x-1$ and $2u\,du=e^x\,dx$. Also, $e^x+8=u^2+9$. Thus,

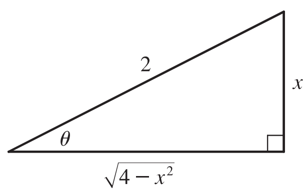
$$\int_0^{\ln 10}\frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx = \int_0^3\frac{u\cdot 2u\,du}{u^2+9} = 2\int_0^3\frac{u^2}{u^2+9}\,du = 2\int_0^3\left(1-\frac{9}{u^2+9}\right)du$$

$$= 2\left[u-\frac{9}{3}\tan^{-1}\!\left(\frac{u}{3}\right)\right]_0^3 = 2\big[(3-3\tan^{-1}1)-0\big] = 2\left(3-3\cdot\frac{\pi}{4}\right) = 6-\frac{3\pi}{2}$$

33. Let $x=2\sin\theta \quad\Rightarrow\quad \big(4-x^2\big)^{3/2}=(2\cos\theta)^3$, $dx=2\cos\theta\,d\theta$, so

$$\int\frac{x^2}{(4-x^2)^{3/2}}\,dx = \int\frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta = \int\tan^2\theta\,d\theta = \int\big(\sec^2\theta-1\big)\,d\theta$$

$$= \tan\theta-\theta+C = \frac{x}{\sqrt{4-x^2}}-\sin^{-1}\!\left(\frac{x}{2}\right)+C$$



35. $\displaystyle\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x\,(1+\sqrt{x}\,)}} = \int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}} \quad \begin{bmatrix} u = 1+\sqrt{x}, \\ du = \dfrac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$

$\displaystyle = 4\sqrt{u} + C = 4\sqrt{1+\sqrt{x}} + C$

37. $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int \left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx = \int (1+\sin 2x)\cos 2x\,dx$

$= \int \cos 2x\,dx + \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$

Or: $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int (\cos x + \sin x)^2(\cos^2 x - \sin^2 x)\,dx$

$= \int (\cos x + \sin x)^3(\cos x - \sin x)\,dx = \frac{1}{4}(\cos x + \sin x)^4 + C_1$

39. We'll integrate $\displaystyle I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $\displaystyle dv = \frac{dx}{(1+2x)^2}$. Then $du = (x\cdot 2e^{2x} + e^{2x}\cdot 1)\,dx$ and $\displaystyle v = -\frac{1}{2}\cdot\frac{1}{1+2x}$, so

$$I = -\frac{1}{2}\cdot\frac{xe^{2x}}{1+2x} - \int \left[-\frac{1}{2}\cdot\frac{e^{2x}(2x+1)}{1+2x}\right]dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C = e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C$$

Thus, $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}.$

41. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t$

$\displaystyle = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}$

43. $\displaystyle\int \frac{dx}{x\ln x} \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} \quad = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\displaystyle\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the integral is divergent.

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$\displaystyle = \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\,\ln t - 4\sqrt{t}\right)\right] \overset{**}{=} (4\ln 4 - 8) - (0 - 0) = 4\ln 4 - 8$

(*) Let $u = \ln x$, $\displaystyle dv = \frac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

(**) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\,\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

47. $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \lim_{t\to 0^+}\int_t^1 (x^{1/2} - x^{-1/2})\,dx = \lim_{t\to 0^+}\left[\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right]_t^1$

$\displaystyle = \lim_{t\to 0^+}\left[\left(\tfrac{2}{3} - 2\right) - \left(\tfrac{2}{3}t^{3/2} - 2t^{1/2}\right)\right] = -\tfrac{4}{3} - 0 = -\tfrac{4}{3}$

49. Let $u = 2x + 1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2 + 4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2 + 4} + \frac{1}{2}\int_{0}^{\infty} \frac{du}{u^2 + 4}$$

$$= \tfrac{1}{2} \lim_{t\to-\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2} \lim_{t\to\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t = \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}.$$

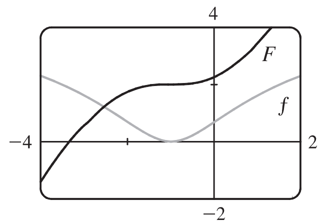
51. We first make the substitution $t = x + 1$, so $\ln(x^2 + 2x + 2) = \ln\left[(x+1)^2 + 1\right] = \ln(t^2 + 1)$. Then we use parts with $u = \ln(t^2 + 1)$, $dv = dt$:

$$\int \ln(t^2+1)\,dt = t\,\ln(t^2+1) - \int \frac{t(2t)\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int \frac{t^2\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int \left(1 - \frac{1}{t^2+1}\right) dt$$

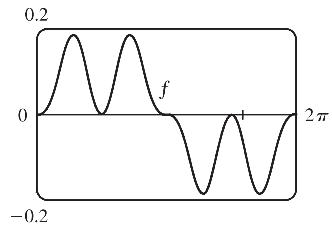
$$= t\,\ln(t^2+1) - 2t + 2\arctan t + C$$

$$= (x+1)\,\ln(x^2+2x+2) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln(x^2 + 2x + 2)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.



53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x\, \sin^3 x\, dx$ is equal to 0. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\, \sin x\, dx$ and let $u = \cos x \quad\Rightarrow$ $du = -\sin x\, dx$. Thus, $I = \int_1^1 u^2(1 - u^2)(-du) = 0$.



55. $\int \sqrt{4x^2 - 4x - 3}\, dx = \int \sqrt{(2x-1)^2 - 4}\, dx \quad \begin{bmatrix} u = 2x - 1, \\ du = 2\,dx \end{bmatrix} \quad = \int \sqrt{u^2 - 2^2}\left(\tfrac{1}{2}\,du\right)$

$$\overset{39}{=} \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2 - 2^2} - \frac{2^2}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right|\right) + C = \tfrac{1}{4}u\sqrt{u^2 - 4} - \ln\left|u + \sqrt{u^2 - 4}\right| + C$$

$$= \tfrac{1}{4}(2x - 1)\sqrt{4x^2 - 4x - 3} - \ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$$

57. Let $u = \sin x$, so that $du = \cos x\, dx$. Then

$$\int \cos x\,\sqrt{4 + \sin^2 x}\, dx = \int \sqrt{2^2 + u^2}\, du \overset{21}{=} \frac{u}{2}\sqrt{2^2 + u^2} + \frac{2^2}{2}\ln\left(u + \sqrt{2^2 + u^2}\right) + C$$

$$= \tfrac{1}{2}\sin x\,\sqrt{4 + \sin^2 x} + 2\ln\left(\sin x + \sqrt{4 + \sin^2 x}\right) + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2 - u^2} + \dfrac{1}{\sqrt{a^2 - u^2}} - \dfrac{1}{\sqrt{1 - u^2/a^2}} \cdot \dfrac{1}{a}$

$$= \left(a^2 - u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2 - u^2\right) + 1 - 1\right] = \frac{\sqrt{a^2 - u^2}}{u^2}$$

(b) Let $u = a\sin\theta \;\Rightarrow\; du = a\cos\theta\,d\theta$, $a^2 - u^2 = a^2\left(1-\sin^2\theta\right) = a^2\cos^2\theta$.

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \int \frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int \left(\csc^2\theta - 1\right)d\theta = -\cot\theta - \theta + C$$
$$= -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

61. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (7.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 7.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

63. $f(x) = \dfrac{1}{\ln x}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + \cdots + f(3.8)] + f(4)\} \approx 1.925444$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + \cdots + f(3.9)] \approx 1.920915$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 1.922470$

65. $f(x) = \dfrac{1}{\ln x} \;\Rightarrow\; f'(x) = -\dfrac{1}{x(\ln x)^2} \;\Rightarrow\; f''(x) = \dfrac{2+\ln x}{x^2(\ln x)^3} = \dfrac{2}{x^2(\ln x)^3} + \dfrac{1}{x^2(\ln x)^2}$. Note that each term of $f''(x)$ decreases on $[2,4]$, so we'll take $K = f''(2) \approx 2.022$. $|E_T| \leq \dfrac{K(b-a)^3}{12n^2} \approx \dfrac{2.022(4-2)^3}{12(10)^2} = 0.01348$ and $|E_M| \leq \dfrac{K(b-a)^3}{24n^2} = 0.00674$. $|E_T| \leq 0.00001 \;\Leftrightarrow\; \dfrac{2.022(8)}{12n^2} \leq \dfrac{1}{10^5} \;\Leftrightarrow\; n^2 \geq \dfrac{10^5(2.022)(8)}{12} \;\Rightarrow\; n \geq 367.2$. Take $n = 368$ for T_n. $|E_M| \leq 0.00001 \;\Leftrightarrow\; n^2 \geq \dfrac{10^5(2.022)(8)}{24} \;\Rightarrow\; n \geq 259.6$. Take $n = 260$ for M_n.

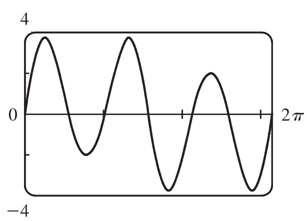
67. $\Delta t = \left(\frac{10}{60} - 0\right)/\,10 = \frac{1}{60}$.

$$\begin{aligned}\text{Distance traveled } &= \int_0^{10} v\,dt \approx S_{10}\\ &= \tfrac{1}{60\cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]\\ &= \tfrac{1}{180}(1544) = 8.5\overline{7}\text{ mi}\end{aligned}$$

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$\begin{aligned}f^{(4)}(x) = {}&\sin(\sin x)[\cos^4 x + 7\cos^2 x - 3]\\ &+ \cos(\sin x)\left[6\cos^2 x\sin x + \sin x\right]\end{aligned}$$

From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0,\pi]$.



(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x)\,dx \approx \tfrac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0,\pi]$, so we use Theorem 7.7.4 with $K = 3.8$, and estimate the error as $|E_S| \leq \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \leq \dfrac{3.8\pi^5}{180n^4} \leq 0.00001$,

so $n^4 \geq \dfrac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \quad \Rightarrow \quad n \geq 28.35$. Since n must be even for Simpson's Rule, we must have $n \geq 30$ to ensure the desired accuracy.

71. $\dfrac{x^3}{x^5+2} \leq \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (7.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \leq \cos^2 x \leq \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \leq 0 \leq \cos^2 x$. Thus,

$$\begin{aligned}\text{area} &= \textstyle\int_0^{\pi/2}(\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi}(\cos^2 x - \cos x)\,dx \\ &= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} = \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2\end{aligned}$$

75. Using the formula for disks, the volume is

$$\begin{aligned} V &= \textstyle\int_0^{\pi/2} \pi\,[f(x)]^2\;dx = \pi\int_0^{\pi/2}(\cos^2 x)^2\,dx = \pi\int_0^{\pi/2}\left[\tfrac{1}{2}(1+\cos 2x)\right]^2\;dx \\ &= \tfrac{\pi}{4}\textstyle\int_0^{\pi/2}(1+\cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4}\int_0^{\pi/2}\left[1 + \tfrac{1}{2}(1+\cos 4x) + 2\cos 2x\right]dx \\ &= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}\end{aligned}$$

77. By the Fundamental Theorem of Calculus,

$$\textstyle\int_0^\infty f'(x)\,dx = \lim\limits_{t\to\infty}\int_0^t f'(x)\,dx = \lim\limits_{t\to\infty}[f(t) - f(0)] = \lim\limits_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

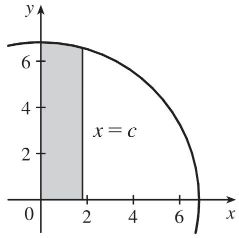
79. Let $u = 1/x \quad \Rightarrow \quad x = 1/u \quad \Rightarrow \quad dx = -(1/u^2)\,du$.

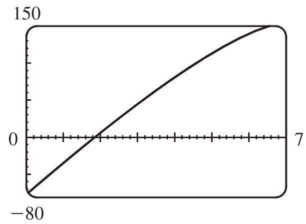
$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}\,(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

☐ PROBLEMS PLUS

1.





By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle.An equation of the semicircle is $y = \sqrt{49 - x^2}$, so we require that $\int_0^c \sqrt{49 - x^2}\,dx = \frac{1}{3} \cdot \frac{1}{4}\pi(7)^2 \quad \Leftrightarrow$

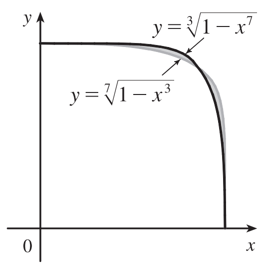
$\left[\frac{1}{2}x\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}(x/7)\right]_0^c = \frac{49}{12}\pi$ [by Formula 30] $\Leftrightarrow \quad \frac{1}{2}c\sqrt{49 - c^2} + \frac{49}{2}\sin^{-1}(c/7) = \frac{49}{12}\pi$.

This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c, and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7} \quad \Rightarrow \quad x = \sqrt[7]{1 - y^3}$ and

$y = \sqrt[7]{1 - x^3} \quad \Rightarrow \quad x = \sqrt[3]{1 - y^7}$, so

$\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right) dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right) dx$. But this equation is of the form $z = -z$. So $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx = 0$.

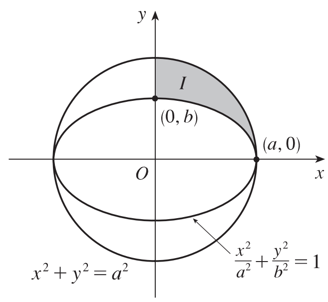


5. The area A of the remaining part of the circle is given by

$$A = 4I = 4\int_0^a \left(\sqrt{a^2 - x^2} - \frac{b}{a}\sqrt{a^2 - x^2}\right) dx = 4\left(1 - \frac{b}{a}\right)\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$\overset{30}{=} \frac{4}{a}(a - b)\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4}{a}(a - b)\left[\left(0 + \frac{a^2}{2}\frac{\pi}{2}\right) - 0\right] = \frac{4}{a}(a - b)\left(\frac{a^2\pi}{4}\right) = \pi a(a - b),$$



which is the area of an ellipse with semiaxes a and $a - b$.

Alternate solution: Subtracting the area of the ellipse from the area of the circle gives us $\pi a^2 - \pi ab = \pi a\,(a - b)$, as calculated above. (The formula for the area of an ellipse was derived in Example 2 in Section 7.3.)

7. Recall that $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. So

$$\begin{aligned} f(x) &= \textstyle\int_0^\pi \cos t \cos(x-t)\,dt = \frac{1}{2}\int_0^\pi [\cos(t+x-t) + \cos(t-x+t)]\,dt = \frac{1}{2}\int_0^\pi [\cos x + \cos(2t-x)]\,dt \\ &= \tfrac{1}{2}\left[t\cos x + \tfrac{1}{2}\sin(2t-x)\right]_0^\pi = \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(2\pi - x) - \tfrac{1}{4}\sin(-x) \\ &= \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(-x) - \tfrac{1}{4}\sin(-x) = \tfrac{\pi}{2}\cos x \end{aligned}$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

9. In accordance with the hint, we let $I_k = \int_0^1 (1-x^2)^k\,dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = (1-x^2)^{k+1} \Rightarrow du = (k+1)(1-x^2)^k(-2x)$, $dv = dx \Rightarrow v = x$, and then split the remaining integral into identifiable quantities:

$$\begin{aligned} I_{k+1} &= x(1-x^2)^{k+1}\Big|_0^1 + 2(k+1)\textstyle\int_0^1 x^2(1-x^2)^k\,dx = (2k+2)\int_0^1 (1-x^2)^k[1-(1-x^2)]\,dx \\ &= (2k+2)(I_k - I_{k+1}) \end{aligned}$$

So $I_{k+1}[1 + (2k+2)] = (2k+2)I_k \Rightarrow I_{k+1} = \dfrac{2k+2}{2k+3}I_k$. Now to complete the proof, we use induction:

$I_0 = 1 = \dfrac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

$$\begin{aligned} I_{k+1} &= \frac{2k+2}{2k+3}I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2}\cdot\frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} \\ &= \frac{[2(k+1)]^2\,2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}\,[(k+1)!]^2}{[2(k+1)+1]!} \end{aligned}$$

So by induction, the formula holds for all integers $n \geq 0$.

11. $0 < a < b$. Now

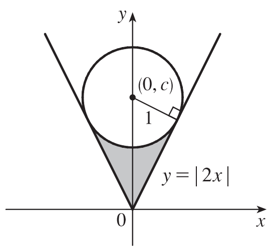
$$\int_0^1 [bx + a(1-x)]^t\,dx = \int_a^b \frac{u^t}{(b-a)}\,du \quad [u = bx + a(1-x)] \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

Now let $y = \lim\limits_{t\to 0}\left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t\to 0}\left[\dfrac{1}{t}\ln\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

13.



An equation of the circle with center $(0, c)$ and radius 1 is $x^2 + (y - c)^2 = 1^2$, so an equation of the lower semicircle is $y = c - \sqrt{1 - x^2}$. At the points of tangency, the slopes of the line and semicircle must be equal. For $x \geq 0$, we must have

$y' = 2 \quad \Rightarrow \quad \dfrac{x}{\sqrt{1 - x^2}} = 2 \quad \Rightarrow \quad x = 2\sqrt{1 - x^2} \quad \Rightarrow \quad x^2 = 4(1 - x^2) \quad \Rightarrow$

$5x^2 = 4 \quad \Rightarrow \quad x^2 = \frac{4}{5} \quad \Rightarrow \quad x = \frac{2}{5}\sqrt{5}$ and so $y = 2\left(\frac{2}{5}\sqrt{5}\right) = \frac{4}{5}\sqrt{5}$.

The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}\left(x - \frac{2}{5}\sqrt{5}\right) \quad \Leftrightarrow$

$y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1 - x^2}$.

Thus, the shaded area is

$$\begin{aligned}
2\int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5} - \sqrt{1 - x^2}\right) - 2x\right] dx &\overset{30}{=} 2\left[\sqrt{5}\,x - \frac{x}{2}\sqrt{1 - x^2} - \frac{1}{2}\sin^{-1} x - x^2\right]_0^{(2/5)\sqrt{5}} \\
&= 2\left[2 - \frac{\sqrt{5}}{5} \cdot \frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5}\right] - 2(0) \\
&= 2\left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)
\end{aligned}$$

15. We integrate by parts with $u = \dfrac{1}{\ln(1 + x + t)}$, $dv = \sin t\,dt$, so $du = \dfrac{-1}{(1 + x + t)[\ln(1 + x + t)]^2}$ and $v = -\cos t$. The integral becomes

$$\begin{aligned}
I = \int_0^\infty \frac{\sin t\,dt}{\ln(1 + x + t)} &= \lim_{b \to \infty}\left(\left[\frac{-\cos t}{\ln(1 + x + t)}\right]_0^b - \int_0^b \frac{\cos t\,dt}{(1 + x + t)[\ln(1 + x + t)]^2}\right) \\
&= \lim_{b \to \infty} \frac{-\cos b}{\ln(1 + x + b)} + \frac{1}{\ln(1 + x)} + \int_0^\infty \frac{-\cos t\,dt}{(1 + x + t)[\ln(1 + x + t)]^2} = \frac{1}{\ln(1 + x)} + J
\end{aligned}$$

where $J = \displaystyle\int_0^\infty \frac{-\cos t\,dt}{(1 + x + t)[\ln(1 + x + t)]^2}$. Now $-1 \leq -\cos t \leq 1$ for all t; in fact, the inequality is strict except at isolated points. So $-\displaystyle\int_0^\infty \frac{dt}{(1 + x + t)[\ln(1 + x + t)]^2} < J < \int_0^\infty \frac{dt}{(1 + x + t)[\ln(1 + x + t)]^2} \quad \Leftrightarrow$

$-\dfrac{1}{\ln(1 + x)} < J < \dfrac{1}{\ln(1 + x)} \quad \Leftrightarrow \quad 0 < I < \dfrac{2}{\ln(1 + x)}$.

8 □ FURTHER APPLICATIONS OF INTEGRATION

8.1 Arc Length

1. $y = 2x - 5 \quad \Rightarrow \quad L = \int_{-1}^{3} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-1}^{3} \sqrt{1 + (2)^2}\, dx = \sqrt{5}\,[3 - (-1)] = 4\sqrt{5}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$L = [\text{distance from } (-1, -7) \text{ to } (3, 1)] = \sqrt{[3 - (-1)]^2 + [1 - (-7)]^2} = \sqrt{80} = 4\sqrt{5}$

3. $y = \cos x \quad \Rightarrow \quad dy/dx = -\sin x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \sin^2 x$. So $L = \int_0^{2\pi} \sqrt{1 + \sin^2 x}\, dx$.

5. $x = y + y^3 \quad \Rightarrow \quad dx/dy = 1 + 3y^2 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + (1 + 3y^2)^2 = 9y^4 + 6y^2 + 2.$

So $L = \int_1^4 \sqrt{9y^4 + 6y^2 + 2}\, dy$.

7. $y = 1 + 6x^{3/2} \quad \Rightarrow \quad dy/dx = 9x^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 81x$. So

$$L = \int_0^1 \sqrt{1 + 81x}\, dx = \int_1^{82} u^{1/2}\left(\tfrac{1}{81}\, du\right) \quad \begin{bmatrix} u = 1 + 81x, \\ du = 81\, dx \end{bmatrix} \quad = \tfrac{1}{81} \cdot \tfrac{2}{3}\left[u^{3/2}\right]_1^{82} = \tfrac{2}{243}\left(82\sqrt{82} - 1\right)$$

9. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{5}{6}x^4 - \dfrac{3}{10}x^{-4} \quad \Rightarrow$

$1 + (dy/dx)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8} = \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$. So

$$L = \int_1^2 \sqrt{\left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right)^2}\, dx = \int_1^2 \left(\tfrac{5}{6}x^4 + \tfrac{3}{10}x^{-4}\right) dx = \left[\tfrac{1}{6}x^5 - \tfrac{1}{10}x^{-3}\right]_1^2 = \left(\tfrac{32}{6} - \tfrac{1}{80}\right) - \left(\tfrac{1}{6} - \tfrac{1}{10}\right)$$
$$= \tfrac{31}{6} + \tfrac{7}{80} = \tfrac{1261}{240}$$

11. $x = \frac{1}{3}\sqrt{y}\,(y - 3) = \frac{1}{3}y^{3/2} - y^{1/2} \quad \Rightarrow \quad dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \quad \Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$$L = \int_1^9 \left(\tfrac{1}{2}y^{1/2} + \tfrac{1}{2}y^{-1/2}\right) dy = \tfrac{1}{2}\left[\tfrac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \tfrac{1}{2}\left[\left(\tfrac{2}{3} \cdot 27 + 2 \cdot 3\right) - \left(\tfrac{2}{3} \cdot 1 + 2 \cdot 1\right)\right]$$
$$= \tfrac{1}{2}\left(24 - \tfrac{8}{3}\right) = \tfrac{1}{2}\left(\tfrac{64}{3}\right) = \tfrac{32}{3}.$$

13. $y = \ln(\sec x) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sec x\, \tan x}{\sec x} = \tan x \quad \Rightarrow \quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so

$$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} |\sec x|\, dx = \int_0^{\pi/4} \sec x\, dx = \Big[\ln(\sec x + \tan x)\Big]_0^{\pi/4}$$
$$= \ln\left(\sqrt{2} + 1\right) - \ln(1 + 0) = \ln\left(\sqrt{2} + 1\right)$$

15. $y = \ln(1 - x^2) \;\Rightarrow\; y' = \dfrac{1}{1 - x^2} \cdot (-2x) \;\Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \frac{1 + 2x^2 + x^4}{(1 - x^2)^2} = \frac{(1 + x^2)^2}{(1 - x^2)^2} \;\Rightarrow$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{1 + x^2}{1 - x^2}\right)^2} = \frac{1 + x^2}{1 - x^2} = -1 + \frac{2}{1 - x^2} \quad \text{[by division]} \quad = -1 + \frac{1}{1 + x} + \frac{1}{1 - x} \quad \text{[partial fractions]}.$$

So $L = \displaystyle\int_0^{1/2} \left(-1 + \frac{1}{1 + x} + \frac{1}{1 - x}\right) dx = \Big[-x + \ln|1 + x| - \ln|1 - x|\Big]_0^{1/2} = \left(-\tfrac{1}{2} + \ln\tfrac{3}{2} - \ln\tfrac{1}{2}\right) - 0 = \ln 3 - \tfrac{1}{2}.$

17. $y = e^x \;\Rightarrow\; y' = e^x \;\Rightarrow\; 1 + (y')^2 = 1 + e^{2x}$. So

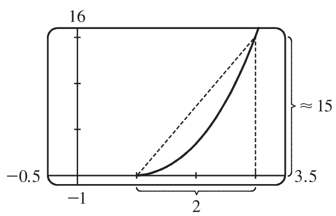
$$\begin{aligned}
L &= \int_0^1 \sqrt{1 + e^{2x}}\, dx = \int_1^e \sqrt{1 + u^2}\, \frac{du}{u} \quad \left[\begin{array}{c} u = e^x, \text{ so} \\ x = \ln u,\ dx = du/u \end{array}\right] \quad = \int_1^e \frac{\sqrt{1 + u^2}}{u^2}\, u\, du \\
&= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2 - 1}\, v\, dv \quad \left[\begin{array}{c} v = \sqrt{1 + u^2}, \text{ so} \\ v^2 = 1 + u^2,\ v\, dv = u\, du \end{array}\right] = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v - 1} - \frac{1/2}{v + 1}\right) dv \\
&= \left[v + \frac{1}{2}\ln\frac{v - 1}{v + 1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}} = \sqrt{1 + e^2} + \frac{1}{2}\ln\frac{\sqrt{1 + e^2} - 1}{\sqrt{1 + e^2} + 1} - \sqrt{2} - \frac{1}{2}\ln\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \\
&= \sqrt{1 + e^2} - \sqrt{2} + \ln\left(\sqrt{1 + e^2} - 1\right) - 1 - \ln\left(\sqrt{2} - 1\right)
\end{aligned}$$

Or: Use Formula 23 for $\int \left(\sqrt{1 + u^2}/u\right) du$, or substitute $u = \tan\theta$.

19. $y = \frac{1}{2}x^2 \;\Rightarrow\; dy/dx = x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + x^2$. So

$$\begin{aligned}
L &= \textstyle\int_{-1}^1 \sqrt{1 + x^2}\, dx = 2\int_0^1 \sqrt{1 + x^2}\, dx \quad \text{[by symmetry]} \quad \overset{21}{=} 2\left[\frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right)\right]_0^1 \quad \left[\begin{array}{c} \text{or substitute} \\ x = \tan\theta \end{array}\right] \\
&= 2\left[\left(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(1 + \sqrt{2}\right)\right) - \left(0 + \tfrac{1}{2}\ln 1\right)\right] = \sqrt{2} + \ln\left(1 + \sqrt{2}\right)
\end{aligned}$$

21.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5.

$y = \frac{2}{3}(x^2 - 1)^{3/2} \;\Rightarrow\; y' = (x^2 - 1)^{1/2}(2x) \;\Rightarrow\; 1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$,

so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\, dx = \int_1^3 \left|2x^2 - 1\right| dx = \int_1^3 (2x^2 - 1)\, dx = \left[\frac{2}{3}x^3 - x\right]_1^3 = (18 - 3) - \left(\frac{2}{3} - 1\right) = \frac{46}{3} = 15.\overline{3}.$

23. $y = xe^{-x} \;\Rightarrow\; dy/dx = e^{-x} - xe^{-x} = e^{-x}(1 - x) \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + e^{-2x}(1 - x)^2$. Let

$f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + e^{-2x}(1 - x)^2}$. Then $L = \int_0^5 f(x)\, dx$. Since $n = 10$, $\Delta x = \frac{5 - 0}{10} = \frac{1}{2}$. Now

$$\begin{aligned}
L \approx S_{10} &= \tfrac{1/2}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + 2f(3) + 4f\left(\tfrac{7}{2}\right) + 2f(4) + 4f\left(\tfrac{9}{2}\right) + f(5)\right] \\
&\approx 5.115840
\end{aligned}$$

The value of the integral produced by a calculator is 5.113568 (to six decimal places).

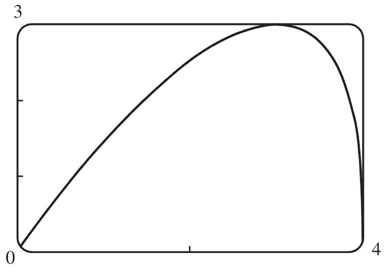
25. $y = \sec x \;\Rightarrow\; dy/dx = \sec x\,\tan x \;\Rightarrow\; L = \int_0^{\pi/3} f(x)\,dx$, where $f(x) = \sqrt{1 + \sec^2 x\,\tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Now

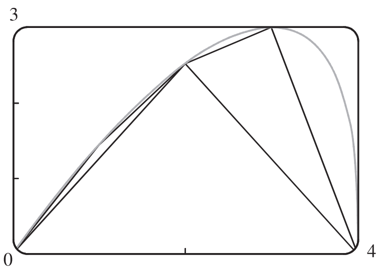
$$L \approx S_{10} = \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + 4f\left(\frac{3\pi}{30}\right) + 2f\left(\frac{4\pi}{30}\right) + 4f\left(\frac{5\pi}{30}\right)\right.$$
$$\left. + 2f\left(\frac{6\pi}{30}\right) + 4f\left(\frac{7\pi}{30}\right) + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 1.569619.$$

The value of the integral produced by a calculator is 1.569259 (to six decimal places).

27. (a)



(b)



Let $f(x) = y = x\,\sqrt[3]{4 - x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(4, f(4)) = (4, 0)$, and its length $L_1 = 4$.

The polygon with two sides joins the points $(0, 0)$, $(2, f(2)) = \left(2, 2\,\sqrt[3]{2}\right)$ and $(4, 0)$. Its length

$$L_2 = \sqrt{(2-0)^2 + \left(2\,\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\,\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0, 0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\,\sqrt[3]{2}\right)$, $(3, 3)$, and $(4, 0)$, so its length

$$L_3 = \sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\,\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\,\sqrt[3]{2}\right)^2} + \sqrt{1 + 9} \approx 7.50$$

(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\tfrac{1}{3}(4 - x)^{-2/3}(-1)\right] + \sqrt[3]{4 - x} = \dfrac{12 - 4x}{3(4 - x)^{2/3}}$, the length of the curve is

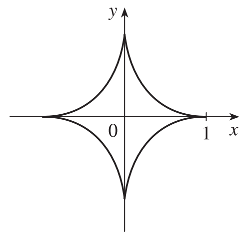
$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1 + \left[\frac{12 - 4x}{3(4 - x)^{2/3}}\right]^2}\,dx.$$

(d) According to a CAS, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

29. $y = \ln x \;\Rightarrow\; dy/dx = 1/x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 1/x^2 = (x^2+1)/x^2 \;\Rightarrow$

$$L = \int_1^2 \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^2 \frac{\sqrt{1+x^2}}{x}\,dx \overset{23}{=} \left[\sqrt{1+x^2} - \ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|\right]_1^2$$
$$= \sqrt{5} - \ln\left(\frac{1+\sqrt{5}}{2}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right)$$

31. $y^{2/3} = 1 - x^{2/3} \;\Rightarrow\; y = (1-x^{2/3})^{3/2} \;\Rightarrow$



$\dfrac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1-x^{2/3})^{1/2} \;\Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2 = x^{-2/3}(1-x^{2/3}) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1 + (x^{-2/3} - 1)}\,dx = 4\int_0^1 x^{-1/3}\,dx = 4\lim\limits_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^1 = 6.$

33. $y = 2x^{3/2} \;\Rightarrow\; y' = 3x^{1/2} \;\Rightarrow\; 1 + (y')^2 = 1 + 9x$. The arc length function with starting point $P_0(1,2)$ is

$s(x) = \int_1^x \sqrt{1+9t}\,dt = \left[\frac{2}{27}(1+9t)^{3/2}\right]_1^x = \frac{2}{27}\left[(1+9x)^{3/2} - 10\sqrt{10}\right].$

35. $y = \sin^{-1}x + \sqrt{1-x^2} \;\Rightarrow\; y' = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1-x^2}} = \dfrac{1-x}{\sqrt{1-x^2}} \;\Rightarrow$

$1 + (y')^2 = 1 + \dfrac{(1-x)^2}{1-x^2} = \dfrac{1-x^2+1-2x+x^2}{1-x^2} = \dfrac{2-2x}{1-x^2} = \dfrac{2(1-x)}{(1+x)(1-x)} = \dfrac{2}{1+x} \;\Rightarrow$

$\sqrt{1+(y')^2} = \sqrt{\dfrac{2}{1+x}}$. Thus, the arc length function with starting point $(0,1)$ is given by

$s(x) = \displaystyle\int_0^x \sqrt{1+[f'(t)]^2}\,dt = \int_0^x \sqrt{\frac{2}{1+t}}\,dt = \sqrt{2}\left[2\sqrt{1+t}\right]_0^x = 2\sqrt{2}\left(\sqrt{1+x} - 1\right).$

37. The prey hits the ground when $y = 0 \;\Leftrightarrow\; 180 - \frac{1}{45}x^2 = 0 \;\Leftrightarrow\; x^2 = 45\cdot 180 \;\Rightarrow\; x = \sqrt{8100} = 90,$

since x must be positive. $y' = -\frac{2}{45}x \;\Rightarrow\; 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$L = \displaystyle\int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\,du\right) \qquad \left[\begin{array}{l} u = \frac{2}{45}x, \\ du = \frac{2}{45}\,dx \end{array}\right]$

$\overset{21}{=} \frac{45}{2}\left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 = \frac{45}{2}\left[2\sqrt{17} + \frac{1}{2}\ln\left(4+\sqrt{17}\right)\right] = 45\sqrt{17} + \frac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m}$

39. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$:

$L = \int_0^{28} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx = 2\int_0^{14} \sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

41. $y = \int_1^x \sqrt{t^3-1}\,dt \;\Rightarrow\; dy/dx = \sqrt{x^3-1}$ [by FTC1] $\Rightarrow\; 1 + (dy/dx)^2 = 1 + \left(\sqrt{x^3-1}\right)^2 = x^3 \;\Rightarrow$

$L = \int_1^4 \sqrt{x^3}\,dx = \int_1^4 x^{3/2}\,dx = \frac{2}{5}\left[x^{5/2}\right]_1^4 = \frac{2}{5}(32-1) = \frac{62}{5} = 12.4$

8.2 Area of a Surface of Revolution

1. $y = x^4 \quad\Rightarrow\quad dy/dx = 4x^3 \quad\Rightarrow\quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + 16x^6}\,dx$

(a) By (7), an integral for the area of the surface obtained by rotating the curve about the x-axis is

$S = \int 2\pi y\,ds = \int_0^1 2\pi x^4\sqrt{1 + 16x^6}\,dx.$

(b) By (8), an integral for the area of the surface obtained by rotating the curve about the y-axis is

$S = \int 2\pi x\,ds = \int_0^1 2\pi x\sqrt{1 + 16x^6}\,dx.$

3. $y = \tan^{-1} x \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{1 + x^2} \quad\Rightarrow\quad ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\,dx.$

(a) By (7), $S = \int 2\pi y\,ds = \int_0^1 2\pi \tan^{-1} x\sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\,dx.$

(b) By (8), $S = \int 2\pi x\,ds = \int_0^1 2\pi x\sqrt{1 + \dfrac{1}{(1 + x^2)^2}}\,dx.$

5. $y = x^3 \quad\Rightarrow\quad y' = 3x^2$. So

$$\begin{aligned} S &= \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx] \\ &= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right) \end{aligned}$$

7. $y = \sqrt{1 + 4x} \quad\Rightarrow\quad y' = \frac{1}{2}(1 + 4x)^{-1/2}(4) = \dfrac{2}{\sqrt{1 + 4x}} \quad\Rightarrow\quad \sqrt{1 + (y')^2} = \sqrt{1 + \dfrac{4}{1 + 4x}} = \sqrt{\dfrac{5 + 4x}{1 + 4x}}$. So

$$\begin{aligned} S &= \int_1^5 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_1^5 \sqrt{1 + 4x}\sqrt{\frac{5 + 4x}{1 + 4x}}\,dx = 2\pi\int_1^5 \sqrt{4x + 5}\,dx \\ &= 2\pi\int_9^{25}\sqrt{u}\left(\tfrac{1}{4}\,du\right) \quad \left[\begin{array}{l} u = 4x + 5, \\ du = 4\,dx \end{array}\right] \quad = \tfrac{2\pi}{4}\left[\tfrac{2}{3}u^{3/2}\right]_9^{25} = \tfrac{\pi}{3}\left(25^{3/2} - 9^{3/2}\right) = \tfrac{\pi}{3}(125 - 27) = \tfrac{98}{3}\pi \end{aligned}$$

9. $y = \sin \pi x \quad\Rightarrow\quad y' = \pi\cos \pi x \quad\Rightarrow\quad 1 + (y')^2 = 1 + \pi^2\cos^2(\pi x)$. So

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^1 \sin \pi x\sqrt{1 + \pi^2\cos^2(\pi x)}\,dx \qquad \left[\begin{array}{l} u = \pi\cos \pi x, \\ du = -\pi^2\sin \pi x\,dx \end{array}\right] \\ &= 2\pi\int_{\pi}^{-\pi}\sqrt{1 + u^2}\left(-\frac{1}{\pi^2}\,du\right) = \frac{2}{\pi}\int_{-\pi}^{\pi}\sqrt{1 + u^2}\,du \\ &= \frac{4}{\pi}\int_0^{\pi}\sqrt{1 + u^2}\,du \overset{21}{=} \frac{4}{\pi}\left[\frac{u}{2}\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^{\pi} \\ &= \frac{4}{\pi}\left[\left(\frac{\pi}{2}\sqrt{1 + \pi^2} + \tfrac{1}{2}\ln\left(\pi + \sqrt{1 + \pi^2}\right)\right) - 0\right] = 2\sqrt{1 + \pi^2} + \frac{2}{\pi}\ln\left(\pi + \sqrt{1 + \pi^2}\right) \end{aligned}$$

11. $x = \frac{1}{3}(y^2 + 2)^{3/2} \quad\Rightarrow\quad dx/dy = \frac{1}{2}(y^2 + 2)^{1/2}(2y) = y\sqrt{y^2 + 2} \quad\Rightarrow\quad 1 + (dx/dy)^2 = 1 + y^2(y^2 + 2) = (y^2 + 1)^2.$

So $S = 2\pi\int_1^2 y(y^2 + 1)\,dy = 2\pi\left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \frac{1}{4} - \frac{1}{2}\right) = \frac{21\pi}{2}.$

13. $y = \sqrt[3]{x} \quad\Rightarrow\quad x = y^3 \quad\Rightarrow\quad 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$\begin{aligned} S &= 2\pi\int_1^2 x\sqrt{1 + (dx/dy)^2}\,dy = 2\pi\int_1^2 y^3\sqrt{1 + 9y^4}\,dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1 + 9y^4}\,36y^3\,dy = \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1 + 9y^4\right)^{3/2}\right]_1^2 \\ &= \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \end{aligned}$$

15. $x = \sqrt{a^2 - y^2} \quad\Rightarrow\quad dx/dy = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \quad\Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \quad\Rightarrow$$

$$S = \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\,\frac{a}{\sqrt{a^2 - y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\big[\,y\,\big]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2.$$

Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

17. $y = \ln x \quad\Rightarrow\quad dy/dx = 1/x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + 1/x^2 \quad\Rightarrow\quad S = \int_1^3 2\pi \ln x\,\sqrt{1 + 1/x^2}\,dx$.

Let $f(x) = \ln x\,\sqrt{1 + 1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3}\,[f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)] \approx 9.023754$.

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

19. $y = \sec x \quad\Rightarrow\quad dy/dx = \sec x\,\tan x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \sec^2 x\,\tan^2 x \quad\Rightarrow$

$S = \int_0^{\pi/3} 2\pi \sec x\,\sqrt{1 + \sec^2 x\,\tan^2 x}\,dx$. Let $f(x) = \sec x\,\sqrt{1 + \sec^2 x\,\tan^2 x}$. Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$.

Then $S \approx S_{10} = 2\pi \cdot \dfrac{\pi/30}{3}\left[f(0) + 4f\left(\dfrac{\pi}{30}\right) + 2f\left(\dfrac{2\pi}{30}\right) + \cdots + 2f\left(\dfrac{8\pi}{30}\right) + 4f\left(\dfrac{9\pi}{30}\right) + f\left(\dfrac{\pi}{3}\right)\right] \approx 13.527296$.

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

21. $y = 1/x \quad\Rightarrow\quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-1/x^2)^2}\,dx = \sqrt{1 + 1/x^4}\,dx \quad\Rightarrow$

$$S = \int_1^2 2\pi \cdot \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi\int_1^2 \frac{\sqrt{x^4 + 1}}{x^3}\,dx = 2\pi\int_1^4 \frac{\sqrt{u^2 + 1}}{u^2}\left(\tfrac{1}{2}\,du\right) \qquad [u = x^2,\ du = 2x\,dx]$$

$$= \pi\int_1^4 \frac{\sqrt{1 + u^2}}{u^2}\,du \overset{24}{=} \pi\left[-\frac{\sqrt{1 + u^2}}{u} + \ln\left(u + \sqrt{1 + u^2}\right)\right]_1^4$$

$$= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4 + \sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1 + \sqrt{2}\right)\right] = \frac{\pi}{4}\left[4\ln\left(\sqrt{17} + 4\right) - 4\ln\left(\sqrt{2} + 1\right) - \sqrt{17} + 4\sqrt{2}\right]$$

23. $y = x^3$ and $0 \le y \le 1 \quad\Rightarrow\quad y' = 3x^2$ and $0 \le x \le 1$.

$$S = \int_0^1 2\pi x\,\sqrt{1 + (3x^2)^2}\,dx = 2\pi\int_0^3 \sqrt{1 + u^2}\,\tfrac{1}{6}\,du \quad \begin{bmatrix} u = 3x^2, \\ du = 6x\,dx \end{bmatrix} \quad = \tfrac{\pi}{3}\int_0^3 \sqrt{1 + u^2}\,du$$

$$\overset{21}{=} \text{[or use CAS]}\ \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^3 = \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10} + \tfrac{1}{2}\ln\left(3 + \sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10} + \ln\left(3 + \sqrt{10}\right)\right]$$

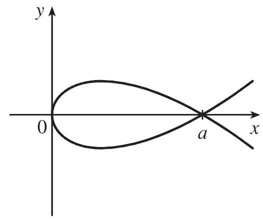
25. $S = 2\pi\displaystyle\int_1^\infty y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi\int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3}\,dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4 + 1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$S = 2\pi\displaystyle\int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3}\,dx > 2\pi\int_1^\infty \frac{x^2}{x^3}\,dx = 2\pi\int_1^\infty \frac{1}{x}\,dx$. But we know that this integral diverges, so the area S is infinite.

27. Since $a > 0$, the curve $3ay^2 = x(a-x)^2$ only has points with $x \ge 0$.

$[3ay^2 \ge 0 \;\Rightarrow\; x(a-x)^2 \ge 0 \;\Rightarrow\; x \ge 0.]$

The curve is symmetric about the x-axis (since the equation is unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a, so the curve's loop extends from $x = 0$ to $x = a$.



$$\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a-x)^2] \;\Rightarrow\; 6ay\,\frac{dy}{dx} = x \cdot 2(a-x)(-1) + (a-x)^2 \;\Rightarrow\; \frac{dy}{dx} = \frac{(a-x)[-2x+a-x]}{6ay} \;\Rightarrow$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-x)^2(a-3x)^2}{36a^2} \cdot \frac{3a}{x(a-x)^2} \quad \left[\begin{array}{c}\text{the last fraction}\\ \text{is } 1/y^2\end{array}\right] = \frac{(a-3x)^2}{12ax} \;\Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2-6ax+9x^2}{12ax} = \frac{a^2+6ax+9x^2}{12ax} = \frac{(a+3x)^2}{12ax} \quad \text{for } x \ne 0.$$

(a) $$S = \int_{x=0}^{a} 2\pi y\,ds = 2\pi \int_0^a \frac{\sqrt{x}\,(a-x)}{\sqrt{3a}} \cdot \frac{a+3x}{\sqrt{12ax}}\,dx = 2\pi \int_0^a \frac{(a-x)(a+3x)}{6a}\,dx$$

$$= \frac{\pi}{3a}\int_0^a (a^2 + 2ax - 3x^2)\,dx = \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a = \frac{\pi}{3a}(a^3 + a^3 - a^3) = \frac{\pi}{3a}\cdot a^3 = \frac{\pi a^2}{3}.$$

Note that we have rotated the top half of the loop about the x-axis. This generates the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$S = 2\cdot 2\pi\int_{x=0}^{a} x\,ds = 4\pi\int_0^a x\,\frac{a+3x}{\sqrt{12ax}}\,dx = \frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx = \frac{2\pi}{\sqrt{3a}}\int_0^a (ax^{1/2} + 3x^{3/2})\,dx$$

$$= \frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2} + \frac{6}{5}x^{5/2}\right]_0^a = \frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2} + \frac{6}{5}a^{5/2}\right) = \frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2 = \frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2$$

$$= \frac{56\pi\sqrt{3}\,a^2}{45}$$

29. (a) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;\Rightarrow\; \frac{y\,(dy/dx)}{b^2} = -\frac{x}{a^2} \;\Rightarrow\; \frac{dy}{dx} = -\frac{b^2x}{a^2y} \;\Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^4x^2}{a^4y^2} = \frac{b^4x^2 + a^4y^2}{a^4y^2} = \frac{b^4x^2 + a^4b^2(1 - x^2/a^2)}{a^4b^2(1 - x^2/a^2)} = \frac{a^4b^2 + b^4x^2 - a^2b^2x^2}{a^4b^2 - a^2b^2x^2}$$

$$= \frac{a^4 + b^2x^2 - a^2x^2}{a^4 - a^2x^2} = \frac{a^4 - (a^2-b^2)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the x-axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx = \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - (a^2-b^2)x^2}\,dx$$

$$= \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}} \quad \left[u = \sqrt{a^2-b^2}\,x\right] \overset{30}{=} \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2} + \frac{a^4}{2}\sin^{-1}\!\left(\frac{u}{a^2}\right)\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4 - a^2(a^2-b^2)} + \frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right] = 2\pi\left[b^2 + \frac{a^2b\sin^{-1}\dfrac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

(b) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{x\,(dx/dy)}{a^2}=-\dfrac{y}{b^2} \;\Rightarrow\; \dfrac{dx}{dy}=-\dfrac{a^2y}{b^2x} \;\Rightarrow$

$$1+\left(\frac{dx}{dy}\right)^2=1+\frac{a^4y^2}{b^4x^2}=\frac{b^4x^2+a^4y^2}{b^4x^2}=\frac{b^4a^2(1-y^2/b^2)+a^4y^2}{b^4a^2(1-y^2/b^2)}=\frac{a^2b^4-a^2b^2y^2+a^4y^2}{a^2b^4-a^2b^2y^2}$$
$$=\frac{b^4-b^2y^2+a^2y^2}{b^4-b^2y^2}=\frac{b^4-(b^2-a^2)y^2}{b^2(b^2-y^2)}$$

The oblate spheroid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the y-axis. Thus,

$$S=2\int_0^b 2\pi\,x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=4\pi\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,\frac{\sqrt{b^4-(b^2-a^2)y^2}}{b\sqrt{b^2-y^2}}\,dy$$
$$=\frac{4\pi a}{b^2}\int_0^b\sqrt{b^4-(b^2-a^2)y^2}\,dy=\frac{4\pi a}{b^2}\int_0^{b\sqrt{b^2-a^2}}\sqrt{b^4-u^2}\,\frac{du}{\sqrt{b^2-a^2}}\qquad\left[u=\sqrt{b^2-a^2}\,y\right]$$
$$\overset{30}{=}\frac{4\pi a}{b^2\sqrt{b^2-a^2}}\left[\frac{u}{2}\sqrt{b^4-u^2}+\frac{b^4}{2}\sin^{-1}\left(\frac{u}{b^2}\right)\right]_0^{b\sqrt{b^2-a^2}}$$
$$=\frac{4\pi a}{b^2\sqrt{b^2-a^2}}\left[\frac{b\sqrt{b^2-a^2}}{2}\sqrt{b^4-b^2(b^2-a^2)}+\frac{b^4}{2}\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}\right]=2\pi\left[a^2+\frac{ab^2\sin^{-1}\dfrac{\sqrt{b^2-a^2}}{b}}{\sqrt{b^2-a^2}}\right]$$

Notice that this result can be obtained from the answer in part (a) by interchanging a and b.

31. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c-f(x_i^*)$, so

$S=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$

33. For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x)=-x/\sqrt{r^2-x^2}$. The surface area generated is

$$S_1=\int_{-r}^{r}2\pi\left(r-\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\pi\int_0^r\left(r-\sqrt{r^2-x^2}\right)\frac{r}{\sqrt{r^2-x^2}}\,dx$$
$$=4\pi\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}-r\right)dx$$

For the lower semicircle, $f(x)=-\sqrt{r^2-x^2}$ and $f'(x)=\dfrac{x}{\sqrt{r^2-x^2}}$, so $S_2=4\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}+r\right)dx.$

Thus, the total area is $S=S_1+S_2=8\pi\displaystyle\int_0^r\left(\frac{r^2}{\sqrt{r^2-x^2}}\right)dx=8\pi\left[r^2\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=8\pi r^2\left(\frac{\pi}{2}\right)=4\pi^2r^2.$

35. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|$,

the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i=f(x_i)\approx f(x_i^*)$ and $y_{i-1}=f(x_{i-1})\approx f(x_i^*)$ must be replaced by $y_i=|f(x_i)|\approx|f(x_i^*)|$ and $y_{i-1}=|f(x_{i-1})|\approx|f(x_i^*)|$. Thus,

$2\pi\,\dfrac{y_{i-1}+y_i}{2}\,|P_{i-1}P_i|\approx 2\pi\,|f(x_i^*)|\sqrt{1+[f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before,

we obtain $S=\int_a^b 2\pi\,|f(x)|\sqrt{1+[f'(x)]^2}\,dx.$

8.3 Applications to Physics and Engineering

1. The weight density of water is $\delta = 62.5\ \text{lb/ft}^3$.

(a) $P = \delta d \approx (62.5\ \text{lb/ft}^3)(3\ \text{ft}) = 187.5\ \text{lb/ft}^2$

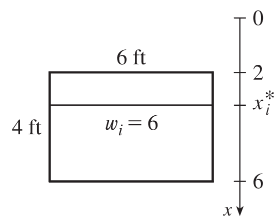
(b) $F = PA \approx (187.5\ \text{lb/ft}^2)(5\ \text{ft})(2\ \text{ft}) = 1875\ \text{lb}.$ (A is the area of the bottom of the tank.)

(c) As in Example 1, the area of the ith strip is $2\,(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x \cdot 2\,dx \approx (62.5)(2)\int_0^3 x\,dx = 125\left[\tfrac{1}{2}x^2\right]_0^3 = 125\left(\tfrac{9}{2}\right) = 562.5\ \text{lb}.$$

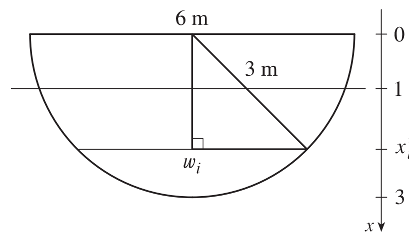
In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

3. Set up a vertical x-axis as shown, with $x = 0$ at the water's surface and x increasing in the downward direction. Then the area of the ith rectangular strip is $6\,\Delta x$ and the pressure on the strip is δx_i^* (where $\delta \approx 62.5\ \text{lb/ft}^3$). Thus, the hydrostatic force on the strip is $\delta x_i^* \cdot 6\,\Delta x$ and the total hydrostatic force $\approx \sum\limits_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x$. The total force



$$F = \lim_{n\to\infty}\sum_{i=1}^{n} \delta x_i^* \cdot 6\,\Delta x = \int_2^6 \delta x \cdot 6\,dx = 6\delta\int_2^6 x\,dx = 6\delta\left[\tfrac{1}{2}x^2\right]_2^6 = 6\delta(18 - 2) = 96\delta \approx 6000\ \text{lb}$$

5. Set up a vertical x-axis as shown. The base of the triangle shown in the figure has length $\sqrt{3^2 - (x_i^*)^2}$, so $w_i = 2\sqrt{9 - (x_i^*)^2}$, and the area of the ith rectangular strip is $2\sqrt{9 - (x_i^*)^2}\,\Delta x$. The ith rectangular strip is $(x_i^* - 1)$ m below the surface level of the water, so the pressure on the strip is $\rho g(x_i^* - 1)$. The hydrostatic force on the strip is $\rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x$ and the total force on the plate $\approx \sum\limits_{i=1}^{n} \rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x$. The total force



$$F = \lim\sum_{i=1}^{n} \rho g(x_i^* - 1)\cdot 2\sqrt{9 - (x_i^*)^2}\,\Delta x = 2\rho g\int_1^3 (x - 1)\sqrt{9 - x^2}\,dx$$

$$= 2\rho g\int_1^3 x\sqrt{9 - x^2}\,dx - 2\rho g\int_1^3 \sqrt{9 - x^2}\,dx \overset{30}{=} 2\rho g\left[-\tfrac{1}{3}(9 - x^2)^{3/2}\right]_1^3 - 2\rho g\left[\frac{x}{2}\sqrt{9 - x^2} + \tfrac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_1^3$$

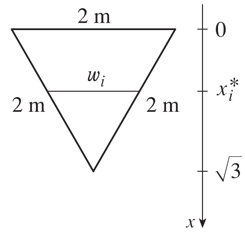
$$= 2\rho g\left[0 + \tfrac{1}{3}\left(8\sqrt{8}\right)\right] - 2\rho g\left[\left(0 + \tfrac{9}{2}\cdot\tfrac{\pi}{2}\right) - \left(\tfrac{1}{2}\sqrt{8} + \tfrac{9}{2}\sin^{-1}\left(\tfrac{1}{3}\right)\right)\right]$$

$$= \tfrac{32}{3}\sqrt{2}\,\rho g - \tfrac{9\pi}{2}\rho g + 2\sqrt{2}\,\rho g + 9\left[\sin^{-1}\left(\tfrac{1}{3}\right)\right]\rho g = \left(\tfrac{38}{3}\sqrt{2} - \tfrac{9\pi}{2} + 9\sin^{-1}\left(\tfrac{1}{3}\right)\right)\rho g$$

$$\approx 6.835\cdot 1000\cdot 9.8 \approx 6.7\times 10^4\ \text{N}$$

Note: If you set up a typical coordinate system with the water level at $y = -1$, then $F = \int_{-3}^{-1} \rho g(-1 - y)2\sqrt{9 - y^2}\,dy$.

7. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is



$\left(2 - \frac{2}{\sqrt{3}}\, x_i^*\right) \Delta x$. $\left[\text{By similar triangles, } \frac{w_i}{2} = \frac{\sqrt{3} - x_i^*}{\sqrt{3}}\text{, so } w_i = 2 - \frac{2}{\sqrt{3}}\, x_i^*.\right]$

The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is

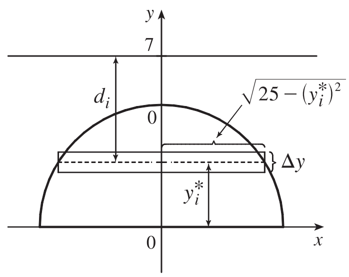
$\rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\, x_i^*\right) \Delta x$ and the hydrostatic force on the plate $\approx \sum\limits_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\, x_i^*\right) \Delta x$.

The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}}\, x_i^*\right) \Delta x = \int_0^{\sqrt{3}} \rho g x \left(2 - \frac{2}{\sqrt{3}}\, x\right) dx = \rho g \int_0^{\sqrt{3}} \left(2x - \frac{2}{\sqrt{3}}\, x^2\right) dx$$

$$= \rho g \left[x^2 - \frac{2}{3\sqrt{3}}\, x^3\right]_0^{\sqrt{3}} = \rho g\,[(3 - 2) - 0] = \rho g \approx 1000 \cdot 9.8 = 9.8 \times 10^3 \text{ N}$$

9. Set up coordinate axes as shown in the figure. The length of the ith strip is



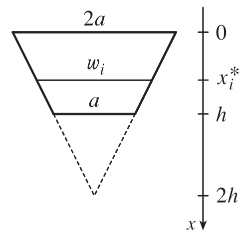
$2\sqrt{25 - (y_i^*)^2}$ and its area is $2\sqrt{25 - (y_i^*)^2}\,\Delta y$. The pressure on this strip is approximately $\delta d_i = 62.5(7 - y_i^*)$ and so the force on the strip is approximately $62.5(7 - y_i^*)2\sqrt{25 - (y_i^*)^2}\,\Delta y$. The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.5(7 - y_i^*)2\sqrt{25 - (y_i^*)^2}\,\Delta y = 125 \int_0^5 (7 - y)\sqrt{25 - y^2}\,dy$$

$$= 125\left\{\int_0^5 7\sqrt{25 - y^2}\,dy - \int_0^5 y\sqrt{25 - y^2}\,dy\right\} = 125\left\{7\int_0^5 \sqrt{25 - y^2}\,dy - \left[-\tfrac{1}{3}(25 - y^2)^{3/2}\right]_0^5\right\}$$

$$= 125\left\{7\left(\tfrac{1}{4}\pi \cdot 5^2\right) + \tfrac{1}{3}(0 - 125)\right\} = 125\left(\tfrac{175\pi}{4} - \tfrac{125}{3}\right) \approx 11{,}972 \approx 1.2 \times 10^4 \text{ lb}$$

11. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is



$\frac{a}{h}(2h - x_i^*)\,\Delta x$. $\left[\text{By similar triangles, } \frac{w_i}{2h - x_i^*} = \frac{2a}{2h}\text{, so } w_i = \frac{a}{h}(2h - x_i^*).\right]$

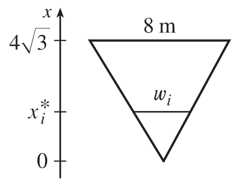
The pressure on the strip is δx_i^*, so the hydrostatic force on the plate

$\approx \sum\limits_{i=1}^{n} \delta x_i^*\, \frac{a}{h}(2h - x_i^*)\,\Delta x$. The total force

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^*\, \frac{a}{h}(2h - x_i^*)\,\Delta x = \delta\,\frac{a}{h}\int_0^h x(2h - x)\,dx = \frac{a\delta}{h}\int_0^h \left(2hx - x^2\right) dx$$

$$= \frac{a\delta}{h}\left[hx^2 - \tfrac{1}{3}x^3\right]_0^h = \frac{a\delta}{h}\left(h^3 - \tfrac{1}{3}h^3\right) = \frac{a\delta}{h}\left(\frac{2h^3}{3}\right) = \tfrac{2}{3}\delta a h^2$$

13. By similar triangles, $\dfrac{8}{4\sqrt{3}} = \dfrac{w_i}{x_i^*} \quad\Rightarrow\quad w_i = \dfrac{2x_i^*}{\sqrt{3}}$. The area of the ith rectangular strip is $\dfrac{2x_i^*}{\sqrt{3}}\,\Delta x$ and the pressure on it is $\rho g\big(4\sqrt{3} - x_i^*\big)$.



$$F = \int_0^{4\sqrt{3}} \rho g\Big(4\sqrt{3} - x\Big)\frac{2x}{\sqrt{3}}\,dx = 8\rho g\int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}} x^2\,dx$$

$$= 4\rho g\big[x^2\big]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\big[x^3\big]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}}\,64\cdot 3\sqrt{3} = 192\rho g - 128\rho g = 64\rho g$$

$$\approx 64(840)(9.8) \approx 5.27\times 10^5\ \mathrm{N}$$

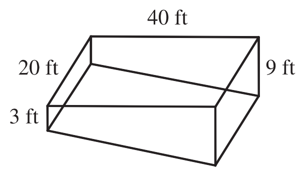
15. (a) The top of the cube has depth $d = 1\ \mathrm{m} - 20\ \mathrm{cm} = 80\ \mathrm{cm} = 0.8\ \mathrm{m}$.

$$F = \rho g d A \approx (1000)(9.8)(0.8)(0.2)^2 = 313.6 \approx 314\ \mathrm{N}$$

(b) The area of a strip is $0.2\,\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F = \int_{0.8}^1 \rho g x(0.2)\,dx = 0.2\rho g\big[\tfrac{1}{2}x^2\big]_{0.8}^1 = (0.2\rho g)(0.18) = 0.036\rho g = 0.036(1000)(9.8) = 352.8 \approx 353\ \mathrm{N}$$

17. (a) The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i.



$$F = \int_0^3 \delta x 20\,dx = 20\delta\big[\tfrac{1}{2}x^2\big]_0^3 = 20\delta\cdot\tfrac{9}{2} = 90\delta$$

$$= 90(62.5) = 5625\ \mathrm{lb} \approx 5.63\times 10^3\ \mathrm{lb}$$

(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\big[\tfrac{1}{2}x^2\big]_0^9 = 20\delta\cdot\tfrac{81}{2} = 810\delta = 810(62.5) = 50{,}625\ \mathrm{lb} \approx 5.06\times 10^4\ \mathrm{lb}.$

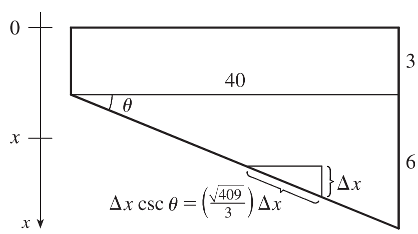
(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a:

$$\frac{a}{40} = \frac{9-x}{6} \quad\Rightarrow\quad a = 40\cdot\frac{9-x}{6}.$$

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x(40)\,\frac{9-x}{6}\,dx = 40\delta\big[\tfrac{1}{2}x^2\big]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx = 180\delta + \tfrac{20}{3}\delta\big[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\big]_3^9$$

$$= 180\delta + \tfrac{20}{3}\delta\big[\big(\tfrac{729}{2} - 243\big) - \big(\tfrac{81}{2} - 9\big)\big] = 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750\ \mathrm{lb} \approx 4.88\times 10^4\ \mathrm{lb}$$

(d) For any right triangle with hypotenuse on the bottom,



$$\sin\theta = \frac{\Delta x}{\text{hypotenuse}} \quad\Rightarrow$$

$$\text{hypotenuse} = \Delta x\csc\theta = \Delta x\,\frac{\sqrt{40^2+6^2}}{6} = \frac{\sqrt{409}}{3}\,\Delta x.$$

$$F = \int_3^9 \delta x 20\,\tfrac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\big(20\sqrt{409}\big)\delta\big[\tfrac{1}{2}x^2\big]_3^9$$

$$= \tfrac{1}{3}\cdot 10\sqrt{409}\,\delta(81-9) \approx 303{,}356\ \mathrm{lb} \approx 3.03\times 10^5\ \mathrm{lb}$$

19. $F = \int_2^5 \rho g x\cdot w(x)\,dx$, where $w(x)$ is the width of the plate at depth x. Since $n = 6$, $\Delta x = \frac{5-2}{6} = \frac{1}{2}$, and

$$F \approx S_6$$

$$= \rho g\cdot\tfrac{1/2}{3}\,[2\cdot w(2) + 4\cdot 2.5\cdot w(2.5) + 2\cdot 3\cdot w(3) + 4\cdot 3.5\cdot w(3.5) + 2\cdot 4\cdot w(4) + 4\cdot 4.5\cdot w(4.5) + 5\cdot w(5)]$$

$$= \tfrac{1}{6}\rho g(2\cdot 0 + 10\cdot 0.8 + 6\cdot 1.7 + 14\cdot 2.4 + 8\cdot 2.9 + 18\cdot 3.3 + 5\cdot 3.6)$$

$$= \tfrac{1}{6}(1000)(9.8)(152.4) \approx 2.5\times 10^5\ \mathrm{N}$$

21. The moment M of the system about the origin is $M = \sum_{i=1}^{2} m_i x_i = m_1 x_1 + m_2 x_2 = 40 \cdot 2 + 30 \cdot 5 = 230$.

The mass m of the system is $m = \sum_{i=1}^{2} m_i = m_1 + m_2 = 40 + 30 = 70$.

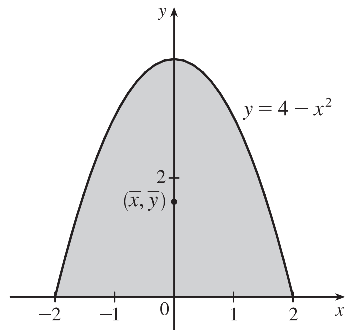
The center of mass of the system is $M/m = \frac{230}{70} = \frac{23}{7}$.

23. $m = \sum_{i=1}^{3} m_i = 6 + 5 + 10 = 21$.

$M_x = \sum_{i=1}^{3} m_i\, y_i = 6(5) + 5(-2) + 10(-1) = 10$; $M_y = \sum_{i=1}^{3} m_i\, x_i = 6(1) + 5(3) + 10(-2) = 1$.

$\overline{x} = \dfrac{M_y}{m} = \dfrac{1}{21}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{10}{21}$, so the center of mass of the system is $\left(\frac{1}{21}, \frac{10}{21}\right)$.

25. Since the region in the figure is symmetric about the y-axis, we know that $\overline{x} = 0$. The region is "bottom-heavy," so we know that $\overline{y} < 2$, and we might guess that $\overline{y} = 1.5$.



$A = \int_{-2}^{2} (4 - x^2)\, dx = 2 \int_0^2 (4 - x^2)\, dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2$

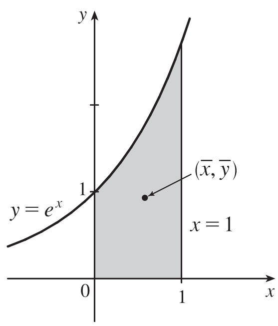
$= 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$.

$\overline{x} = \frac{1}{A} \int_{-2}^{2} x(4 - x^2)\, dx = 0$ since $f(x) = x(4 - x^2)$ is an odd function (or since the region is symmetric about the y-axis).

$$\overline{y} = \tfrac{1}{A} \int_{-2}^{2} \tfrac{1}{2}(4 - x^2)^2\, dx = \tfrac{3}{32} \cdot \tfrac{1}{2} \cdot 2 \int_0^2 (16 - 8x^2 + x^4)\, dx = \tfrac{3}{32}\left[16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right]_0^2$$
$$= \tfrac{3}{32}\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 3\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 3\left(\tfrac{8}{15}\right) = \tfrac{8}{5}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{8}{5}\right)$.

27. The region in the figure is "right-heavy" and "bottom-heavy," so we know $\overline{x} > 0.5$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 0.6$ and $\overline{y} = 0.9$.



$A = \int_0^1 e^x\, dx = [e^x]_0^1 = e - 1$.

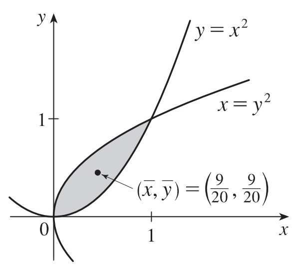
$\overline{x} = \frac{1}{A} \int_0^1 xe^x\, dx = \frac{1}{e-1}[xe^x - e^x]_0^1$ [by parts]

$= \frac{1}{e-1}[0 - (-1)] = \frac{1}{e-1}$.

$\overline{y} = \frac{1}{A} \int_0^1 \frac{1}{2}(e^x)^2\, dx = \frac{1}{e-1} \cdot \frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e-1)}\left(e^2 - 1\right) = \frac{e+1}{4}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.

29. $A = \int_0^1 (x^{1/2} - x^2)\, dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3}$.



$\overline{x} = \frac{1}{A} \int_0^1 x(x^{1/2} - x^2)\, dx = 3 \int_0^1 (x^{3/2} - x^3)\, dx$

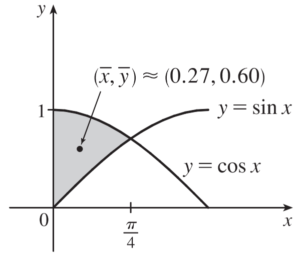
$= 3\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 = 3\left(\frac{2}{5} - \frac{1}{4}\right) = 3\left(\frac{3}{20}\right) = \frac{9}{20}$.

$\overline{y} = \frac{1}{A} \int_0^1 \frac{1}{2}\left[(x^{1/2})^2 - (x^2)^2\right] dx = 3\left(\frac{1}{2}\right) \int_0^1 (x - x^4)\, dx$

$= \frac{3}{2}\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}\left(\frac{3}{10}\right) = \frac{9}{20}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{20}, \frac{9}{20}\right)$.

31. $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \big[\sin x + \cos x\big]_0^{\pi/4} = \sqrt{2} - 1.$

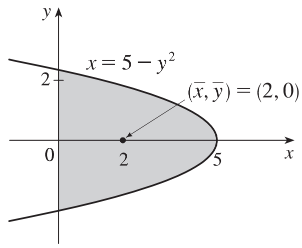


$$\begin{aligned}\overline{x} &= A^{-1} \int_0^{\pi/4} x(\cos x - \sin x)\, dx \\ &= A^{-1}\big[x(\sin x + \cos x) + \cos x - \sin x\big]_0^{\pi/4} \qquad \text{[integration by parts]} \\ &= A^{-1}\left(\tfrac{\pi}{4}\sqrt{2} - 1\right) = \frac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1}.\end{aligned}$$

$$\overline{y} = A^{-1} \int_0^{\pi/4} \tfrac{1}{2}(\cos^2 x - \sin^2 x)\, dx = \tfrac{1}{2A} \int_0^{\pi/4} \cos 2x\, dx = \tfrac{1}{4A}\big[\sin 2x\big]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4\left(\sqrt{2} - 1\right)}.$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{\pi\sqrt{2} - 4}{4(\sqrt{2} - 1)}, \dfrac{1}{4(\sqrt{2} - 1)}\right) \approx (0.27, 0.60)$.

33. From the figure we see that $\overline{y} = 0$. Now



$A = \int_0^5 2\sqrt{5 - x}\, dx = 2\Big[-\tfrac{2}{3}(5 - x)^{3/2}\Big]_0^5 = 2\Big(0 + \tfrac{2}{3} \cdot 5^{3/2}\Big) = \tfrac{20}{3}\sqrt{5}$,
so

$$\begin{aligned}\overline{x} &= \tfrac{1}{A} \int_0^5 x\big[\sqrt{5 - x} - \big(-\sqrt{5 - x}\big)\big]\, dx = \tfrac{1}{A} \int_0^5 2x\sqrt{5 - x}\, dx \\ &= \tfrac{1}{A} \int_{\sqrt{5}}^0 2(5 - u^2)u(-2u)\, du \qquad \begin{bmatrix} u = \sqrt{5 - x},\ x = 5 - u^2, \\ u^2 = 5 - x,\ dx = -2u\, du \end{bmatrix} \\ &= \tfrac{4}{A} \int_0^{\sqrt{5}} u^2(5 - u^2)\, du = \tfrac{4}{A}\big[\tfrac{5}{3}u^3 - \tfrac{1}{5}u^5\big]_0^{\sqrt{5}} = \tfrac{3}{5\sqrt{5}}\big(\tfrac{25}{3}\sqrt{5} - 5\sqrt{5}\big) = 5 - 3 = 2.\end{aligned}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = (2, 0)$.

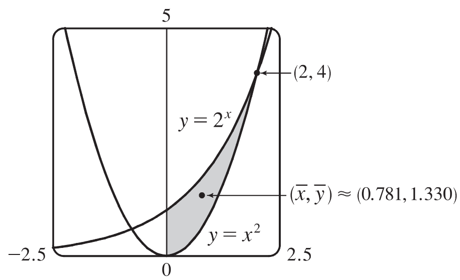
35. The line has equation $y = \frac{3}{4}x$. $A = \frac{1}{2}(4)(3) = 6$, so $m = \rho A = 10(6) = 60$.

$$M_x = \rho \int_0^4 \tfrac{1}{2}\big(\tfrac{3}{4}x\big)^2\, dx = 10 \int_0^4 \tfrac{9}{32}x^2\, dx = \tfrac{45}{16}\big[\tfrac{1}{3}x^3\big]_0^4 = \tfrac{45}{16}\big(\tfrac{64}{3}\big) = 60$$

$$M_y = \rho \int_0^4 x\big(\tfrac{3}{4}x\big)\, dx = \tfrac{15}{2} \int_0^4 x^2\, dx = \tfrac{15}{2}\big[\tfrac{1}{3}x^3\big]_0^4 = \tfrac{15}{2}\big(\tfrac{64}{3}\big) = 160$$

$\overline{x} = \dfrac{M_y}{m} = \dfrac{160}{60} = \dfrac{8}{3}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{60}{60} = 1$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{3}, 1\right)$.

37. $$\begin{aligned}A &= \int_0^2 (2^x - x^2)\, dx = \left[\frac{2^x}{\ln 2} - \frac{x^3}{3}\right]_0^2 \\ &= \left(\frac{4}{\ln 2} - \frac{8}{3}\right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{8}{3} \approx 1.661418.\end{aligned}$$



$$\begin{aligned}\overline{x} &= \frac{1}{A} \int_0^2 x(2^x - x^2)\, dx = \frac{1}{A} \int_0^2 (x2^x - x^3)\, dx \\ &= \frac{1}{A}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - \frac{x^4}{4}\right]_0^2 \qquad \text{[use parts]} \\ &= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} - 4 + \frac{1}{(\ln 2)^2}\right] = \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2} - 4\right] \approx \frac{1}{A}(1.297453) \approx 0.781\end{aligned}$$

[continued]

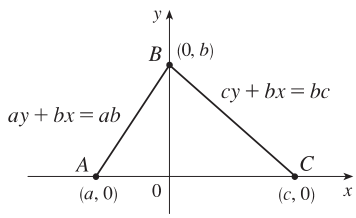
$$\overline{y} = \frac{1}{A}\int_0^2 \tfrac{1}{2}[(2^x)^2 - (x^2)^2]\,dx = \frac{1}{A}\int_0^2 \tfrac{1}{2}(2^{2x} - x^4)\,dx = \frac{1}{A}\cdot\frac{1}{2}\left[\frac{2^{2x}}{2\ln 2} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{A}\cdot\frac{1}{2}\left(\frac{16}{2\ln 2} - \frac{32}{5} - \frac{1}{2\ln 2}\right) = \frac{1}{A}\left(\frac{15}{4\ln 2} - \frac{16}{5}\right) \approx \frac{1}{A}(2.210106) \approx 1.330$$

Thus, the centroid is $(\overline{x},\overline{y}) \approx (0.781, 1.330)$.

Since the position of a centroid is independent of density when the density is constant, we will assume for convenience that $\rho = 1$ in Exercises 38 and 39.

39. Choose x- and y-axes so that the base (one side of the triangle) lies along the x-axis with the other vertex along the positive y-axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a+c), 0\right)$ of side AC, so the point of intersection of the medians is $\left(\frac{2}{3}\cdot\frac{1}{2}(a+c), \frac{1}{3}b\right) = \left(\frac{1}{3}(a+c), \frac{1}{3}b\right)$.



This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c-a)b$.

$$\overline{x} = \frac{1}{A}\left[\int_a^0 x\cdot\frac{b}{a}(a-x)\,dx + \int_0^c x\cdot\frac{b}{c}(c-x)\,dx\right] = \frac{1}{A}\left[\frac{b}{a}\int_a^0 (ax - x^2)\,dx + \frac{b}{c}\int_0^c (cx - x^2)\,dx\right]$$

$$= \frac{b}{Aa}\left[\frac{1}{2}ax^2 - \frac{1}{3}x^3\right]_a^0 + \frac{b}{Ac}\left[\frac{1}{2}cx^2 - \frac{1}{3}x^3\right]_0^c = \frac{b}{Aa}\left[-\frac{1}{2}a^3 + \frac{1}{3}a^3\right] + \frac{b}{Ac}\left[\frac{1}{2}c^3 - \frac{1}{3}c^3\right]$$

$$= \frac{2}{a\,(c-a)}\cdot\frac{-a^3}{6} + \frac{2}{c\,(c-a)}\cdot\frac{c^3}{6} = \frac{1}{3\,(c-a)}(c^2 - a^2) = \frac{a+c}{3}$$

and
$$\overline{y} = \frac{1}{A}\left[\int_a^0 \frac{1}{2}\left(\frac{b}{a}(a-x)\right)^2 dx + \int_0^c \frac{1}{2}\left(\frac{b}{c}(c-x)\right)^2 dx\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\int_a^0 (a^2 - 2ax + x^2)\,dx + \frac{b^2}{2c^2}\int_0^c (c^2 - 2cx + x^2)\,dx\right]$$

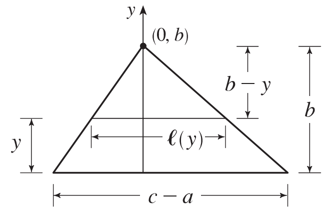
$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left[a^2x - ax^2 + \tfrac{1}{3}x^3\right]_a^0 + \frac{b^2}{2c^2}\left[c^2x - cx^2 + \tfrac{1}{3}x^3\right]_0^c\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left(-a^3 + a^3 - \tfrac{1}{3}a^3\right) + \frac{b^2}{2c^2}\left(c^3 - c^3 + \tfrac{1}{3}c^3\right)\right] = \frac{1}{A}\left[\frac{b^2}{6}\,(-a+c)\right] = \frac{2}{(c-a)\,b}\cdot\frac{(c-a)b^2}{6} = \frac{b}{3}$$

Thus, the centroid is $(\overline{x},\overline{y}) = \left(\dfrac{a+c}{3}, \dfrac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\overline{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\overline{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\,\Delta y$, its mass is $\rho\ell(y)\,\Delta y$, and its moment about the x-axis is $\Delta M_x = \rho y \ell(y)\,\Delta y$. Thus,



$$M_x = \int \rho y \ell(y)\,dy \qquad \text{and} \qquad \overline{y} = \frac{\int \rho y \ell(y)\,dy}{\rho A} = \frac{1}{A}\int y\ell(y)\,dy$$

In this problem, $\ell(y) = \dfrac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y} = \frac{1}{A}\int_0^b \frac{c-a}{b}\,y(b-y)\,dy = \frac{2}{b^2}\int_0^b (by - y^2)\,dy = \frac{2}{b^2}\left[\tfrac{1}{2}by^2 - \tfrac{1}{3}y^3\right]_0^b = \frac{2}{b^2}\cdot\frac{b^3}{6} = \frac{b}{3}$$

Notice that only one integral is needed when this method is used.

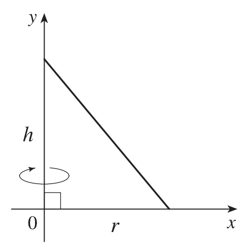
41. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 39, the triangles have centroids $\left(-1, \frac{2}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0, -\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have $\overline{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{3} m_i\, y_i = \frac{1}{8}\left[2\left(\tfrac{2}{3}\right) + 2\left(\tfrac{2}{3}\right) + 4\left(-\tfrac{1}{2}\right)\right] = \frac{1}{8}\left(\tfrac{2}{3}\right) = \frac{1}{12}$, and $\overline{x} = 0$, since the lamina is symmetric about the line $x = 0$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{1}{12}\right)$.

43. $\int_a^b (cx+d)\,f(x)\,dx = \int_a^b cx\,f(x)\,dx + \int_a^b d\,f(x)\,dx = c\int_a^b x\,f(x)\,dx + d\int_a^b f(x)\,dx = c\overline{x}A + d\int_a^b f(x)\,dx$ [by (8)]

$$= c\overline{x}\int_a^b f(x)\,dx + d\int_a^b f(x)\,dx = (c\overline{x} + d)\int_a^b f(x)\,dx$$

45. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 39, $\overline{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is

$$V = Ad = \left(\tfrac{1}{2}\cdot\text{base}\cdot\text{height}\right)\cdot(2\pi\overline{x}) = \tfrac{1}{2}rh\cdot 2\pi\left(\tfrac{1}{3}r\right) = \tfrac{1}{3}\pi r^2 h.$$



47. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the ith subinterval; that is, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$. Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$. Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x\cdot\overline{x}_i = \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and $M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x\cdot\frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho\cdot\frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x$. Summing over i and taking the limit as $n\to\infty$, we get $M_y = \lim\limits_{n\to\infty}\sum_i \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and

$M_x = \lim\limits_{n\to\infty}\sum_i \rho\cdot\frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx.$

Thus, $\overline{x} = \dfrac{M_y}{m} = \dfrac{M_y}{\rho A} = \dfrac{1}{A}\displaystyle\int_a^b x[f(x) - g(x)]\,dx$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{M_x}{\rho A} = \dfrac{1}{A}\displaystyle\int_a^b \tfrac{1}{2}\left[f(x)^2 - g(x)^2\right]dx.$

8.4 Applications to Economics and Biology

1. By the Net Change Theorem, $C(2000) - C(0) = \int_0^{2000} C'(x)\,dx \quad \Rightarrow$

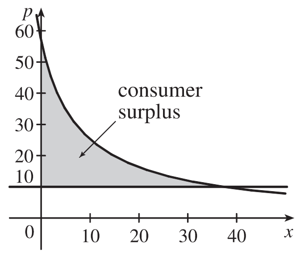
$$\begin{aligned} C(2000) &= 20{,}000 + \int_0^{2000} (5 - 0.008x + 0.000009x^2)\,dx = 20{,}000 + \left[5x - 0.004x^2 + 0.000003x^3\right]_0^{2000} \\ &= 20{,}000 + 10{,}000 - 0.004(4{,}000{,}000) + 0.000003(8{,}000{,}000{,}000) = 30{,}000 - 16{,}000 + 24{,}000 \\ &= \$38{,}000 \end{aligned}$$

3. If the production level is raised from 1200 units to 1600 units, then the increase in cost is

$$\begin{aligned} C(1600) - C(1200) &= \int_{1200}^{1600} C'(x)\,dx = \int_{1200}^{1600} (74 + 1.1x - 0.002x^2 + 0.00004x^3)\,dx \\ &= \left[74x + 0.55x^2 - \tfrac{0.002}{3}x^3 + 0.00001x^4\right]_{1200}^{1600} = 64{,}331{,}733.33 - 20{,}464{,}800 = \$43{,}866{,}933.33 \end{aligned}$$

5. $p(x) = 10 \quad \Rightarrow \quad \dfrac{450}{x+8} = 10 \quad \Rightarrow \quad x + 8 = 45 \quad \Rightarrow \quad x = 37.$

$$\begin{aligned} \text{Consumer surplus} &= \int_0^{37} [p(x) - 10]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10\right) dx \\ &= \left[450 \ln(x+8) - 10x\right]_0^{37} = (450 \ln 45 - 370) - 450 \ln 8 \\ &= 450 \ln\left(\tfrac{45}{8}\right) - 370 \approx \$407.25 \end{aligned}$$

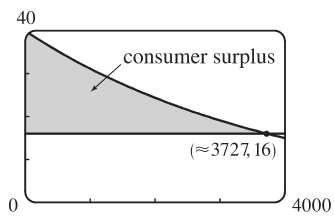


7. $P = p_S(x) \quad \Rightarrow \quad 400 = 200 + 0.2x^{3/2} \quad \Rightarrow \quad 200 = 0.2x^{3/2} \quad \Rightarrow \quad 1000 = x^{3/2} \quad \Rightarrow \quad x = 1000^{2/3} = 100.$

$$\begin{aligned} \text{Producer surplus} &= \int_0^{100} [P - p_S(x)]\,dx = \int_0^{100} [400 - (200 + 0.2x^{3/2})]\,dx = \int_0^{100} \left(200 - \tfrac{1}{5}x^{3/2}\right) dx \\ &= \left[200x - \tfrac{2}{25}x^{5/2}\right]_0^{100} = 20{,}000 - 8{,}000 = \$12{,}000 \end{aligned}$$

9. $p(x) = \dfrac{800{,}000e^{-x/5000}}{x + 20{,}000} = 16 \quad \Rightarrow \quad x = x_1 \approx 3727.04.$

Consumer surplus $= \int_0^{x_1} [p(x) - 16]\,dx \approx \$37{,}753$



11. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\tfrac{2}{3}t^{3/2}\right]_4^8 = \tfrac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

13. $N = \displaystyle\int_a^b Ax^{-k}\,dx = A\left[\frac{x^{-k+1}}{-k+1}\right]_a^b = \frac{A}{1-k}\,(b^{1-k} - a^{1-k}).$

Similarly, $\displaystyle\int_a^b Ax^{1-k}\,dx = A\left[\frac{x^{2-k}}{2-k}\right]_a^b = \frac{A}{2-k}\,(b^{2-k} - a^{2-k}).$

Thus, $\overline{x} = \displaystyle\frac{1}{N}\int_a^b Ax^{1-k}\,dx = \frac{[A/(2-k)](b^{2-k} - a^{2-k})}{[A/(1-k)](b^{1-k} - a^{1-k})} = \frac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}.$

15. $F = \dfrac{\pi P R^4}{8\eta l} = \dfrac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

17. From (3), $F = \dfrac{A}{\int_0^T c(t)\,dt} = \dfrac{6}{20I}$, where

$$I = \int_0^{10} te^{-0.6t}\,dt = \left[\frac{1}{(-0.6)^2}(-0.6t-1)\,e^{-0.6t}\right]_0^{10} \quad \begin{bmatrix}\text{integrating}\\ \text{by parts}\end{bmatrix} \quad = \tfrac{1}{0.36}(-7e^{-6}+1)$$

Thus, $F = \dfrac{6(0.36)}{20(1-7e^{-6})} = \dfrac{0.108}{1-7e^{-6}} \approx 0.1099$ L/s or 6.594 L/min.

19. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = (16-0)/8 = 2$.

$$\begin{aligned}\textstyle\int_0^{16} c(t)\,dt &\approx \tfrac{2}{3}[c(0)+4c(2)+2c(4)+4c(6)+2c(8)+4c(10)+2c(12)+4c(14)+c(16)]\\ &\approx \tfrac{2}{3}[0+4(6.1)+2(7.4)+4(6.7)+2(5.4)+4(4.1)+2(3.0)+4(2.1)+1.5]\\ &= \tfrac{2}{3}(109.1) = 72.7\overline{3}\ \text{mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{72.7\overline{3}} = \dfrac{7}{72.7\overline{3}} \approx 0.0962$ L/s or 5.77 L/min.

8.5 Probability

1. (a) $\int_{30,000}^{40,000} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime between 30,000 and 40,000 miles.

(b) $\int_{25,000}^{\infty} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime of at least 25,000 miles.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $0 \leq x \leq 4$, we have $f(x) = \frac{3}{64}x\sqrt{16-x^2} \geq 0$, so $f(x) \geq 0$ for all x. Also,

$$\begin{aligned}\textstyle\int_{-\infty}^{\infty} f(x)\,dx &= \textstyle\int_0^4 \frac{3}{64}x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^4 (16-x^2)^{1/2}(-2x)\,dx = -\frac{3}{128}\left[\frac{2}{3}(16-x^2)^{3/2}\right]_0^4\\ &= -\tfrac{1}{64}\left[\left(16-x^2\right)^{3/2}\right]_0^4 = -\tfrac{1}{64}(0-64) = 1.\end{aligned}$$

Therefore, f is a probability density function.

(b)
$$\begin{aligned}P(X<2) &= \textstyle\int_{-\infty}^{2} f(x)\,dx = \int_0^2 \frac{3}{64}x\sqrt{16-x^2}\,dx = -\frac{3}{128}\int_0^2 (16-x^2)^{1/2}(-2x)\,dx\\ &= -\tfrac{3}{128}\left[\tfrac{2}{3}(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}\left[(16-x^2)^{3/2}\right]_0^2 = -\tfrac{1}{64}(12^{3/2}-16^{3/2})\\ &= \tfrac{1}{64}\left(64-12\sqrt{12}\right) = \tfrac{1}{64}\left(64-24\sqrt{3}\right) = 1-\tfrac{3}{8}\sqrt{3} \approx 0.350481\end{aligned}$$

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. If $c \geq 0$, then $f(x) \geq 0$, so condition (1) is satisfied. For condition (2), we see that

$\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx$ and

$$\int_0^{\infty} \frac{c}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{c}{1+x^2}\,dx = c\lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = c\lim_{t\to\infty}\tan^{-1}t = c\left(\frac{\pi}{2}\right)$$

Similarly, $\displaystyle\int_{-\infty}^{0} \frac{c}{1+x^2}\,dx = c\left(\frac{\pi}{2}\right)$, so $\displaystyle\int_{-\infty}^{\infty} \frac{c}{1+x^2}\,dx = 2c\left(\frac{\pi}{2}\right) = c\pi$.

Since $c\pi$ must equal 1, we must have $c = 1/\pi$ so that f is a probability density function.

(b) $P\,(-1 < X < 1) = \displaystyle\int_{-1}^{1} \frac{1/\pi}{1+x^2}\,dx = \frac{2}{\pi}\int_0^1 \frac{1}{1+x^2}\,dx = \frac{2}{\pi}\Big[\tan^{-1} x\Big]_0^1 = \frac{2}{\pi}\Big(\frac{\pi}{4} - 0\Big) = \frac{1}{2}$

7. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, (1) $f(x) \geq 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Since $f(x) = 0$ or $f(x) = 0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.1\,dx = \left[\frac{1}{10}x\right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{10} x(0.1)\,dx = \left[\tfrac{1}{20}x^2\right]_0^{10} = \tfrac{100}{20} = 5, \quad \text{as expected.}$$

9. We need to find m so that $\int_m^{\infty} f(t)\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty} \int_m^x \frac{1}{5}e^{-t/5}\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty} \left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x = \frac{1}{2} \;\Rightarrow$ $(-1)(0 - e^{-m/5}) = \frac{1}{2} \;\Rightarrow\; e^{-m/5} = \frac{1}{2} \;\Rightarrow\; -m/5 = \ln\frac{1}{2} \;\Rightarrow\; m = -5\ln\frac{1}{2} = 5\ln 2 \approx 3.47$ min.

11. We use an exponential density function with $\mu = 2.5$ min.

(a) $P(X > 4) = \int_4^{\infty} f(t)\,dt = \lim\limits_{x\to\infty} \int_4^x \frac{1}{2.5}e^{-t/2.5}\,dt = \lim\limits_{x\to\infty} \left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$

(b) $P(0 \leq X \leq 2) = \int_0^2 f(t)\,dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$

(c) We need to find a value a so that $P(X \geq a) = 0.02$, or, equivalently, $P(0 \leq X \leq a) = 0.98 \;\Leftrightarrow$ $\int_0^a f(t)\,dt = 0.98 \;\Leftrightarrow\; \left[-e^{-t/2.5}\right]_0^a = 0.98 \;\Leftrightarrow\; -e^{-a/2.5} + 1 = 0.98 \;\Leftrightarrow\; e^{-a/2.5} = 0.02 \;\Leftrightarrow$ $-a/2.5 = \ln 0.02 \;\Leftrightarrow\; a = -2.5\ln\frac{1}{50} = 2.5\ln 50 \approx 9.78 \text{ min} \approx 10$ min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

13. $P(X \geq 10) = \displaystyle\int_{10}^{\infty} \frac{1}{4.2\sqrt{2\pi}} \exp\!\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right) dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X \geq 10) \approx \displaystyle\int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\!\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right) dx \approx 0.443$ (using a calculator or computer to estimate the integral), so about 44 percent of the households throw out at least 10 lb of paper a week.

Note: We can't evaluate $1 - P(0 \leq X \leq 10)$ for this problem since a significant amount of area lies to the left of $X = 0$.

15. (a) $P(0 \leq X \leq 100) = \displaystyle\int_0^{100} \frac{1}{8\sqrt{2\pi}} \exp\!\left(-\frac{(x-112)^2}{2\cdot 8^2}\right) dx \approx 0.0668$ (using a calculator or computer to estimate the integral), so there is about a 6.68% chance that a randomly chosen vehicle is traveling at a legal speed.

(b) $P(X \geq 125) = \displaystyle\int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} \exp\!\left(-\frac{(x-112)^2}{2\cdot 8^2}\right) dx = \int_{125}^{\infty} f(x)\,dx$. In this case, we could use a calculator or computer to estimate either $\int_{125}^{300} f(x)\,dx$ or $1 - \int_0^{125} f(x)\,dx$. Both are approximately 0.0521, so about 5.21% of the motorists are targeted.

17. $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$. Substituting $t = \frac{x-\mu}{\sigma}$ and $dt = \frac{1}{\sigma}\,dx$ gives us

$\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\,dt) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-t^2/2}\,dt \approx 0.9545$.

19. (a) First $p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \ge 0$ for $r \ge 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^2 e^{-2r/a_0}\,dr$$

By using parts, tables, or a CAS , we find that $\int x^2 e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$. $(\star)$

Next, we use $(\star)$ (with $b = -2/a_0$) and l'Hospital's Rule to get $\frac{4}{a_0^3}\left[\frac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\frac{4}{a_0^3} \lim_{r\to\infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r\to\infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r\to\infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0$.

To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

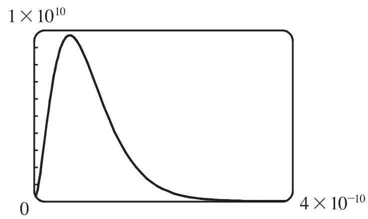
$p'(r) = 0 \quad \Leftrightarrow \quad r = 0$ or $1 = \frac{r}{a_0} \quad \Leftrightarrow \quad r = a_0 \quad [a_0 \approx 5.59 \times 10^{-11}\text{ m}]$.

$p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3} a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0} e^{-2} \approx 9{,}684{,}098{,}979$$

With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.



(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds \quad\Rightarrow\quad P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\,ds$. Using $(\star)$ from part (a) [with $b = -2/a_0$],

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)[e^{-8}(64+16+2) - 1(2)] = -\tfrac{1}{2}(82e^{-8} - 2)$$
$$= 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} r p(r)\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0}\,dr$. Integrating by parts three times or using a CAS, we find that

$\int x^3 e^{bx}\,dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get

$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0$.

8 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 8.1.

(b) See Equation 8.1.2.

(c) See Equation 8.1.4.

2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$

(b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y\,\sqrt{1 + [g'(y)]^2}\,dy$.

(c) $S = \int_a^b 2\pi x\,\sqrt{1 + [f'(x)]^2}\,dx$ or $S = \int_c^d 2\pi g(y)\,\sqrt{1 + [g'(y)]^2}\,dy$

3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\,dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.

4. (a) The center of mass is the point at which the plate balances horizontally.

(b) See Equations 8.3.8.

5. If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

6. See Figure 3 in Section 8.4, and the discussion which precedes it.

7. (a) See the definition in the first paragraph of the subsection *Cardiac Output* in Section 8.4.

(b) See the discussion in the second paragraph of the subsection *Cardiac Output* in Section 8.4.

8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\,dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\,dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{130} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 130 pounds.

(b) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x f(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

10. See the discussion near Equation 3 in Section 8.5.

EXERCISES

1. $y = \frac{1}{6}(x^2+4)^{3/2} \quad\Rightarrow\quad dy/dx = \frac{1}{4}(x^2+4)^{1/2}(2x) \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + \left[\frac{1}{2}x(x^2+4)^{1/2}\right]^2 = 1 + \frac{1}{4}x^2(x^2+4) = \frac{1}{4}x^4 + x^2 + 1 = \left(\frac{1}{2}x^2+1\right)^2.$

Thus, $L = \int_0^3 \sqrt{\left(\frac{1}{2}x^2+1\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}x^2+1\right)dx = \left[\frac{1}{6}x^3 + x\right]_0^3 = \frac{15}{2}.$

3. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \quad\Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2.$

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right)dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}.$

(b) $S = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right)dx = 2\pi \int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right)dx = 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$

$= 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

5. $y = e^{-x^2} \quad\Rightarrow\quad dy/dx = -2xe^{-x^2} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}.$ Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right] \approx 3.292287$$

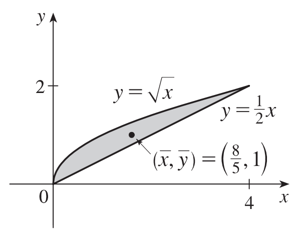
7. $y = \int_1^x \sqrt{\sqrt{t} - 1}\,dt \quad\Rightarrow\quad dy/dx = \sqrt{\sqrt{x} - 1} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}.$

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}.$

9. As in Example 1 of Section 8.3, $\dfrac{a}{2-x} = \dfrac{1}{2} \quad\Rightarrow\quad 2a = 2 - x$ and $w = 2(1.5 + a) = 3 + 2a = 3 + 2 - x = 5 - x.$

Thus, $F = \int_0^2 \rho gx(5-x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \quad [\rho g = \delta] \quad \approx \frac{22}{3} \cdot 62.5 \approx 458$ lb.

11. $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right)dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$



$\overline{x} = \frac{1}{A}\int_0^4 x\left(\sqrt{x} - \frac{1}{2}x\right)dx = \frac{3}{4}\int_0^4 \left(x^{3/2} - \frac{1}{2}x^2\right)dx$

$= \frac{3}{4}\left[\frac{2}{5}x^{5/2} - \frac{1}{6}x^3\right]_0^4 = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{3}{4}\left(\frac{64}{30}\right) = \frac{8}{5}$

$\overline{y} = \frac{1}{A}\int_0^4 \frac{1}{2}\left[\left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2\right]dx = \frac{3}{4}\int_0^4 \frac{1}{2}\left(x - \frac{1}{4}x^2\right)dx = \frac{3}{8}\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \frac{3}{8}\left(8 - \frac{16}{3}\right) = \frac{3}{8}\left(\frac{8}{3}\right) = 1$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, 1\right)$.

13. An equation of the line passing through $(0, 0)$ and $(3, 2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2} \cdot 3 \cdot 2 = 3$. Therefore, using Equations 8.3.8,

$\overline{x} = \frac{1}{3}\int_0^3 x\left(\frac{2}{3}x\right)dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$ and $\overline{y} = \frac{1}{3}\int_0^3 \frac{1}{2}\left(\frac{2}{3}x\right)^2 dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(2, \frac{2}{3}\right)$.

15. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

17. $x = 100 \quad\Rightarrow\quad P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\begin{aligned}\text{Consumer surplus} &= \int_0^{100} [p(x) - P]\,dx = \int_0^{100} \left(2000 - 0.1x - 0.01x^2 - 1890\right) dx \\ &= \left[110x - 0.05x^2 - \tfrac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \tfrac{10{,}000}{3} \approx \$7166.67\end{aligned}$$

19. $f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f(x) \ge 0$ for all real numbers x and

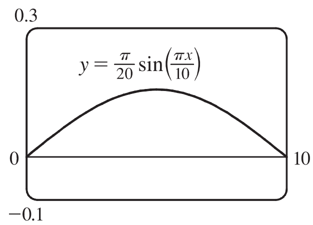
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \tfrac{\pi}{20}\sin\left(\tfrac{\pi}{10}x\right) dx = \tfrac{\pi}{20} \cdot \tfrac{10}{\pi}\left[-\cos\left(\tfrac{\pi}{10}x\right)\right]_0^{10} = \tfrac{1}{2}(-\cos\pi + \cos 0) = \tfrac{1}{2}(1 + 1) = 1$$

Therefore, f is a probability density function.

(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos\frac{2\pi}{5} + \cos 0\right)$

$$\approx \tfrac{1}{2}(-0.309017 + 1) \approx 0.3455$$

(c) $$\begin{aligned}\mu &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} \tfrac{\pi}{20}x\sin\left(\tfrac{\pi}{10}x\right) dx \\ &= \int_0^{\pi} \tfrac{\pi}{20} \cdot \tfrac{10}{\pi}u(\sin u)\left(\tfrac{10}{\pi}\right) du \qquad \left[u = \tfrac{\pi}{10}x,\ du = \tfrac{\pi}{10}\,dx\right] \\ &= \tfrac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \tfrac{5}{\pi}\left[\sin u - u\cos u\right]_0^{\pi} = \tfrac{5}{\pi}[0 - \pi(-1)] = 5\end{aligned}$$



This answer is expected because the graph of f is symmetric about the line $x = 5$.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$$P(0 \le X \le 3) = \int_0^3 \tfrac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$$

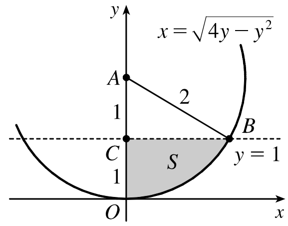
(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P(X \ge m) = \frac{1}{2} \quad\Rightarrow\quad \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \quad\Rightarrow\quad \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \quad\Rightarrow$

$\lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \quad\Rightarrow\quad e^{-m/8} = \frac{1}{2} \quad\Rightarrow\quad -m/8 = \ln\frac{1}{2} \quad\Rightarrow\quad m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

PROBLEMS PLUS

1. $x^2 + y^2 \le 4y \quad \Leftrightarrow \quad x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is

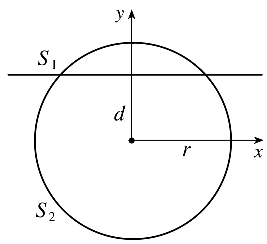


$$\int_0^1 \sqrt{4y - y^2}\,dy \overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y-y^2} + 2\cos^{-1}\!\left(\tfrac{2-y}{2}\right)\right]_0^1 \quad [a = 2]$$
$$= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\!\left(\tfrac{1}{2}\right) - 2\cos^{-1} 1$$
$$= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}\left(2^2\right)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



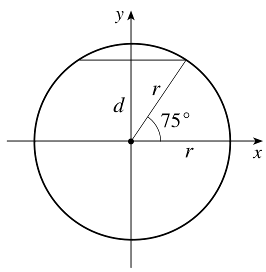
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy = \sqrt{\frac{r^2}{r^2 - y^2}}\,dy = \frac{r\,dy}{\sqrt{r^2 - y^2}}$$

From Formula 8.2.8 we have

$$S_1 = \int_d^r 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_d^r 2\pi\sqrt{r^2 - y^2}\,\frac{r\,dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\,dy = 2\pi r(r - d)$$

Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x\sqrt{1 + (dx/dy)^2}\,dy = \int_{-r}^{d} 2\pi r\,dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

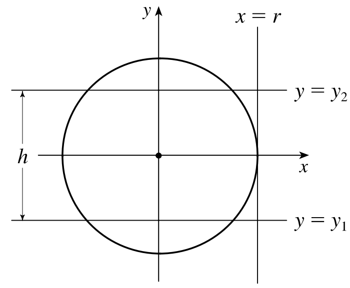
(b) $r = 3960$ mi and $d = r\,(\sin 75^\circ) \approx 3825$ mi, so the surface area of the Arctic Ocean is about $2\pi r(r - d) \approx 2\pi(3960)(135) \approx 3.36 \times 10^6\ \text{mi}^2$.



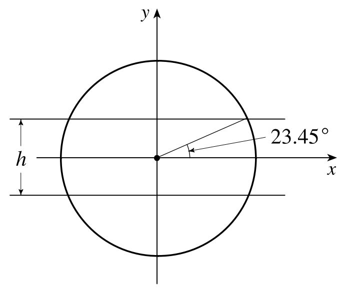
(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r\, dy = 2\pi r(y_2 - y_1) = 2\pi rh$.

This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is

$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\, dy$$
$$= 2\pi r y\Big|_{y=y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi rh$$



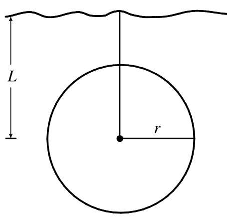
(d) $h = 2r \sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7\ \text{mi}^2$.



5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$.

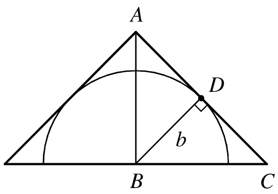
More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

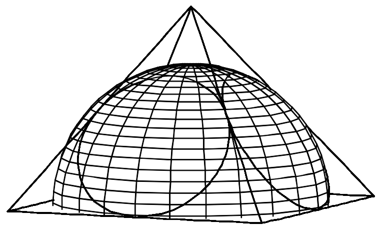
(b)

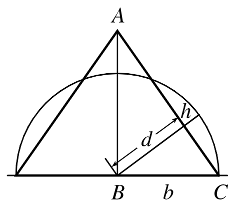


$$\begin{aligned}
F &= \textstyle\int_{-r}^{r} P(L + x) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= \textstyle\int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\, dz\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= P_0 \textstyle\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 gH \int_{-r}^{r} \left(e^{(L+x)/H} - 1\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= (P_0 - \rho_0 gH) \textstyle\int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 gH \int_{-r}^{r} e^{(L+x)/H} \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= (P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gH e^{L/H} \textstyle\int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\, dx
\end{aligned}$$

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}\,b$.





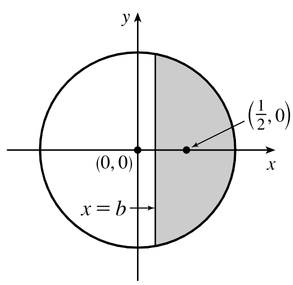


We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.51 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\,b}{\sqrt{b^2 + \left(\sqrt{2}\,b\right)^2}} \quad\Rightarrow\quad d = \frac{\sqrt{2}\,b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\,b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 6.2.51 with $r = b$, we find that the volume of each of the caps is $\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3\cdot 3}b\right) = \frac{15-6\sqrt{6}}{9}\cdot\frac{6+\sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

9. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have $\dfrac{1}{2} = \overline{x} = \dfrac{\int_b^1 x \cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the



numerator gives us $-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0 - \left(1-b^2\right)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$.

Using Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_b^1 = \left(0 + \frac{\pi}{2}\right) - \left(b\sqrt{1-b^2} + \sin^{-1}b\right)$. Thus, we have $\dfrac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}$, or, equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

11. If $h = L$, then $P = \dfrac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \dfrac{\int_0^\pi L\sin\theta\,d\theta}{\pi L} = \dfrac{[-\cos\theta]_0^\pi}{\pi} = \dfrac{-(-1)+1}{\pi} = \dfrac{2}{\pi}$.

If $h = L/2$, then $P = \dfrac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}} = \dfrac{\int_0^\pi \frac{1}{2}L\sin\theta\,d\theta}{\pi L} = \dfrac{[-\cos\theta]_0^\pi}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$.

9 ☐ DIFFERENTIAL EQUATIONS

9.1 Modeling with Differential Equations

1. $y = x - x^{-1} \quad \Rightarrow \quad y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\text{LHS} = xy' + y = x(1 + x^{-2}) + (x - x^{-1}) = x + x^{-1} + x - x^{-1} = 2x = \text{RHS}$$

3. (a) $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2 e^{rx}$. Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get $2r^2 e^{rx} + re^{rx} - e^{rx} = 0 \quad \Rightarrow \quad (2r^2 + r - 1)e^{rx} = 0 \quad \Rightarrow$ $(2r - 1)(r + 1) = 0 \quad$ [since e^{rx} is never zero] $\quad \Rightarrow \quad r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = ae^{x/2} + be^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$y = ae^{x/2} + be^{-x} \quad \Rightarrow \quad y' = \frac{1}{2}ae^{x/2} - be^{-x} \quad \Rightarrow \quad y'' = \frac{1}{4}ae^{x/2} + be^{-x}$.

$$\begin{aligned}
\text{LHS} = 2y'' + y' - y &= 2\left(\tfrac{1}{4}ae^{x/2} + be^{-x}\right) + \left(\tfrac{1}{2}ae^{x/2} - be^{-x}\right) - (ae^{x/2} + be^{-x}) \\
&= \tfrac{1}{2}ae^{x/2} + 2be^{-x} + \tfrac{1}{2}ae^{x/2} - be^{-x} - ae^{x/2} - be^{-x} \\
&= \left(\tfrac{1}{2}a + \tfrac{1}{2}a - a\right)e^{x/2} + (2b - b - b)e^{-x} \\
&= 0 = \text{RHS}
\end{aligned}$$

5. (a) $y = \sin x \quad \Rightarrow \quad y' = \cos x \quad \Rightarrow \quad y'' = -\sin x$.

$\text{LHS} = y'' + y = -\sin x + \sin x = 0 \neq \sin x$, so $y = \sin x$ **is not** a solution of the differential equation.

(b) $y = \cos x \quad \Rightarrow \quad y' = -\sin x \quad \Rightarrow \quad y'' = -\cos x$.

$\text{LHS} = y'' + y = -\cos x + \cos x = 0 \neq \sin x$, so $y = \cos x$ **is not** a solution of the differential equation.

(c) $y = \frac{1}{2}x\sin x \quad \Rightarrow \quad y' = \frac{1}{2}(x\cos x + \sin x) \quad \Rightarrow \quad y'' = \frac{1}{2}(-x\sin x + \cos x + \cos x)$.

$\text{LHS} = y'' + y = \frac{1}{2}(-x\sin x + 2\cos x) + \frac{1}{2}x\sin x = \cos x \neq \sin x$, so $y = \frac{1}{2}x\sin x$ **is not** a solution of the differential equation.

(d) $y = -\frac{1}{2}x\cos x \quad \Rightarrow \quad y' = -\frac{1}{2}(-x\sin x + \cos x) \quad \Rightarrow \quad y'' = -\frac{1}{2}(-x\cos x - \sin x - \sin x)$.

$\text{LHS} = y'' + y = -\frac{1}{2}(-x\cos x - 2\sin x) + \left(-\frac{1}{2}x\cos x\right) = \sin x = \text{RHS}$, so $y = -\frac{1}{2}x\cos x$ **is** a solution of the differential equation.

7. (a) Since the derivative $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \dfrac{1}{x + C} \quad \Rightarrow \quad y' = -\dfrac{1}{(x + C)^2}$. $\quad \text{LHS} = y' = -\dfrac{1}{(x + C)^2} = -\left(\dfrac{1}{x + C}\right)^2 = -y^2 = \text{RHS}$

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

(d) If $y(x) = \dfrac{1}{x+C}$, then $y(0) = \dfrac{1}{0+C} = \dfrac{1}{C}$. Since $y(0) = 0.5$, $\dfrac{1}{C} = \dfrac{1}{2} \;\Rightarrow\; C = 2$, so $y = \dfrac{1}{x+2}$.

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \;\Rightarrow\; 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow\; \dfrac{P}{4200} < 1 \;\Rightarrow$ $P < 4200 \;\Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \;\Rightarrow\; P > 4200$

(c) $\dfrac{dP}{dt} = 0 \;\Rightarrow\; P = 4200$ or $P = 0$

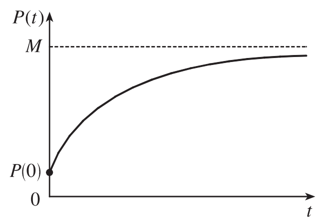
11. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y-1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

13. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

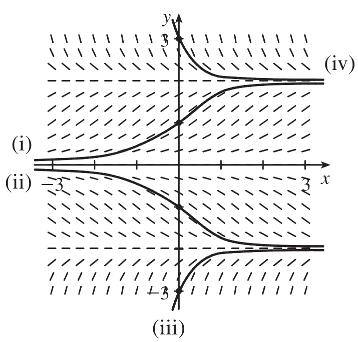
(b) $\dfrac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(c)



9.2 Direction Fields and Euler's Method

1. (a)



(b) It appears that the constant functions $y = 0$, $y = -2$, and $y = 2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$.

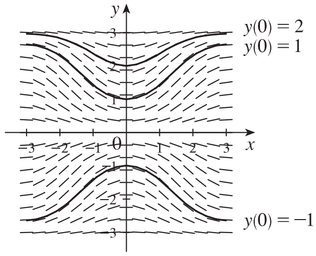
3. $y' = 2 - y$. The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, III is the direction field for this equation. Note that for $y = 2$, $y' = 0$.

5. $y' = x + y - 1 = 0$ on the line $y = -x + 1$. Direction field IV satisfies this condition. Notice also that on the line $y = -x$ we have $y' = -1$, which is true in IV.

7. (a) $y(0) = 1$

(b) $y(0) = 2$

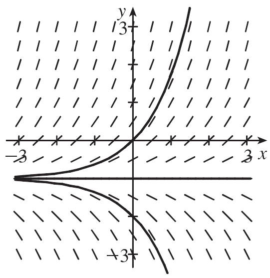
(c) $y(0) = -1$



9.

x	y	$y' = 1 + y$
0	0	1
0	1	2
0	2	3
0	-3	-2
0	-2	-1

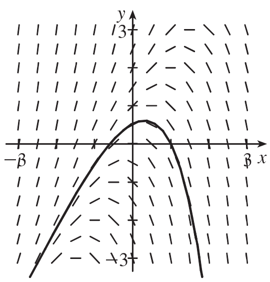
Note that for $y = -1$, $y' = 0$. The three solution curves sketched go through $(0, 0)$, $(0, -1)$, and $(0, -2)$.



11.

x	y	$y' = y - 2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

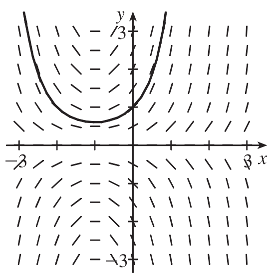
Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.



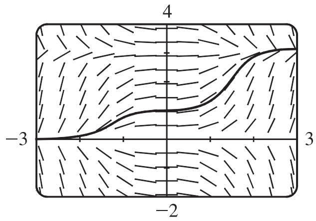
13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

Note that $y' = y(x + 1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x + 1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



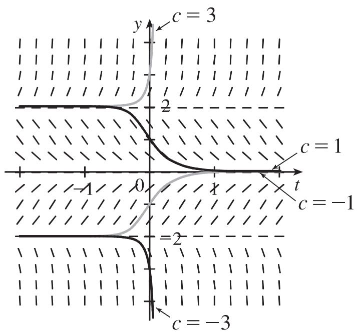
15. In Maple, we can use either `directionfield` (in Maple's share library) or `DEtools[DEplot]` to plot the direction field. To plot the solution, we can either use the initial-value option in `directionfield`, or actually solve the equation.



In Mathematica, we use `PlotVectorField` for the direction field, and the `Plot[Evaluate[...]]` construction to plot the solution, which is $y = 2\arctan\left(e^{x^3/3} \cdot \tan\frac{1}{2}\right)$.

In Derive, use `Direction_Field` (in utility file `ODE_APPR`) to plot the direction field. Then use `DSOLVE1(-x^2*SIN(y),1,x,y,0,1)` (in utility file `ODE1`) to solve the equation. Simplify each result.

17.



The direction field is for the differential equation $y' = y^3 - 4y$.

$L = \lim\limits_{t\to\infty} y(t)$ exists for $-2 \le c \le 2$;

$L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$.

For other values of c, L does not exist.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad\Rightarrow\quad x_0 = 0,\, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad\Rightarrow\quad y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$, so $y_1 = y(0.4) = 1.4$.

(ii) $h = 0.2 \quad\Rightarrow\quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

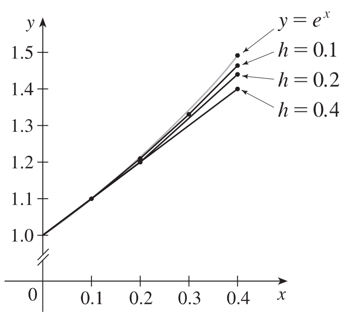
(iii) $h = 0.1 \quad\Rightarrow\quad x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

(c) (i) For $h = 0.4$: (exact value) − (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) − (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) − (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$

$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$

$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$

$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$

$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$

$$\begin{aligned} y_4 = y_3 + hF(x_3, y_3) &= 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)] \\ &= 1.5452976. \end{aligned}$$

$$\begin{aligned} y_5 = y_4 + hF(x_4, y_4) &= 1.5452976 + 0.1F(0.4, 1.5452976) \\ &= 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264. \end{aligned}$$

Thus, $y(0.5) \approx 1.7616$.

25. (a) $dy/dx + 3x^2y = 6x^2 \quad \Rightarrow \quad y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $\mathrm{H} = h = 1$ and $\mathrm{N} = 1$ for part (i), $\mathrm{H} = 0.1$ and $\mathrm{N} = 10$ for part (ii), and so forth.

$h \to$ H: $0 \to$ X: $3 \to$ Y:

For(I, 1, N): $\mathrm{Y} + \mathrm{H} \times \mathrm{Y}_1 \to$ Y: $\mathrm{X} + \mathrm{H} \to$ X:

End(loop):

Display Y. [To see all iterations, include this statement in the loop.]

(i) $\mathrm{H} = 1, \mathrm{N} = 1 \quad \Rightarrow \quad y(1) = 3$

(ii) $\mathrm{H} = 0.1, \mathrm{N} = 10 \quad \Rightarrow \quad y(1) \approx 2.3928$

(iii) $\mathrm{H} = 0.01, \mathrm{N} = 100 \quad \Rightarrow \quad y(1) \approx 2.3701$

(iv) $\mathrm{H} = 0.001, \mathrm{N} = 1000 \quad \Rightarrow \quad y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \quad \Rightarrow \quad y' = -3x^2e^{-x^3}$

$$\text{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \text{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.

(i) For $h = 1$: (exact value) − (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$

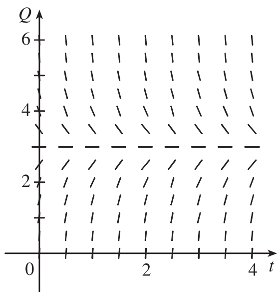
(ii) For $h = 0.1$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$

(iii) For $h = 0.01$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) − (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

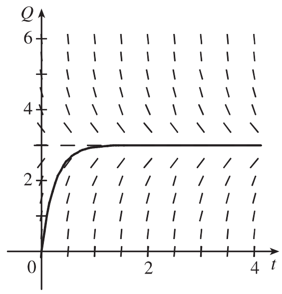
27. (a) $R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ becomes $5Q' + \dfrac{1}{0.05}Q = 60$ or $Q' + 4Q = 12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \;\Rightarrow\; Q = 3$ is an equilibrium solution.

(d)



(e) $Q' + 4Q = 12 \;\Rightarrow\; Q' = 12 - 4Q$. Now $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$

$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$

$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$

$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$

$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

9.3 Separable Equations

1. $\dfrac{dy}{dx} = \dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{dx}{x}$ $[y \neq 0]$ $\;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \;\Rightarrow\; \ln|y| = \ln|x| + C \;\Rightarrow$

$|y| = e^{\ln|x| + C} = e^{\ln|x|}e^C = e^C\,|x| \;\Rightarrow\; y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

3. $(x^2 + 1)y' = xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{xy}{x^2 + 1} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{x\,dx}{x^2 + 1}$ $[y \neq 0]$ $\;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{x^2 + 1} \;\Rightarrow$

$\ln|y| = \frac{1}{2}\ln(x^2 + 1) + C$ $[u = x^2 + 1,\ du = 2x\,dx]$ $= \ln(x^2 + 1)^{1/2} + \ln e^C = \ln\!\left(e^C\sqrt{x^2 + 1}\right) \;\Rightarrow$

$|y| = e^C\sqrt{x^2 + 1} \;\Rightarrow\; y = K\sqrt{x^2 + 1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

5. $(1+\tan y)\,y' = x^2+1 \quad\Rightarrow\quad (1+\tan y)\,\dfrac{dy}{dx} = x^2+1 \quad\Rightarrow\quad \left(1+\dfrac{\sin y}{\cos y}\right)dy = (x^2+1)\,dx \quad\Rightarrow$

$\displaystyle\int\left(1-\frac{-\sin y}{\cos y}\right)dy = \int(x^2+1)\,dx \quad\Rightarrow\quad y-\ln|\cos y| = \tfrac{1}{3}x^3+x+C.$

Note: The left side is equivalent to $y+\ln|\sec y|$.

7. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \quad\Rightarrow\quad y\sqrt{1+y^2}\,dy = te^t\,dt \quad\Rightarrow\quad \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \quad\Rightarrow\quad \tfrac{1}{3}\left(1+y^2\right)^{3/2} = te^t-e^t+C$

[where the first integral is evaluated by substitution and the second by parts] $\quad\Rightarrow\quad 1+y^2 = [3(te^t-e^t+C)]^{2/3} \quad\Rightarrow$

$y = \pm\sqrt{[3(te^t-e^t+C)]^{2/3}-1}$

9. $\dfrac{du}{dt} = 2+2u+t+tu \quad\Rightarrow\quad \dfrac{du}{dt} = (1+u)(2+t) \quad\Rightarrow\quad \displaystyle\int\frac{du}{1+u} = \int(2+t)dt \quad [u\neq -1] \quad\Rightarrow$

$\ln|1+u| = \tfrac{1}{2}t^2+2t+C \quad\Rightarrow\quad |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$, where $K=e^C \quad\Rightarrow\quad 1+u = \pm Ke^{t^2/2+2t} \quad\Rightarrow$

$u = -1\pm Ke^{t^2/2+2t}$ where $K>0$. $u=-1$ is also a solution, so $u=-1+Ae^{t^2/2+2t}$, where A is an arbitrary constant.

11. $\dfrac{dy}{dx} = \dfrac{x}{y} \quad\Rightarrow\quad y\,dy = x\,dx \quad\Rightarrow\quad \int y\,dy = \int x\,dx \quad\Rightarrow\quad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2+C. \quad y(0)=-3 \quad\Rightarrow$

$\tfrac{1}{2}(-3)^2 = \tfrac{1}{2}(0)^2+C \quad\Rightarrow\quad C=\tfrac{9}{2}$, so $\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2+\tfrac{9}{2} \quad\Rightarrow\quad y^2 = x^2+9 \quad\Rightarrow\quad y = -\sqrt{x^2+9}$ since $y(0)=-3<0$.

13. $x\cos x = (2y+e^{3y})\,y' \quad\Rightarrow\quad x\cos x\,dx = (2y+e^{3y})\,dy \quad\Rightarrow\quad \int(2y+e^{3y})\,dy = \int x\cos x\,dx \quad\Rightarrow$

$y^2+\tfrac{1}{3}e^{3y} = x\sin x+\cos x+C$ [where the second integral is evaluated using integration by parts].

Now $y(0)=0 \quad\Rightarrow\quad 0+\tfrac{1}{3} = 0+1+C \quad\Rightarrow\quad C=-\tfrac{2}{3}$. Thus, a solution is $y^2+\tfrac{1}{3}e^{3y} = x\sin x+\cos x-\tfrac{2}{3}$.

We cannot solve explicitly for y.

15. $\dfrac{du}{dt} = \dfrac{2t+\sec^2 t}{2u}$, $u(0)=-5$. $\int 2u\,du = \int\left(2t+\sec^2 t\right)dt \quad\Rightarrow\quad u^2 = t^2+\tan t+C$,

where $[u(0)]^2 = 0^2+\tan 0+C \quad\Rightarrow\quad C=(-5)^2=25$. Therefore, $u^2 = t^2+\tan t+25$, so $u=\pm\sqrt{t^2+\tan t+25}$.

Since $u(0)=-5$, we must have $u=-\sqrt{t^2+\tan t+25}$.

17. $y'\tan x = a+y,\ 0<x<\pi/2 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \quad\Rightarrow\quad \dfrac{dy}{a+y} = \cot x\,dx \quad [a+y\neq 0] \quad\Rightarrow$

$\displaystyle\int\frac{dy}{a+y} = \int\frac{\cos x}{\sin x}\,dx \quad\Rightarrow\quad \ln|a+y| = \ln|\sin x|+C \quad\Rightarrow\quad |a+y| = e^{\ln|\sin x|+C} = e^{\ln|\sin x|}\cdot e^C = e^C\,|\sin x| \quad\Rightarrow$

$a+y = K\sin x$, where $K=\pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case

$y=-a$ by allowing K to be zero.) $\quad y(\pi/3)=a \quad\Rightarrow\quad a+a = K\sin\left(\dfrac{\pi}{3}\right) \quad\Rightarrow\quad 2a = K\dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad K=\dfrac{4a}{\sqrt{3}}$.

Thus, $a+y = \dfrac{4a}{\sqrt{3}}\sin x$ and so $y = \dfrac{4a}{\sqrt{3}}\sin x - a$.

19. If the slope at the point (x,y) is xy, then we have $\dfrac{dy}{dx} = xy \quad\Rightarrow\quad \dfrac{dy}{y} = x\,dx \quad [y\neq 0] \quad\Rightarrow\quad \displaystyle\int\frac{dy}{y} = \int x\,dx \quad\Rightarrow$

$\ln|y| = \tfrac{1}{2}x^2+C. \quad y(0)=1 \quad\Rightarrow\quad \ln 1 = 0+C \quad\Rightarrow\quad C=0$. Thus, $|y| = e^{x^2/2} \quad\Rightarrow\quad y=\pm e^{x^2/2}$, so $y=e^{x^2/2}$

since $y(0)=1>0$. Note that $y=0$ is not a solution because it doesn't satisfy the initial condition $y(0)=1$.

21. $u = x + y \quad \Rightarrow \quad \dfrac{d}{dx}(u) = \dfrac{d}{dx}(x + y) \quad \Rightarrow \quad \dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$, but $\dfrac{dy}{dx} = x + y = u$, so $\dfrac{du}{dx} = 1 + u \quad \Rightarrow$

$\dfrac{du}{1+u} = dx \quad [u \neq -1] \quad \Rightarrow \quad \displaystyle\int \frac{du}{1+u} = \int dx \quad \Rightarrow \quad \ln|1+u| = x + C \quad \Rightarrow \quad |1+u| = e^{x+C} \quad \Rightarrow$

$1 + u = \pm e^C e^x \quad \Rightarrow \quad u = \pm e^C e^x - 1 \quad \Rightarrow \quad x + y = \pm e^C e^x - 1 \quad \Rightarrow \quad y = Ke^x - x - 1$, where $K = \pm e^C \neq 0$.

If $u = -1$, then $-1 = x + y \quad \Rightarrow \quad y = -x - 1$, which is just $y = Ke^x - x - 1$ with $K = 0$. Thus, the general solution is $y = Ke^x - x - 1$, where $K \in \mathbb{R}$.

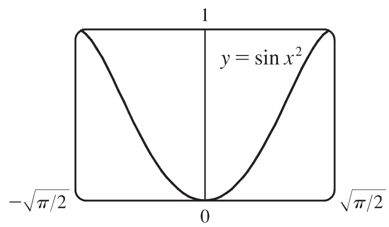
23. (a) $y' = 2x\sqrt{1-y^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x\sqrt{1-y^2} \quad \Rightarrow \quad \dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx \quad \Rightarrow \quad \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \quad \Rightarrow$

$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \leq x^2 + C \leq \frac{\pi}{2}$.

(b) $y(0) = 0 \quad \Rightarrow \quad \sin^{-1} 0 = 0^2 + C \quad \Rightarrow \quad C = 0$,

so $\sin^{-1} y = x^2$ and $y = \sin(x^2)$ for

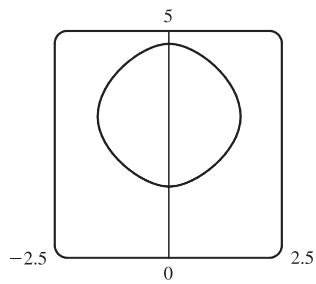
$-\sqrt{\pi/2} \leq x \leq \sqrt{\pi/2}$.



(c) For $\sqrt{1-y^2}$ to be a real number, we must have $-1 \leq y \leq 1$; that is, $-1 \leq y(0) \leq 1$. Thus, the initial-value problem $y' = 2x\sqrt{1-y^2}$, $y(0) = 2$ does *not* have a solution.

25. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sin y}, \quad y(0) = \dfrac{\pi}{2}$. So $\int \sin y\,dy = \int \sin x\,dx \quad \Leftrightarrow$

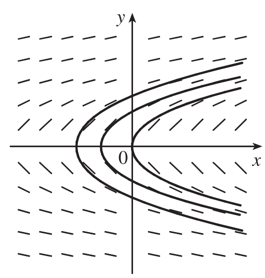
$-\cos y = -\cos x + C \quad \Leftrightarrow \quad \cos y = \cos x - C$. From the initial condition, we need $\cos\frac{\pi}{2} = \cos 0 - C \quad \Rightarrow \quad 0 = 1 - C \quad \Rightarrow \quad C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \leq \cos x - 1 \quad \Leftrightarrow \quad 0 \leq \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[···]]`.



27. (a)

x	y	$y' = 1/y$
0	0.5	2
0	-0.5	-2
0	1	1
0	-1	-1
0	2	0.5

x	y	$y' = 1/y$
0	-2	-0.5
0	4	0.25
0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3

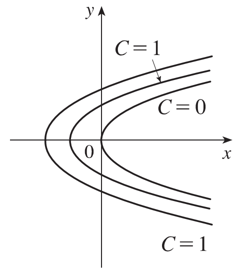


(b) $y' = 1/y \quad\Rightarrow\quad dy/dx = 1/y \quad\Rightarrow$

$y\,dy = dx \quad\Rightarrow\quad \int y\,dy = \int dx \quad\Rightarrow\quad \frac{1}{2}y^2 = x + C \quad\Rightarrow$

$y^2 = 2(x + C)$ or $y = \pm\sqrt{2(x + C)}$.

(c)

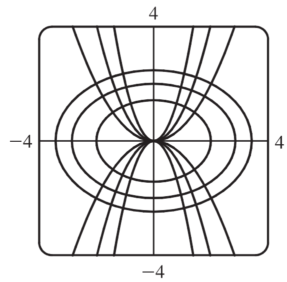


29. The curves $x^2 + 2y^2 = k^2$ form a family of ellipses with major axis on the x-axis. Differentiating gives

$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(k^2) \quad\Rightarrow\quad 2x + 4yy' = 0 \quad\Rightarrow\quad 4yy' = -2x \quad\Rightarrow\quad y' = \frac{-x}{2y}$. Thus, the slope of the tangent line at any point (x, y) on one of the ellipses is $y' = \frac{-x}{2y}$, so the orthogonal trajectories must satisfy $y' = \frac{2y}{x} \quad\Leftrightarrow\quad \frac{dy}{dx} = \frac{2y}{x} \quad\Leftrightarrow\quad \frac{dy}{y} = 2 = \frac{dx}{x} \quad\Leftrightarrow$

$\int \frac{dy}{y} = 2\int \frac{dx}{x} \quad\Leftrightarrow\quad \ln|y| = 2\ln|x| + C_1 \quad\Leftrightarrow\quad \ln|y| = \ln|x|^2 + C_1 \quad\Leftrightarrow$

$|y| = e^{\ln x^2 + C_1} \quad\Leftrightarrow\quad y = \pm x^2 \cdot e^{C_1} = Cx^2$. This is a family of parabolas.

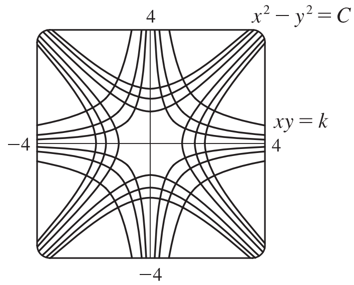


31. The curves $y = k/x$ form a family of hyperbolas with asymptotes $x = 0$ and $y = 0$. Differentiating gives

$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{k}{x}\right) \quad\Rightarrow\quad y' = -\frac{k}{x^2} \quad\Rightarrow\quad y' = -\frac{xy}{x^2}$ [since $y = k/x \quad\Rightarrow\quad xy = k$] $\quad\Rightarrow\quad y' = -\frac{y}{x}$. Thus, the slope of the tangent line at any point (x, y) on one of the hyperbolas is $y' = -y/x$, so the orthogonal trajectories must satisfy $y' = x/y \quad\Leftrightarrow\quad \frac{dy}{dx} = \frac{x}{y} \quad\Leftrightarrow$

$y\,dy = x\,dx \quad\Leftrightarrow\quad \int y\,dy = \int x\,dx \quad\Leftrightarrow\quad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \quad\Leftrightarrow$

$y^2 = x^2 + C_2 \quad\Leftrightarrow\quad x^2 - y^2 = C$. This is a family of hyperbolas with asymptotes $y = \pm x$.



33. From Exercise 9.2.27, $\frac{dQ}{dt} = 12 - 4Q \quad\Leftrightarrow\quad \int \frac{dQ}{12 - 4Q} = \int dt \quad\Leftrightarrow\quad -\frac{1}{4}\ln|12 - 4Q| = t + C \quad\Leftrightarrow$

$\ln|12 - 4Q| = -4t - 4C \quad\Leftrightarrow\quad |12 - 4Q| = e^{-4t-4C} \quad\Leftrightarrow\quad 12 - 4Q = Ke^{-4t}$ $[K = \pm e^{-4C}] \quad\Leftrightarrow$

$4Q = 12 - Ke^{-4t} \quad\Leftrightarrow\quad Q = 3 - Ae^{-4t}$ $[A = K/4]$. $Q(0) = 0 \quad\Leftrightarrow\quad 0 = 3 - A \quad\Leftrightarrow\quad A = 3 \quad\Leftrightarrow$

$Q(t) = 3 - 3e^{-4t}$. As $t \to \infty$, $Q(t) \to 3 - 0 = 3$ (the limiting value).

35. $\frac{dP}{dt} = k(M - P) \quad\Leftrightarrow\quad \int \frac{dP}{P - M} = \int (-k)\,dt \quad\Leftrightarrow\quad \ln|P - M| = -kt + C \quad\Leftrightarrow\quad |P - M| = e^{-kt+C} \quad\Leftrightarrow$

$P - M = Ae^{-kt}$ $[A = \pm e^C] \quad\Leftrightarrow\quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then

$P(0) = 0 \quad\Leftrightarrow\quad 0 = M + A \quad\Leftrightarrow\quad A = -M \quad\Leftrightarrow\quad P(t) = M - Me^{-kt}$. $\lim\limits_{t\to\infty} P(t) = M - M \cdot 0 = M$.

37. (a) If $a = b$, then $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2}$ becomes $\dfrac{dx}{dt} = k(a-x)^{3/2} \;\Rightarrow\; (a-x)^{-3/2}\,dx = k\,dt \;\Rightarrow$

$\int (a-x)^{-3/2}\,dx = \int k\,dt \;\Rightarrow\; 2(a-x)^{-1/2} = kt + C$ [by substitution] $\Rightarrow\; \dfrac{2}{kt+C} = \sqrt{a-x} \;\Rightarrow$

$\left(\dfrac{2}{kt+C}\right)^2 = a - x \;\Rightarrow\; x(t) = a - \dfrac{4}{(kt+C)^2}$. The initial concentration of HBr is 0, so $x(0) = 0 \;\Rightarrow$

$0 = a - \dfrac{4}{C^2} \;\Rightarrow\; \dfrac{4}{C^2} = a \;\Rightarrow\; C^2 = \dfrac{4}{a} \;\Rightarrow\; C = 2/\sqrt{a}$ [C is positive since $kt + C = 2(a-x)^{-1/2} > 0$].

Thus, $x(t) = a - \dfrac{4}{\left(kt + 2/\sqrt{a}\,\right)^2}$.

(b) $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2} \;\Rightarrow\; \dfrac{dx}{(a-x)\sqrt{b-x}} = k\,dt \;\Rightarrow\; \displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int k\,dt \;\; (\star)$.

From the hint, $u = \sqrt{b-x} \;\Rightarrow\; u^2 = b - x \;\Rightarrow\; 2u\,du = -dx$, so

$$\int \frac{dx}{(a-x)\sqrt{b-x}} = \int \frac{-2u\,du}{[a-(b-u^2)]u} = -2\int \frac{du}{a-b+u^2} = -2\int \frac{du}{\left(\sqrt{a-b}\,\right)^2 + u^2}$$

$$\overset{17}{=} -2\left(\frac{1}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}}\right)$$

So $(\star)$ becomes $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt + C$. Now $x(0) = 0 \;\Rightarrow\; C = \dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$ and we have

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}} \;\Rightarrow\; \dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right) = kt \;\Rightarrow$

$t(x) = \dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$.

39. (a) $\dfrac{dC}{dt} = r - kC \;\Rightarrow\; \dfrac{dC}{dt} = -(kC - r) \;\Rightarrow\; \displaystyle\int \frac{dC}{kC-r} = \int -dt \;\Rightarrow\; (1/k)\ln|kC - r| = -t + M_1 \;\Rightarrow$

$\ln|kC - r| = -kt + M_2 \;\Rightarrow\; |kC - r| = e^{-kt+M_2} \;\Rightarrow\; kC - r = M_3e^{-kt} \;\Rightarrow\; kC = M_3e^{-kt} + r \;\Rightarrow$

$C(t) = M_4e^{-kt} + r/k$. $C(0) = C_0 \;\Rightarrow\; C_0 = M_4 + r/k \;\Rightarrow\; M_4 = C_0 - r/k \;\Rightarrow$

$C(t) = (C_0 - r/k)e^{-kt} + r/k$.

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim\limits_{t\to\infty} C(t) = r/k$.

As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

41. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and $\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\,\dfrac{\text{kg}}{\text{L}}\right]\left(10\,\dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\,\dfrac{\text{kg}}{\text{min}}$.

$\displaystyle\int \frac{dy}{y} = -\frac{1}{100}\int dt \;\Rightarrow\; \ln y = -\frac{t}{100} + C$, and $y(0) = 15 \;\Rightarrow\; \ln 15 = C$, so $\ln y = \ln 15 - \dfrac{t}{100}$.

It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

43. Let $y(t)$ be the amount of alcohol in the vat after t minutes. Then $y(0) = 0.04(500) = 20$ gal. The amount of beer in the vat is 500 gallons at all times, so the percentage at time t (in minutes) is $y(t)/500 \times 100$, and the change in the amount of alcohol with respect to time t is $\dfrac{dy}{dt} = \text{rate in} - \text{rate out} = 0.06\left(5\,\dfrac{\text{gal}}{\text{min}}\right) - \dfrac{y(t)}{500}\left(5\,\dfrac{\text{gal}}{\text{min}}\right) = 0.3 - \dfrac{y}{100} = \dfrac{30-y}{100}\,\dfrac{\text{gal}}{\text{min}}$.

Hence, $\displaystyle\int \frac{dy}{30-y} = \int \frac{dt}{100}$ and $-\ln|30-y| = \frac{1}{100}t + C$. Because $y(0) = 20$, we have $-\ln 10 = C$, so $-\ln|30-y| = \frac{1}{100}t - \ln 10 \quad\Rightarrow\quad \ln|30-y| = -t/100 + \ln 10 \quad\Rightarrow\quad \ln|30-y| = \ln e^{-t/100} + \ln 10 \quad\Rightarrow$ $\ln|30-y| = \ln(10e^{-t/100}) \quad\Rightarrow\quad |30-y| = 10e^{-t/100}$. Since y is continuous, $y(0) = 20$, and the right-hand side is never zero, we deduce that $30-y$ is always positive. Thus, $30 - y = 10e^{-t/100} \quad\Rightarrow\quad y = 30 - 10e^{-t/100}$. The percentage of alcohol is $p(t) = y(t)/500 \times 100 = y(t)/5 = 6 - 2e^{-t/100}$. The percentage of alcohol after one hour is $p(60) = 6 - 2e^{-60/100} \approx 4.9$.

45. Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \quad\Rightarrow\quad mv' + vm' = gm \quad\Rightarrow$ $mv' + v(km) = gm \quad\Rightarrow\quad v' + vk = g \quad\Rightarrow\quad \dfrac{dv}{dt} = g - kv \quad\Rightarrow\quad \displaystyle\int \frac{dv}{g-kv} = \int dt \quad\Rightarrow$ $-(1/k)\ln|g-kv| = t + C \quad\Rightarrow\quad \ln|g-kv| = -kt - kC \quad\Rightarrow\quad g - kv = Ae^{-kt}$. $v(0) = 0 \quad\Rightarrow\quad A = g$. So $kv = g - ge^{-kt} \quad\Rightarrow\quad v = (g/k)(1-e^{-kt})$. Since $k > 0$, as $t \to \infty$, $e^{-kt} \to 0$ and therefore, $\displaystyle\lim_{t\to\infty} v(t) = g/k$.

47. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k, $dA/dt = k\sqrt{A}\,(M-A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{dA}{dt}\right) &= k\left[\sqrt{A}\,(-1)\frac{dA}{dt} + (M-A)\cdot\tfrac{1}{2}A^{-1/2}\frac{dA}{dt}\right] = \tfrac{1}{2}kA^{-1/2}\frac{dA}{dt}\,[-2A + (M-A)] \\ &= \tfrac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M-A)\right][M-3A] = \tfrac{1}{2}k^2(M-A)(M-3A)\end{aligned}$$

This is 0 when $M - A = 0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y = M$, as in the logistic model] and when $M - 3A = 0 \quad\Leftrightarrow\quad A(t) = M/3$. This represents a maximum by the First Derivative Test, since $\dfrac{d}{dt}\left(\dfrac{dA}{dt}\right)$ goes from positive to negative when $A(t) = M/3$.

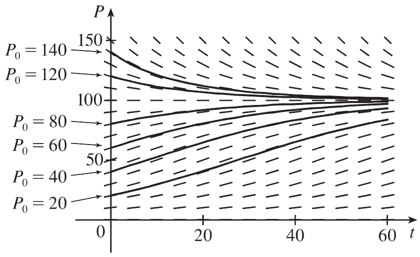
(b) From the CAS, we get $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M, we substitute $t = 0$ and $A = A_0 = A(0)$: $A_0 = M\left(\dfrac{C-1}{C+1}\right)^2 \quad\Leftrightarrow\quad (C+1)\sqrt{A_0} = (C-1)\sqrt{M} \quad\Leftrightarrow$ $C\sqrt{A_0} + \sqrt{A_0} = C\sqrt{M} - \sqrt{M} \quad\Leftrightarrow\quad \sqrt{M} + \sqrt{A_0} = C\sqrt{M} - C\sqrt{A_0} \quad\Leftrightarrow$ $\sqrt{M} + \sqrt{A_0} = C\left(\sqrt{M} - \sqrt{A_0}\right) \quad\Leftrightarrow\quad C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$. [Notice that if $A_0 = 0$, then $C = 1$.]

9.4 Models for Population Growth

1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 4, $dP/dt = kP(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)



All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

3. (a) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{K}\right) \;\Rightarrow\; y(t) = \dfrac{K}{1 + Ae^{-kt}}$ with $A = \dfrac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \;\Rightarrow\; \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \;\Rightarrow\; 2 = 1 + 3e^{-0.71t} \;\Rightarrow\; e^{-0.71t} = \frac{1}{3} \;\Rightarrow$

$-0.71t = \ln \frac{1}{3} \;\Rightarrow\; t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

5. (a) We will assume that the difference in the birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 1990 and use a unit of 1 billion for all calculations. $k \approx \dfrac{1}{P}\dfrac{dP}{dt} = \dfrac{1}{5.3}(0.02) = \dfrac{1}{265}$, so

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{1}{265}P\left(1 - \frac{P}{100}\right), \qquad P \text{ in billions}$$

(b) $A = \dfrac{K - P_0}{P_0} = \dfrac{100 - 5.3}{5.3} = \dfrac{947}{53} \approx 17.8679$. $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + \frac{947}{53}e^{-(1/265)t}}$, so $P(10) \approx 5.49$ billion.

(c) $P(110) \approx 7.81$, and $P(510) \approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

(d) If $K = 50$, then $P(t) = \dfrac{50}{1 + \frac{447}{53}e^{-(1/265)t}}$. So $P(10) \approx 5.48$, $P(110) \approx 7.61$, and $P(510) \approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

7. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1 - y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (4), $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$, we substitute $y = \dfrac{P}{K}$, $P = Ky$, and $\dfrac{dP}{dt} = K\,\dfrac{dy}{dt}$,

to obtain $K\,\dfrac{dy}{dt} = k(Ky)(1-y) \quad\Leftrightarrow\quad \dfrac{dy}{dt} = ky(1-y)$, our equation in part (a).

Now the solution to (4) is $P(t) = \dfrac{K}{1 + Ae^{-kt}}$, where $A = \dfrac{K - P_0}{P_0}$.

We use the same substitution to obtain $Ky = \dfrac{K}{1 + \dfrac{K - Ky_0}{Ky_0}e^{-kt}} \quad\Rightarrow\quad y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in the solution of Equation 4.

(c) Let t be the number of hours since 8 AM. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$,

so $y = \dfrac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \dfrac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \quad\Rightarrow\quad \left(\frac{2}{23}\right)^{t/4} = \frac{2 - 2y}{23y} \quad\Rightarrow\quad \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23}\cdot\frac{1-y}{y} \quad\Rightarrow\quad \left(\frac{2}{23}\right)^{t/4-1} = \frac{1-y}{y}.$$

It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$.

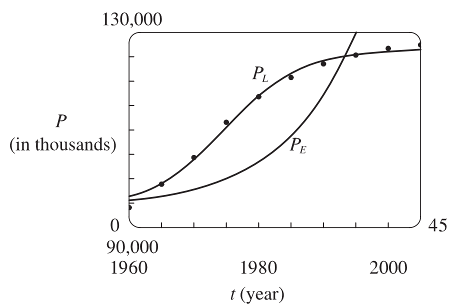
When $y = 0.9$, $\dfrac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

9. (a) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right) \quad\Rightarrow$

$$\frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\,\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$
$$= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \quad\Leftrightarrow\quad P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \quad\Leftrightarrow\quad P = K/2$.

11. Following the hint, we choose $t = 0$ to correspond to 1960 and subtract 94,000 from each of the population figures. We then use a calculator to obtain the models and add 94,000 to get the exponential function $P_E(t) = 1578.3(1.0933)^t + 94{,}000$ and the logistic function $P_L(t) = \dfrac{32{,}658.5}{1 + 12.75e^{-0.1706t}} + 94{,}000$. P_L is a reasonably accurate model, while P_E is not, since an exponential model would only be used for the first few data points.



13. (a) $\dfrac{dP}{dt} = kP - m = k\left(P - \dfrac{m}{k}\right)$. Let $y = P - \dfrac{m}{k}$, so $\dfrac{dy}{dt} = \dfrac{dP}{dt}$ and the differential equation becomes $\dfrac{dy}{dt} = ky$.

The solution is $y = y_0e^{kt} \quad\Rightarrow\quad P - \dfrac{m}{k} = \left(P_0 - \dfrac{m}{k}\right)e^{kt} \quad\Rightarrow\quad P(t) = \dfrac{m}{k} + \left(P_0 - \dfrac{m}{k}\right)e^{kt}$.

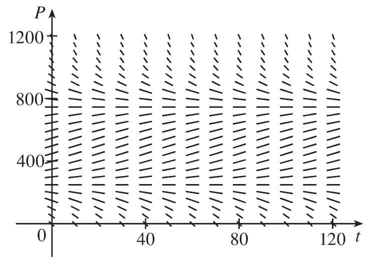
(b) Since $k > 0$, there will be an exponential expansion $\Leftrightarrow$ $P_0 - \dfrac{m}{k} > 0$ $\Leftrightarrow$ $m < kP_0$.

(c) The population will be constant if $P_0 - \dfrac{m}{k} = 0$ $\Leftrightarrow$ $m = kP_0$. It will decline if $P_0 - \dfrac{m}{k} < 0$ $\Leftrightarrow$ $m > kP_0$.

(d) $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000$ $\Rightarrow$ $m > kP_0$ ($= 128{,}000$), so by part (c), the population was declining.

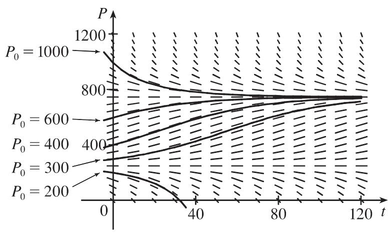
15. (a) The term -15 represents a harvesting of fish at a constant rate—in this case, 15 fish/week. This is the rate at which fish are caught.

(b)



(c) From the graph in part (b), it appears that $P(t) = 250$ and $P(t) = 750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt = 0$ as follows: $0.08P(1 - P/1000) - 15 = 0$ $\Rightarrow$ $0.08P - 0.00008P^2 - 15 = 0$ $\Rightarrow$ $-0.00008(P^2 - 1000P + 187{,}500) = 0$ $\Rightarrow$ $(P - 250)(P - 750) = 0$ $\Rightarrow$ $P = 250$ or 750.

(d)



For $0 < P_0 < 250$, $P(t)$ decreases to 0. For $P_0 = 250$, $P(t)$ remains constant. For $250 < P_0 < 750$, $P(t)$ increases and approaches 750. For $P_0 = 750$, $P(t)$ remains constant. For $P_0 > 750$, $P(t)$ decreases and approaches 750.

(e) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right) - 15$ $\Leftrightarrow$ $-\dfrac{100{,}000}{8} \cdot \dfrac{dP}{dt} = (0.08P - 0.00008P^2 - 15) \cdot \left(-\dfrac{100{,}000}{8}\right)$ $\Leftrightarrow$

$-12{,}500\,\dfrac{dP}{dt} = P^2 - 1000P + 187{,}500$ $\Leftrightarrow$ $\dfrac{dP}{(P - 250)(P - 750)} = -\dfrac{1}{12{,}500}\,dt$ $\Leftrightarrow$

$\displaystyle\int\left(\frac{-1/500}{P - 250} + \frac{1/500}{P - 750}\right) dP = -\frac{1}{12{,}500}\,dt$ $\Leftrightarrow$ $\displaystyle\int\left(\frac{1}{P - 250} - \frac{1}{P - 750}\right) dP = \tfrac{1}{25}\,dt$ $\Leftrightarrow$

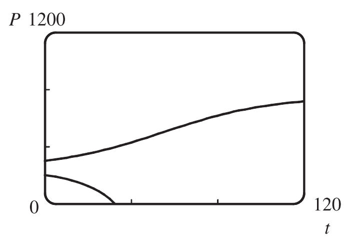
$\ln|P - 250| - \ln|P - 750| = \frac{1}{25}t + C$ $\Leftrightarrow$ $\ln\left|\dfrac{P - 250}{P - 750}\right| = \frac{1}{25}t + C$ $\Leftrightarrow$ $\left|\dfrac{P - 250}{P - 750}\right| = e^{t/25 + C} = ke^{t/25}$ $\Leftrightarrow$

$\dfrac{P - 250}{P - 750} = ke^{t/25}$ $\Leftrightarrow$ $P - 250 = Pke^{t/25} - 750ke^{t/25}$ $\Leftrightarrow$ $P - Pke^{t/25} = 250 - 750ke^{t/25}$ $\Leftrightarrow$

$P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$. If $t = 0$ and $P = 200$, then $200 = \dfrac{250 - 750k}{1 - k}$ $\Leftrightarrow$ $200 - 200k = 250 - 750k$ $\Leftrightarrow$

$550k = 50$ $\Leftrightarrow$ $k = \frac{1}{11}$. Similarly, if $t = 0$ and $P = 300$, then $k = -\frac{1}{9}$. Simplifying P with these two values of k gives us

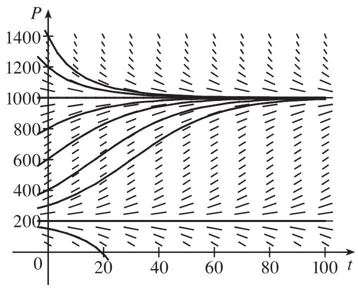
$P(t) = \dfrac{250(3e^{t/25} - 11)}{e^{t/25} - 11}$ and $P(t) = \dfrac{750(e^{t/25} + 3)}{e^{t/25} + 9}$.



17. (a) $\dfrac{dP}{dt} = (kP)\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right)$. If $m < P < K$, then $dP/dt = (+)(+)(+) = + \;\Rightarrow\; P$ is increasing.

If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \;\Rightarrow\; P$ is decreasing.

(b)



$k = 0.08$, $K = 1000$, and $m = 200 \;\Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right) = kP\left(\dfrac{K-P}{K}\right)\left(\dfrac{P-m}{P}\right) = \dfrac{k}{K}(K-P)(P-m) \;\Leftrightarrow$

$\displaystyle\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K}\,dt$. By partial fractions, $\dfrac{1}{(K-P)(P-m)} = \dfrac{A}{K-P} + \dfrac{B}{P-m}$, so

$A(P-m) + B(K-P) = 1$.

If $P = m$, $B = \dfrac{1}{K-m}$; if $P = K$, $A = \dfrac{1}{K-m}$, so $\dfrac{1}{K-m}\displaystyle\int\left(\frac{1}{K-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{K}\,dt \;\Rightarrow$

$\dfrac{1}{K-m}\left(-\ln|K-P| + \ln|P-m|\right) = \dfrac{k}{K}t + M \;\Rightarrow\; \dfrac{1}{K-m}\ln\left|\dfrac{P-m}{K-P}\right| = \dfrac{k}{K}t + M \;\Rightarrow$

$\ln\left|\dfrac{P-m}{K-P}\right| = (K-m)\dfrac{k}{K}t + M_1 \;\Leftrightarrow\; \dfrac{P-m}{K-P} = De^{(K-m)(k/K)t} \quad [D = \pm e^{M_1}]$.

Let $t = 0$: $\dfrac{P_0 - m}{K - P_0} = D$. So $\dfrac{P-m}{K-P} = \dfrac{P_0 - m}{K - P_0}e^{(K-m)(k/K)t}$. Solving for P, we get

$$P(t) = \frac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}.$$

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then

$N(0) = P_0(K-m) > 0$, and $P_0 - m < 0 \;\Leftrightarrow\; \displaystyle\lim_{t\to\infty} K(P_0 - m)e^{(K-m)(k/K)t} = -\infty \;\Rightarrow\; \lim_{t\to\infty} N(t) = -\infty$.

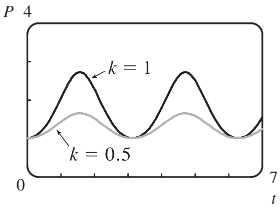
Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

19. (a) $dP/dt = kP\cos(rt - \phi) \;\Rightarrow\; (dP)/P = k\cos(rt-\phi)\,dt \;\Rightarrow\; \int(dP)/P = k\int\cos(rt-\phi)\,dt \;\Rightarrow$

$\ln P = (k/r)\sin(rt - \phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \;\Rightarrow\; C = \ln P_0 + (k/r)\sin\phi$. Thus,

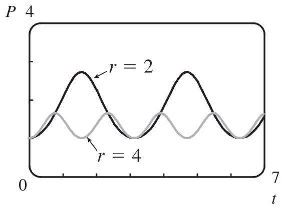
$\ln P = (k/r)\sin(rt - \phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt-\phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0 e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

(b) As k increases, the amplitude increases, but the minimum value stays the same.



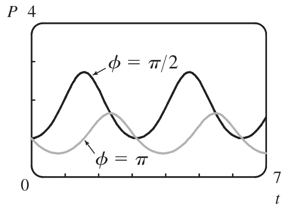
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin\phi)}$ and $P_0 e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim\limits_{t\to\infty} P(t)$ does not exist.

21. By Equation (7), $P(t) = \dfrac{K}{1 + Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t - c)$, then

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^u + e^{-u}}{e^u + e^{-u}} + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^u}{e^u + e^{-u}} \cdot \frac{e^{-u}}{e^{-u}} = \frac{2}{1 + e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc}e^{-kt} = e^{\ln A}e^{-kt} = Ae^{-kt}$, so

$$\tfrac{1}{2}K\left[1 + \tanh\left(\tfrac{1}{2}k(t-c)\right)\right] = \frac{K}{2}[1 + \tanh u] = \frac{K}{2} \cdot \frac{2}{1 + e^{-2u}} = \frac{K}{1 + e^{-2u}} = \frac{K}{1 + Ae^{-kt}} = P(t).$$

9.5 Linear Equations

1. $y' + \cos x = y \quad \Rightarrow \quad y' + (-1)y = -\cos x$ is linear since it can be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

3. $yy' + xy = x^2 \quad \Rightarrow \quad y' + x = x^2/y \quad \Rightarrow \quad y' - x^2/y = -x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2\,dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives

$e^{2x}y' + 2e^{2x}y = 2e^{3x} \quad \Rightarrow \quad (e^{2x}y)' = 2e^{3x} \quad \Rightarrow \quad e^{2x}y = \int 2e^{3x}\,dx \quad \Rightarrow \quad e^{2x}y = \frac{2}{3}e^{3x} + C \quad \Rightarrow \quad y = \frac{2}{3}e^x + Ce^{-2x}$.

7. $xy' - 2y = x^2 \quad$ [divide by x] $\quad \Rightarrow \quad y' + \left(-\dfrac{2}{x}\right)y = x \quad (\star)$.

$I(x) = e^{\int P(x)\,dx} = e^{\int(-2/x)\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = e^{\ln(1/x^2)} = 1/x^2$. Multiplying the differential equation $(\star)$ by $I(x)$ gives $\dfrac{1}{x^2}\,y' - \dfrac{2}{x^3}\,y = \dfrac{1}{x} \quad \Rightarrow \quad \left(\dfrac{1}{x^2}\,y\right)' = \dfrac{1}{x} \quad \Rightarrow \quad \dfrac{1}{x^2}\,y = \ln|x| + C \quad \Rightarrow$

$y = x^2(\ln|x| + C) = x^2\ln|x| + Cx^2$.

9. Since $P(x)$ is the derivative of the coefficient of y' $[P(x) = 1$ and the coefficient is $x]$, we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \quad\Rightarrow\quad xy = \frac{2}{3}x^{3/2} + C \quad\Rightarrow\quad y = \frac{2}{3}\sqrt{x} + C/x$.

11. $\sin x\,\dfrac{dy}{dx} + (\cos x)\,y = \sin(x^2) \quad\Rightarrow\quad [(\sin x)\,y]' = \sin(x^2) \quad\Rightarrow\quad (\sin x)\,y = \int \sin(x^2)\,dx \quad\Rightarrow\quad y = \dfrac{\int \sin(x^2)\,dx + C}{\sin x}$.

13. $(1+t)\,\dfrac{du}{dt} + u = 1 + t,\ t > 0$ [divide by $1+t$] $\Rightarrow\quad \dfrac{du}{dt} + \dfrac{1}{1+t}\,u = 1$ $(\star)$, which has the form $u' + P(t)\,u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)\,dt} = e^{\int [1/(1+t)]\,dt} = e^{\ln(1+t)} = 1 + t$.

Multiplying $(\star)$ by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]' = 1 + t$. Thus,

$(1+t)u = \int (1+t)\,dt = t + \frac{1}{2}t^2 + C \quad\Rightarrow\quad u = \dfrac{t + \frac{1}{2}t^2 + C}{1+t}$ or $u = \dfrac{t^2 + 2t + 2C}{2(t+1)}$.

15. $y' = x + y \quad\Rightarrow\quad y' + (-1)y = x$. $\ I(x) = e^{\int(-1)\,dx} = e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y' - e^{-x}y = xe^{-x} \quad\Rightarrow$ $(e^{-x}y)' = xe^{-x} \quad\Rightarrow\quad e^{-x}y = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$ [integration by parts with $u = x$, $dv = e^{-x}\,dx$] $\Rightarrow$ $y = -x - 1 + Ce^x$. $\ y(0) = 2 \quad\Rightarrow\quad -1 + C = 2 \quad\Rightarrow\quad C = 3$, so $y = -x - 1 + 3e^x$.

17. $\dfrac{dv}{dt} - 2tv = 3t^2e^{t^2},\ v(0) = 5$. $\ I(t) = e^{\int(-2t)\,dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get

$e^{-t^2}\dfrac{dv}{dt} - 2te^{-t^2}v = 3t^2 \quad\Rightarrow\quad \left(e^{-t^2}v\right)' = 3t^2 \quad\Rightarrow\quad e^{-t^2}v = \int 3t^2\,dt = t^3 + C \quad\Rightarrow\quad v = t^3e^{t^2} + Ce^{t^2}$.

$5 = v(0) = 0 \cdot 1 + C \cdot 1 = C$, so $v = t^3e^{t^2} + 5e^{t^2}$.

19. $xy' = y + x^2\sin x \quad\Rightarrow\quad y' - \dfrac{1}{x}\,y = x\sin x$. $\ I(x) = e^{\int(-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$.

Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x}\,y' - \dfrac{1}{x^2}\,y = \sin x \quad\Rightarrow\quad \left(\dfrac{1}{x}\,y\right)' = \sin x \quad\Rightarrow\quad \dfrac{1}{x}\,y = -\cos x + C \quad\Rightarrow\quad y = -x\cos x + Cx$.

$y(\pi) = 0 \quad\Rightarrow\quad -\pi \cdot (-1) + C\pi = 0 \quad\Rightarrow\quad C = -1$, so $y = -x\cos x - x$.

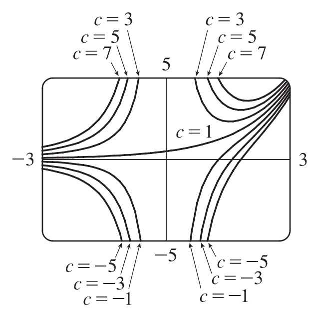
21. $xy' + 2y = e^x \quad\Rightarrow\quad y' + \dfrac{2}{x}\,y = \dfrac{e^x}{x}$.

$I(x) = e^{\int(2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$.

Multiplying by $I(x)$ gives $x^2\,y' + 2xy = xe^x \quad\Rightarrow\quad (x^2y)' = xe^x \quad\Rightarrow$

$x^2y = \int xe^x\,dx = (x-1)e^x + C$ [by parts] $\Rightarrow$

$y = [(x-1)e^x + C]/x^2$. The graphs for $C = -5, -3, -1, 1, 3, 5$, and 7 are shown. $C = 1$ is a transitional value. For $C < 1$, there is an inflection point and for $C > 1$, there is a local minimum. As $|C|$ gets larger, the "branches" get further from the origin.



23. Setting $u = y^{1-n}$, $\dfrac{du}{dx} = (1-n)\,y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{y^n}{1-n}\dfrac{du}{dx} = \dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

25. Here $y' + \dfrac{2}{x}\,y = \dfrac{y^3}{x^2}$, so $n = 3$, $P(x) = \dfrac{2}{x}$ and $Q(x) = \dfrac{1}{x^2}$. Setting $u = y^{-2}$, u satisfies $u' - \dfrac{4u}{x} = -\dfrac{2}{x^2}$.

Then $I(x) = e^{\int(-4/x)\,dx} = x^{-4}$ and $u = x^4\left(\displaystyle\int -\frac{2}{x^6}\,dx + C\right) = x^4\left(\dfrac{2}{5x^5} + C\right) = Cx^4 + \dfrac{2}{5x}$.

Thus, $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$.

27. (a) $2\,\dfrac{dI}{dt} + 10I = 40$ or $\dfrac{dI}{dt} + 5I = 20$. Then the integrating factor is $e^{\int 5\,dt} = e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\,\dfrac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \;\Rightarrow\; (e^{5t}I)' = 20e^{5t} \;\Rightarrow$

$I(t) = e^{-5t}\left[\int 20e^{5t}\,dt + C\right] = 4 + Ce^{-5t}$. But $0 = I(0) = 4 + C$, so $I(t) = 4 - 4e^{-5t}$.

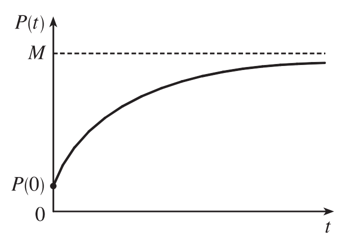
(b) $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$ A

29. $5\,\dfrac{dQ}{dt} + 20Q = 60$ with $Q(0) = 0$ C. Then the integrating factor is $e^{\int 4\,dt} = e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\,\dfrac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \;\Rightarrow\; (e^{4t}Q)' = 12e^{4t} \;\Rightarrow$

$Q(t) = e^{-4t}\left[\int 12e^{4t}\,dt + C\right] = 3 + Ce^{-4t}$. But $0 = Q(0) = 3 + C$ so $Q(t) = 3(1 - e^{-4t})$ is the charge at time t and $I = dQ/dt = 12e^{-4t}$ is the current at time t.

31. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\,\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \;\Rightarrow\; (e^{kt}P)' = kMe^{kt} \;\Rightarrow$

$P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.



33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\,\dfrac{\text{kg}}{\text{L}}\right)\left(5\,\dfrac{\text{L}}{\text{min}}\right) = 2\,\dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of

$\left(\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}\right)\left(3\,\dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100 + 2t}\,\dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get

$\dfrac{dy}{dt} = 2 - \dfrac{3y}{100 + 2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100 + 2t}\right) y = 2$, we see that it is linear.

$$I(t) = \exp\left(\int \frac{3\,dt}{100 + 2t}\right) = \exp\left(\tfrac{3}{2}\ln(100 + 2t)\right) = (100 + 2t)^{3/2}$$

Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2}\dfrac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \quad \Rightarrow$

$[(100 + 2t)^{3/2}y]' = 2(100 + 2t)^{3/2} \quad \Rightarrow \quad (100 + 2t)^{3/2}y = \frac{2}{5}(100 + 2t)^{5/2} + C \quad \Rightarrow$

$y = \frac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$. Now $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \quad \Rightarrow \quad C = -40{,}000$, so

$y = \left[\frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration

at time t to be $C(t) = \dfrac{y(t)}{100 + 2t} = \left[\dfrac{-40{,}000}{(100 + 2t)^{5/2}} + \dfrac{2}{5}\right]\dfrac{\text{kg}}{\text{L}}$. In particular, $C(20) = \dfrac{-40{,}000}{140^{5/2}} + \dfrac{2}{5} \approx 0.2275\ \dfrac{\text{kg}}{\text{L}}$

and $y(20) = \frac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

35. (a) $\dfrac{dv}{dt} + \dfrac{c}{m}v = g$ and $I(t) = e^{\int (c/m)\,dt} = e^{(c/m)t}$, and multiplying the differential equation by

$I(t)$ gives $e^{(c/m)t}\dfrac{dv}{dt} + \dfrac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \quad \Rightarrow \quad \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence,

$v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$ and

$K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)[1 - e^{-(c/m)t}]$.

(b) $\lim\limits_{t\to\infty} v(t) = mg/c$

(c) $s(t) = \int v(t)\,dt = (mg/c)[t + (m/c)e^{-(c/m)t}] + c_1$ where $c_1 = s(0) - m^2g/c^2$.

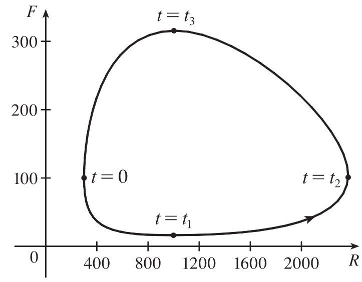
$s(0)$ is the initial position, so $s(0) = 0$ and $s(t) = (mg/c)[t + (m/c)e^{-(c/m)t}] - m^2g/c^2$.

9.6 Predator-Prey Systems

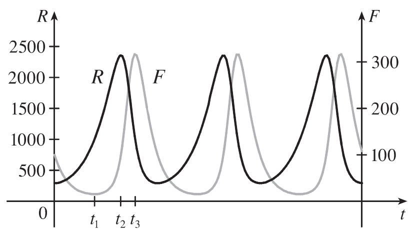
1. (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y, x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x, y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

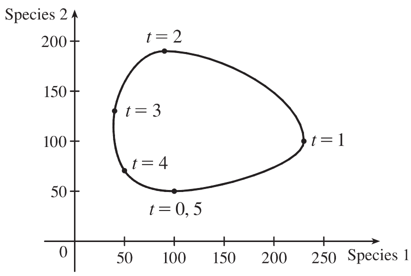
3. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



(b)



5.



7. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \quad \Leftrightarrow \quad (0.08 - 0.001W)R\,dW = (-0.02 + 0.00002R)W\,dR \quad \Leftrightarrow$

$$\frac{0.08 - 0.001W}{W}\,dW = \frac{-0.02 + 0.00002R}{R}\,dR \quad \Leftrightarrow \quad \int\left(\frac{0.08}{W} - 0.001\right)dW = \int\left(-\frac{0.02}{R} + 0.00002\right)dR \quad \Leftrightarrow$$

$$0.08\,\ln|W| - 0.001W = -0.02\,\ln|R| + 0.00002R + K \quad \Leftrightarrow \quad 0.08\,\ln W + 0.02\,\ln R = 0.001W + 0.00002R + K \quad \Leftrightarrow$$

$$\ln\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \quad \Leftrightarrow \quad W^{0.08}R^{0.02} = e^{0.00002R + 0.001W + K} \quad \Leftrightarrow$$

$R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \quad \Leftrightarrow \quad \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$. In general, if $\dfrac{dy}{dx} = \dfrac{-ry + bxy}{kx - axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

9. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \quad \Leftrightarrow \quad R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\quad \Rightarrow \quad R' = 0$ and $W' = 0 \quad \Rightarrow$

$$\left\{\begin{array}{l} 0 = 0.08R(1 - 0.0002R) - 0.001RW \\ 0 = -0.02W + 0.00002RW \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} 0 = R[0.08(1 - 0.0002R) - 0.001W] \\ 0 = W(-0.02 + 0.00002R) \end{array}\right.$$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or

$R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \Leftrightarrow$ $0 = 80(1 - 0.2) - W \Leftrightarrow W = 64$.

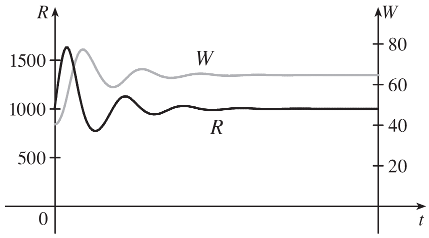
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



9 Review

CONCEPT CHECK

1. (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

(b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c) An initial condition is a condition of the form $y(t_0) = y_0$.

2. $y' = x^2 + y^2 \geq 0$ for all x and y. $y' = 0$ only at the origin, so there is a horizontal tangent at $(0, 0)$, but nowhere else. The graph of the solution is increasing on every interval.

3. See the paragraph preceding Example 1 in Section 9.2.

4. See the paragraph next to Figure 14 in Section 9.2.

5. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)dx$ and solving for y.

6. A first-order linear differential equation is a differential equation that can be put in the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, where P and Q are continuous functions on a given interval. To solve such an equation, multiply it by the integrating factor $I(x) = e^{\int P(x)dx}$ to put it in the form $[I(x)\,y]' = I(x)\,Q(x)$ and then integrate both sides to get $I(x)\,y = \int I(x)\,Q(x)\,dx$, that is, $e^{\int P(x)\,dx}y = \int e^{\int P(x)\,dx}Q(x)\,dx$. Solving for y gives us $y = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx}Q(x)\,dx$.

7. (a) $\dfrac{dy}{dt} = ky$; the relative growth rate, $\dfrac{1}{y}\dfrac{dy}{dt}$, is constant.

(b) The equation in part (a) is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.

(c) If $y(0) = y_0$, then the solution is $y(t) = y_0 e^{kt}$.

8. (a) $dP/dt = kP(1 - P/K)$, where K is the carrying capacity.

(b) The equation in part (a) is an appropriate model for population growth, assuming that the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.

9. (a) $dF/dt = kF - aFS$ and $dS/dt = -rS + bFS$.

(b) In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is, $dS/dt = -rS$, where r is a positive constant.

TRUE-FALSE QUIZ

1. True. Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions.

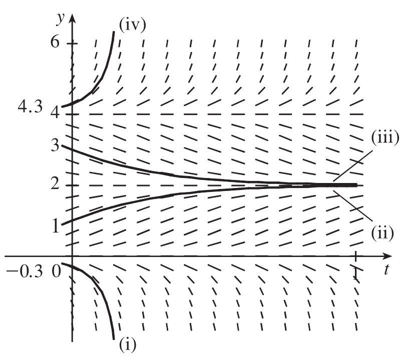
3. False. $x + y$ cannot be written in the form $g(x)f(y)$.

5. True. $e^x y' = y \;\Rightarrow\; y' = e^{-x}y \;\Rightarrow\; y' + (-e^{-x})y = 0$, which is of the form $y' + P(x)\,y = Q(x)$, so the equation is linear.

7. True. By comparing $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{5}\right)$ with the logistic differential equation (9.4.4), we see that the carrying capacity is 5; that is, $\lim\limits_{t\to\infty} y = 5$.

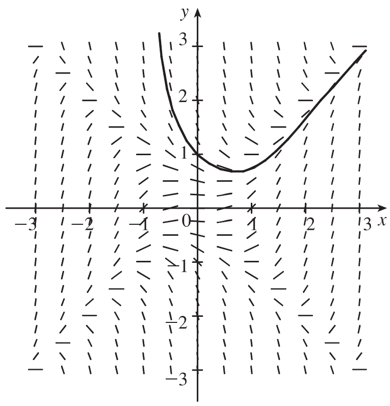
EXERCISES

1. (a)



(b) $\lim\limits_{t\to\infty} y(t)$ appears to be finite for $0 \leq c \leq 4$. In fact $\lim\limits_{t\to\infty} y(t) = 4$ for $c = 4$, $\lim\limits_{t\to\infty} y(t) = 2$ for $0 < c < 4$, and $\lim\limits_{t\to\infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - y_{n-1}^2\right)$. Thus,

$y_1 = 1 + 0.1\left(0^2 - 1^2\right) = 0.9$, $y_2 = 0.9 + 0.1\left(0.1^2 - 0.9^2\right) = 0.82$, $y_3 = 0.82 + 0.1\left(0.2^2 - 0.82^2\right) = 0.75676$.

This is close to our graphical estimate of $y(0.3) \approx 0.8$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$.

When a solution curve crosses one of these lines, it has a local maximum or minimum.

5. $y' = xe^{-\sin x} - y\cos x \quad\Rightarrow\quad y' + (\cos x)\,y = xe^{-\sin x}$ $(\star)$. This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying $(\star)$ by $e^{\sin x}$ gives $e^{\sin x}\,y' + e^{\sin x}(\cos x)\,y = x \quad\Rightarrow\quad (e^{\sin x}\,y)' = x \quad\Rightarrow$ $e^{\sin x}\,y = \frac{1}{2}x^2 + C \quad\Rightarrow\quad y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$.

7. $2ye^{y^2}y' = 2x + 3\sqrt{x} \quad\Rightarrow\quad 2ye^{y^2}\dfrac{dy}{dx} = 2x + 3\sqrt{x} \quad\Rightarrow\quad 2ye^{y^2}\,dy = \left(2x + 3\sqrt{x}\right)dx \quad\Rightarrow$

$\int 2ye^{y^2}\,dy = \int\left(2x + 3\sqrt{x}\right)dx \quad\Rightarrow\quad e^{y^2} = x^2 + 2x^{3/2} + C \quad\Rightarrow\quad y^2 = \ln(x^2 + 2x^{3/2} + C) \quad\Rightarrow$

$y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

9. $\dfrac{dr}{dt} + 2tr = r \quad\Rightarrow\quad \dfrac{dr}{dt} = r - 2tr = r(1 - 2t) \quad\Rightarrow\quad \displaystyle\int\frac{dr}{r} = \int(1 - 2t)\,dt \quad\Rightarrow\quad \ln|r| = t - t^2 + C \quad\Rightarrow$

$|r| = e^{t - t^2 + C} = ke^{t - t^2}$. Since $r(0) = 5$, $5 = ke^0 = k$. Thus, $r(t) = 5e^{t - t^2}$.

11. $xy' - y = x\ln x \quad\Rightarrow\quad y' - \dfrac{1}{x}y = \ln x.\quad I(x) = e^{\int(-1/x)\,dx} = e^{-\ln|x|} = \left(e^{\ln|x|}\right)^{-1} = |x|^{-1} = 1/x$ since the condition $y(1) = 2$ implies that we want a solution with $x > 0$. Multiplying the last differential equation by $I(x)$ gives

$\dfrac{1}{x}y' - \dfrac{1}{x^2}y = \dfrac{1}{x}\ln x \quad\Rightarrow\quad \left(\dfrac{1}{x}y\right)' = \dfrac{1}{x}\ln x \quad\Rightarrow\quad \dfrac{1}{x}y = \displaystyle\int\frac{\ln x}{x}\,dx \quad\Rightarrow\quad \dfrac{1}{x}y = \frac{1}{2}(\ln x)^2 + C \quad\Rightarrow$

$y = \frac{1}{2}x(\ln x)^2 + Cx$. Now $y(1) = 2 \quad\Rightarrow\quad 2 = 0 + C \quad\Rightarrow\quad C = 2$, so $y = \frac{1}{2}x(\ln x)^2 + 2x$.

13. $\dfrac{d}{dx}(y) = \dfrac{d}{dx}(ke^x) \quad\Rightarrow\quad y' = ke^x = y$, so the orthogonal trajectories must have $y' = -\dfrac{1}{y} \quad\Rightarrow\quad \dfrac{dy}{dx} = -\dfrac{1}{y} \quad\Rightarrow$

$y\,dy = -dx \quad\Rightarrow\quad \int y\,dy = -\int dx \quad\Rightarrow\quad \frac{1}{2}y^2 = -x + C \quad\Rightarrow\quad x = C - \frac{1}{2}y^2$, which are parabolas with a horizontal axis.

15. (a) Using (4) and (7) in Section 9.4, we see that for $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{2000}\right)$ with $P(0) = 100$, we have $k = 0.1$,

$K = 2000$, $P_0 = 100$, and $A = \dfrac{2000 - 100}{100} = 19$. Thus, the solution of the initial-value problem is

$P(t) = \dfrac{2000}{1 + 19e^{-0.1t}}$ and $P(20) = \dfrac{2000}{1 + 19e^{-2}} \approx 560$.

(b) $P = 1200 \quad\Leftrightarrow\quad 1200 = \dfrac{2000}{1 + 19e^{-0.1t}} \quad\Leftrightarrow\quad 1 + 19e^{-0.1t} = \dfrac{2000}{1200} \quad\Leftrightarrow\quad 19e^{-0.1t} = \dfrac{5}{3} - 1 \quad\Leftrightarrow$

$e^{-0.1t} = \left(\frac{2}{3}\right)/19 \quad\Leftrightarrow\quad -0.1t = \ln\frac{2}{57} \quad\Leftrightarrow\quad t = -10\ln\frac{2}{57} \approx 33.5$.

17. (a) $\dfrac{dL}{dt} \propto L_\infty - L \quad\Rightarrow\quad \dfrac{dL}{dt} = k(L_\infty - L) \quad\Rightarrow\quad \displaystyle\int \frac{dL}{L_\infty - L} = \int k\,dt \quad\Rightarrow\quad -\ln|L_\infty - L| = kt + C \quad\Rightarrow$

$\ln|L_\infty - L| = -kt - C \quad\Rightarrow\quad |L_\infty - L| = e^{-kt-C} \quad\Rightarrow\quad L_\infty - L = Ae^{-kt} \quad\Rightarrow\quad L = L_\infty - Ae^{-kt}$.

At $t = 0$, $L = L(0) = L_\infty - A \quad\Rightarrow\quad A = L_\infty - L(0) \quad\Rightarrow\quad L(t) = L_\infty - [L_\infty - L(0)]e^{-kt}$.

(b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \quad\Rightarrow\quad L(t) = 53 - (53 - 10)e^{-0.2t} = 53 - 43e^{-0.2t}$.

19. Let P represent the population and I the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $\dfrac{dI}{dt} = kI(P - I) = (kP)I\left(1 - \dfrac{I}{P}\right)$. From Equation 9.4.7 with $K = P$ and k replaced by kP, we have $I(t) = \dfrac{P}{1 + Ae^{-kPt}} = \dfrac{I_0 P}{I_0 + (P - I_0)e^{-kPt}}$.

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k:

$1200 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000\cdot 7\cdot k}} \quad\Leftrightarrow\quad 3 = \dfrac{2000}{160 + 4840e^{-35{,}000k}} \quad\Leftrightarrow\quad 480 + 14{,}520e^{-35{,}000k} = 2000 \quad\Leftrightarrow$

$e^{-35{,}000k} = \dfrac{2000 - 480}{14{,}520} \quad\Leftrightarrow\quad -35{,}000k = \ln\dfrac{38}{363} \quad\Leftrightarrow\quad k = \dfrac{-1}{35{,}000}\ln\dfrac{38}{363} \approx 0.00006448$. Next, let

$I = 5000 \times 80\% = 4000$, and solve for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-k\cdot 5000\cdot t}} \quad\Leftrightarrow\quad 1 = \dfrac{200}{160 + 4840e^{-5000kt}} \quad\Leftrightarrow$

$160 + 4840e^{-5000kt} = 200 \quad\Leftrightarrow\quad e^{-5000kt} = \dfrac{200 - 160}{4840} \quad\Leftrightarrow\quad -5000kt = \ln\dfrac{1}{121} \quad\Leftrightarrow$

$t = \dfrac{-1}{5000k}\ln\dfrac{1}{121} = \dfrac{1}{\frac{1}{7}\ln\frac{38}{363}} \cdot \ln\dfrac{1}{121} = 7 \cdot \dfrac{\ln 121}{\ln\frac{363}{38}} \approx 14.875$. So it takes about 15 days for 80% of the population to be infected.

21. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k + h}\right) \quad\Rightarrow\quad \displaystyle\int \frac{k + h}{h}\,dh = \int\left(-\frac{R}{V}\right)dt \quad\Rightarrow\quad \int\left(1 + \frac{k}{h}\right)dh = -\frac{R}{V}\int 1\,dt \quad\Rightarrow$

$h + k\ln h = -\dfrac{R}{V}t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

23. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0 \quad\Leftrightarrow\quad x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left.\begin{cases} 0 = 0.4x(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{cases}\right\} \Rightarrow \begin{cases} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{cases}$$

The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4(25{,}000)[(1 - 0.000005 \cdot 25{,}000) - 0.005y] \Rightarrow$ $0 = 10{,}000[(1 - 0.125) - 0.005y] \Rightarrow 0 = 8750 - 50y \Rightarrow y = 175$.

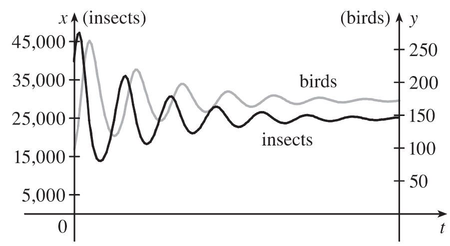
Case (i): $y = 0$, $x = 0$: Zero populations

Case (ii): $y = 0$, $x = 200{,}000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: The predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d)



25. (a) $\dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Setting $z = \dfrac{dy}{dx}$, we get $\dfrac{dz}{dx} = k\sqrt{1 + z^2} \Rightarrow \dfrac{dz}{\sqrt{1+z^2}} = k\,dx$. Using Formula 25 gives

$\ln\left(z + \sqrt{1+z^2}\right) = kx + c \Rightarrow z + \sqrt{1+z^2} = Ce^{kx}$ [where $C = e^c$] $\Rightarrow \sqrt{1+z^2} = Ce^{kx} - z \Rightarrow$

$1 + z^2 = C^2e^{2kx} - 2Ce^{kx}z + z^2 \Rightarrow 2Ce^{kx}z = C^2e^{2kx} - 1 \Rightarrow z = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx}$. Now

$\dfrac{dy}{dx} = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx} \Rightarrow y = \dfrac{C}{2k}e^{kx} + \dfrac{1}{2Ck}e^{-kx} + C'$. From the diagram in the text, we see that $y(0) = a$

and $y(\pm b) = h$. $a = y(0) = \dfrac{C}{2k} + \dfrac{1}{2Ck} + C' \Rightarrow C' = a - \dfrac{C}{2k} - \dfrac{1}{2Ck} \Rightarrow$

$y = \dfrac{C}{2k}(e^{kx} - 1) + \dfrac{1}{2Ck}(e^{-kx} - 1) + a$. From $h = y(\pm b)$, we find $h = \dfrac{C}{2k}(e^{kb} - 1) + \dfrac{1}{2Ck}(e^{-kb} - 1) + a$

and $h = \dfrac{C}{2k}(e^{-kb} - 1) + \dfrac{1}{2Ck}(e^{kb} - 1) + a$. Subtracting the second equation from the first, we get

$$0 = \frac{C}{k}\frac{e^{kb} - e^{-kb}}{2} - \frac{1}{Ck}\frac{e^{kb} - e^{-kb}}{2} = \frac{1}{k}\left(C - \frac{1}{C}\right)\sinh kb.$$

Now $k > 0$ and $b > 0$, so $\sinh kb > 0$ and $C = \pm 1$. If $C = 1$, then

$y = \dfrac{1}{2k}(e^{kx} - 1) + \dfrac{1}{2k}(e^{-kx} - 1) + a = \dfrac{1}{k}\dfrac{e^{kx} + e^{-kx}}{2} - \dfrac{1}{k} + a = a + \dfrac{1}{k}(\cosh kx - 1)$. If $C = -1$,

then $y = -\dfrac{1}{2k}(e^{kx} - 1) - \dfrac{1}{2k}(e^{-kx} - 1) + a = \dfrac{-1}{k}\dfrac{e^{kx} + e^{-kx}}{2} + \dfrac{1}{k} + a = a - \dfrac{1}{k}(\cosh kx - 1)$.

Since $k > 0$, $\cosh kx \geq 1$, and $y \geq a$, we conclude that $C = 1$ and $y = a + \dfrac{1}{k}(\cosh kx - 1)$, where

$h = y(b) = a + \dfrac{1}{k}(\cosh kb - 1)$. Since $\cosh(kb) = \cosh(-kb)$, there is no further information to extract from the condition that $y(b) = y(-b)$. However, we could replace a with the expression $h - \dfrac{1}{k}(\cosh kb - 1)$, obtaining $y = h + \dfrac{1}{k}(\cosh kx - \cosh kb)$. It would be better still to keep a in the expression for y, and use the expression for h to solve for k in terms of a, b, and h. That would enable us to express y in terms of x and the given parameters a, b, and h. Sadly, it is not possible to solve for k in closed form. That would have to be done by numerical methods when specific parameter values are given.

(b) The length of the cable is

$$L = \int_{-b}^{b} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-b}^{b} \sqrt{1 + \sinh^2 kx}\, dx = \int_{-b}^{b} \cosh kx\, dx = 2\int_0^b \cosh kx\, dx$$

$$= 2\Big[(1/k)\sinh kx\Big]_0^b = (2/k)\sinh kb$$

☐ PROBLEMS PLUS

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \quad\Rightarrow\quad 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \quad\Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \quad\Rightarrow\quad [f(x) - f'(x)]^2 = 0 \quad\Leftrightarrow\quad f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 9.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

3. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)\,[f(h) - 1]}{h} \qquad [\text{since } f(x+h) = f(x)f(h)]$$
$$= f(x)\lim_{h\to 0} \frac{f(h) - 1}{h} = f(x)\lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 9.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \quad\Rightarrow\quad A = 1 \quad\Rightarrow$ $f(x) = e^x$.

5. "The area under the graph of f from 0 to x is proportional to the $(n+1)$st power of $f(x)$" translates to

$\int_0^x f(t)\,dt = k[f(x)]^{n+1}$ for some constant k. By FTC1, $\dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt = \dfrac{d}{dx}\left\{k[f(x)]^{n+1}\right\} \quad\Rightarrow$

$$f(x) = k(n+1)[f(x)]^n f'(x) \quad\Rightarrow\quad 1 = k(n+1)[f(x)]^{n-1}f'(x) \quad\Rightarrow\quad 1 = k(n+1)y^{n-1}\frac{dy}{dx} \quad\Rightarrow$$

$$k(n+1)y^{n-1}\,dy = dx \quad\Rightarrow\quad \int k(n+1)y^{n-1}\,dy = \int dx \quad\Rightarrow\quad k(n+1)\frac{1}{n}y^n = x + C.$$

Now $f(0) = 0 \quad\Rightarrow\quad 0 = 0 + C \quad\Rightarrow\quad C = 0$ and then $f(1) = 1 \quad\Rightarrow\quad k(n+1)\dfrac{1}{n} = 1 \quad\Rightarrow\quad k = \dfrac{n}{n+1}$,

so $y^n = x$ and $y = f(x) = x^{1/n}$.

7. Let $y(t)$ denote the temperature of the peach pie t minutes after 5:00 PM and R the temperature of the room. Newton's Law of Cooling gives us $dy/dt = k(y - R)$. Solving for y we get $\dfrac{dy}{y - R} = k\,dt \quad\Rightarrow\quad \ln|y - R| = kt + C \quad\Rightarrow$

$|y - R| = e^{kt+C} \quad\Rightarrow\quad y - R = \pm e^{kt}\cdot e^C \quad\Rightarrow\quad y = Me^{kt} + R$, where M is a nonzero constant. We are given temperatures at three times.

$$\begin{aligned}
y(0) &= 100 &\Rightarrow\quad 100 &= M + R &\Rightarrow\quad R = 100 - M\\
y(10) &= 80 &\Rightarrow\quad 80 &= Me^{10k} + R \quad \textbf{(1)}\\
y(20) &= 65 &\Rightarrow\quad 65 &= Me^{20k} + R \quad \textbf{(2)}
\end{aligned}$$

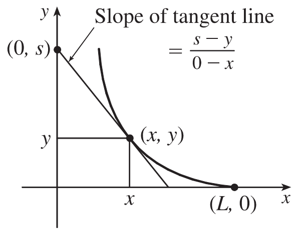
Substituting $100 - M$ for R in **(1)** and **(2)** gives us

$$-20 = Me^{10k} - M \;\; \textbf{(3)} \quad\text{and}\quad -35 = Me^{20k} - M \;\; \textbf{(4)}$$

Dividing **(3)** by **(4)** gives us $\dfrac{-20}{-35} = \dfrac{M\left(e^{10k} - 1\right)}{M\left(e^{20k} - 1\right)} \quad\Rightarrow\quad \dfrac{4}{7} = \dfrac{e^{10k} - 1}{e^{20k} - 1} \quad\Rightarrow\quad 4e^{20k} - 4 = 7e^{10k} - 7 \quad\Rightarrow$

$4e^{20k} - 7e^{10k} + 3 = 0$. This is a quadratic equation in e^{10k}. $\left(4e^{10k} - 3\right)\left(e^{10k} - 1\right) = 0 \Rightarrow e^{10k} = \frac{3}{4}$ or $1 \Rightarrow$ $10k = \ln\frac{3}{4}$ or $\ln 1 \Rightarrow k = \frac{1}{10}\ln\frac{3}{4}$ since k is a nonzero constant of proportionality. Substituting $\frac{3}{4}$ for e^{10k} in **(3)** gives us $-20 = M \cdot \frac{3}{4} - M \Rightarrow -20 = -\frac{1}{4}M \Rightarrow M = 80$. Now $R = 100 - M$ so $R = 20°\text{C}$.

9. (a) While running from $(L, 0)$ to (x, y), the dog travels a distance $s = \int_x^L \sqrt{1 + (dy/dx)^2}\,dx = -\int_L^x \sqrt{1 + (dy/dx)^2}\,dx$, so $\dfrac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx} = \dfrac{s - y}{0 - x}$ (see the figure).



Thus, $s = y - x\dfrac{dy}{dx} \Rightarrow \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\dfrac{d^2y}{dx^2} + 1\dfrac{dy}{dx}\right) = -x\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$ gives us $x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $x\dfrac{dz}{dx} = \sqrt{1 + z^2}$, or $\dfrac{dz}{\sqrt{1 + z^2}} = \dfrac{dx}{x}$. Integrating:

$\ln x = \displaystyle\int \frac{dz}{\sqrt{1 + z^2}} \overset{25}{=} \ln\left(z + \sqrt{1 + z^2}\right) + C$. When $x = L$, $z = dy/dx = 0$, so $\ln L = \ln 1 + C$. Therefore, $C = \ln L$, so $\ln x = \ln\left(\sqrt{1 + z^2} + z\right) + \ln L = \ln\left[L\left(\sqrt{1 + z^2} + z\right)\right] \Rightarrow x = L\left(\sqrt{1 + z^2} + z\right) \Rightarrow$

$\sqrt{1 + z^2} = \dfrac{x}{L} - z \Rightarrow 1 + z^2 = \left(\dfrac{x}{L}\right)^2 - \dfrac{2xz}{L} + z^2 \Rightarrow \left(\dfrac{x}{L}\right)^2 - 2z\left(\dfrac{x}{L}\right) - 1 = 0 \Rightarrow$

$z = \dfrac{(x/L)^2 - 1}{2(x/L)} = \dfrac{x^2 - L^2}{2Lx} = \dfrac{x}{2L} - \dfrac{L}{2}\dfrac{1}{x}$ [for $x > 0$]. Since $z = \dfrac{dy}{dx}$, $y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + C_1$.

Since $y = 0$ when $x = L$, $0 = \dfrac{L}{4} - \dfrac{L}{2}\ln L + C_1 \Rightarrow C_1 = \dfrac{L}{2}\ln L - \dfrac{L}{4}$. Thus,

$y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + \dfrac{L}{2}\ln L - \dfrac{L}{4} = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$.

(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.

11. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi\ \text{ft}^3/\text{h}$, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3 \Rightarrow$

$\dfrac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2\dfrac{dh}{dt} = \frac{9}{4}\pi h^2\dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\star) \Rightarrow$

$\int 3h^2\,dh = \int 80{,}000\,dt \Rightarrow h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so

$h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000 \Rightarrow$

$80{,}000t = 784{,}000 \Rightarrow t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi\ \text{ft}^2$, and the area of the base of the pile is

$A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is

$F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217\ \text{ft}^2$. Now $A = \frac{9\pi}{4}h^2 \quad\Rightarrow\quad dA/dt = \frac{9\pi}{4} \cdot 2h\,(dh/dt)$,

and from ($\star$) in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore,

$dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283\ \text{ft}^2/\text{h}$.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi\ \text{ft}^3/\text{h}$. From ($\star$) in part (a),

$\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{4(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2} \quad\Rightarrow\quad \int 9h^2\,dh = \int 160{,}000\,dt \quad\Rightarrow\quad 3h^3 = 160{,}000t + C$. When $t = 0$,

$h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top, $h = 100 \quad\Rightarrow$

$3(100)^3 = 160{,}000t + 2{,}187{,}000 \quad\Rightarrow\quad t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about 5.1 h.

13. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P, then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\dfrac{1}{m}(x - a)$, or equivalently, $y = -\dfrac{1}{m}x + b + \dfrac{a}{m}$. The y-intercept is always 6, so

$b + \dfrac{a}{m} = 6 \quad\Rightarrow\quad \dfrac{a}{m} = 6 - b \quad\Rightarrow\quad m = \dfrac{a}{6 - b}$. We will solve the equivalent differential equation $\dfrac{dy}{dx} = \dfrac{x}{6 - y} \quad\Rightarrow$

$(6 - y)\,dy = x\,dx \quad\Rightarrow\quad \displaystyle\int (6 - y)\,dy = \int x\,dx \quad\Rightarrow\quad 6y - \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \quad\Rightarrow\quad 12y - y^2 = x^2 + K$.

Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \quad\Rightarrow\quad K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \quad\Rightarrow$

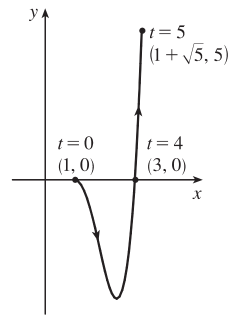
$x^2 + y^2 - 12y + 36 = -11 + 36 \quad\Rightarrow\quad x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.

10 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

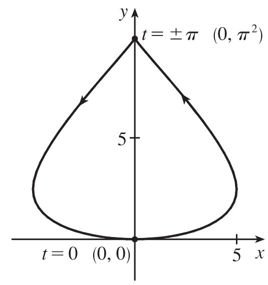
1. $x = 1 + \sqrt{t}, \quad y = t^2 - 4t, \quad 0 \le t \le 5$

t	0	1	2	3	4	5
x	1	2	$1+\sqrt{2}$ 2.41	$1+\sqrt{3}$ 2.73	3	$1+\sqrt{5}$ 3.24
y	0	-3	-4	-3	0	5



3. $x = 5\sin t, \quad y = t^2, \quad -\pi \le t \le \pi$

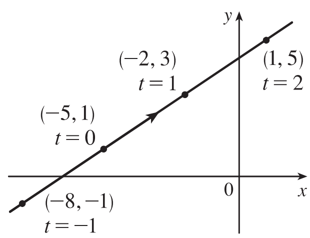
t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	0	-5	0	5	0
y	π^2 9.87	$\pi^2/4$ 2.47	0	$\pi^2/4$ 2.47	π^2 9.87



5. $x = 3t - 5, \quad y = 2t + 1$

(a)

t	-2	-1	0	1	2	3	4
x	-11	-8	-5	-2	1	4	7
y	-3	-1	1	3	5	7	9



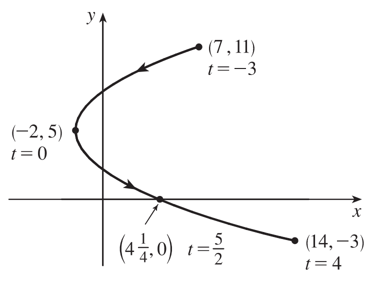
(b) $x = 3t - 5 \quad\Rightarrow\quad 3t = x + 5 \quad\Rightarrow\quad t = \frac{1}{3}(x+5) \quad\Rightarrow$

$y = 2 \cdot \frac{1}{3}(x+5) + 1$, so $y = \frac{2}{3}x + \frac{13}{3}$.

7. $x = t^2 - 2, \quad y = 5 - 2t, \quad -3 \le t \le 4$

(a)

t	-3	-2	-1	0	1	2	3	4
x	7	2	-1	-2	-1	2	7	14
y	11	9	7	5	3	1	-1	-3



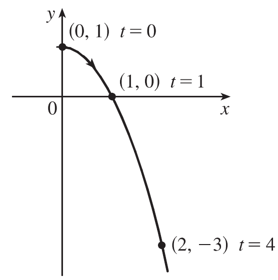
(b) $y = 5 - 2t \quad\Rightarrow\quad 2t = 5 - y \quad\Rightarrow\quad t = \frac{1}{2}(5-y) \quad\Rightarrow$

$x = \left[\frac{1}{2}(5-y)\right]^2 - 2$, so $x = \frac{1}{4}(5-y)^2 - 2, \;\; -3 \le y \le 11$.

9. $x = \sqrt{t},\ y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3

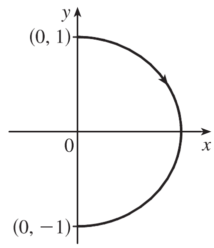


(b) $x = \sqrt{t} \;\Rightarrow\; t = x^2 \;\Rightarrow\; y = 1 - t = 1 - x^2$. Since $t \geq 0$, $x \geq 0$.

So the curve is the right half of the parabola $y = 1 - x^2$.

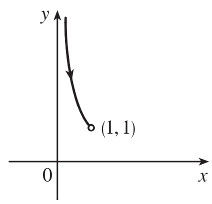
11. (a) $x = \sin\theta$, $y = \cos\theta$, $0 \leq \theta \leq \pi$. $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$. Since $0 \leq \theta \leq \pi$, we have $\sin\theta \geq 0$, so $x \geq 0$. Thus, the curve is the right half of the circle $x^2 + y^2 = 1$.

(b)



13. (a) $x = \sin t$, $y = \csc t$, $0 < t < \frac{\pi}{2}$. $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$. For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

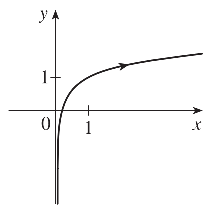
(b)



15. (a) $x = e^{2t} \;\Rightarrow\; 2t = \ln x \;\Rightarrow\; t = \frac{1}{2}\ln x$.

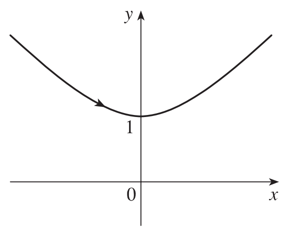
$y = t + 1 = \frac{1}{2}\ln x + 1$.

(b)



17. (a) $x = \sinh t$, $y = \cosh t \;\Rightarrow\; y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$. Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola $y^2 - x^2 = 1$.

(b)

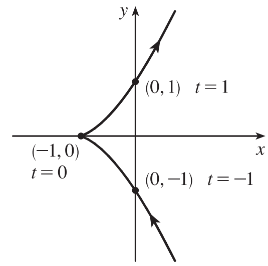


19. $x = 3 + 2\cos t$, $y = 1 + 2\sin t$, $\pi/2 \leq t \leq 3\pi/2$. By Example 4 with $r = 2$, $h = 3$, and $k = 1$, the motion of the particle takes place on a circle centered at $(3, 1)$ with a radius of 2. As t goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the particle starts at the point $(3, 3)$ and moves counterclockwise to $(3, -1)$ [one-half of a circle].

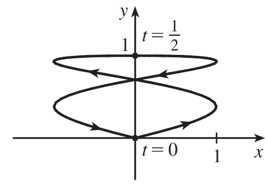
21. $x = 5\sin t$, $y = 2\cos t \;\Rightarrow\; \sin t = \dfrac{x}{5}$, $\cos t = \dfrac{y}{2}$. $\sin^2 t + \cos^2 t = 1 \;\Rightarrow\; \left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.

23. We must have $1 \le x \le 4$ and $2 \le y \le 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

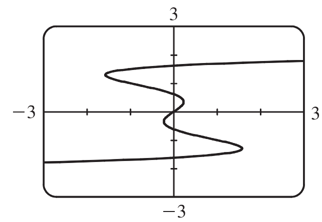
25. When $t = -1$, $(x, y) = (0, -1)$. As t increases to 0, x decreases to -1 and y increases to 0. As t increases from 0 to 1, x increases to 0 and y increases to 1. As t increases beyond 1, both x and y increase. For $t < -1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t = 0$ we see that $x = 0$ and $y = 0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1, decreases to 0 and to -1, then increases back to 0, so we arrive at the point $(0, 1)$. Similarly, as t increases from $\frac{1}{2}$ to 1, y decreases from 1 to 0 while x repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



29. As in Example 6, we let $y = t$ and $x = t - 3t^3 + t^5$ and use a t-interval of $[-3, 3]$.



31. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \le t \le 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \le t \le 1$.

33. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

35. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \le t \le 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

(left) $\quad x = 1 + 0.1\cos t, \quad y = 3 + 0.1\sin t, \quad 0 \le t \le 2\pi$

and (right) $\quad x = 3 + 0.1\cos t, \quad y = 3 + 0.1\sin t, \quad 0 \le t \le 2\pi$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

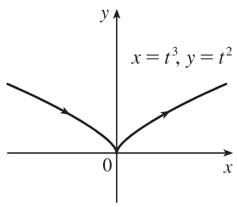
$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \le t \le 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to $[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "+" to a "−" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \le t \le 2\pi$$

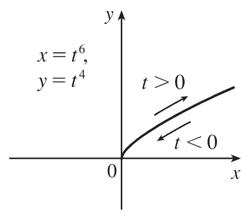
37. (a) $x = t^3 \;\Rightarrow\; t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \;\Rightarrow\; t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

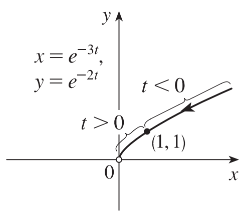
Since $x = t^6 \ge 0$, we only get the right half of the curve $y = x^{2/3}$.



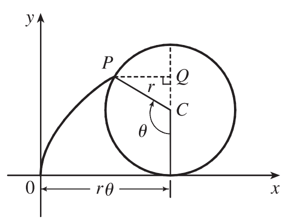
(c) $x = e^{-3t} = (e^{-t})^3 \quad [\text{so } e^{-t} = x^{1/3}]$,

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

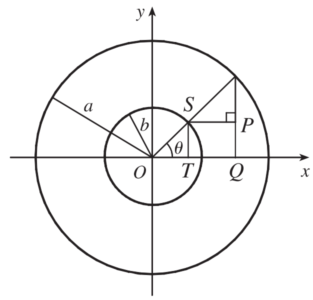
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



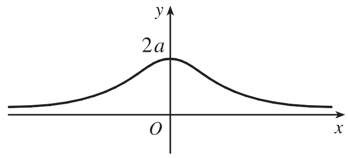
39. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



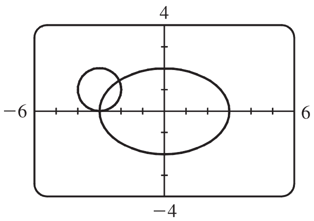
41. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \;\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



43. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



45. (a)



There are 2 points of intersection: $(-3, 0)$ and approximately $(-2.1, 1.4)$.

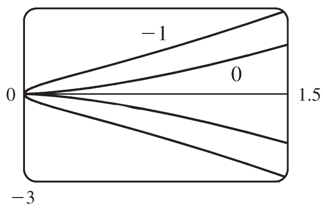
(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

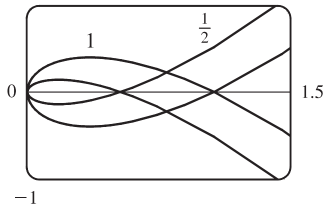
$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \;\Rightarrow\; 5\cos t = 0 \;\; (\star) \;\Rightarrow\; \cos t = 0 \;\Rightarrow\; t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

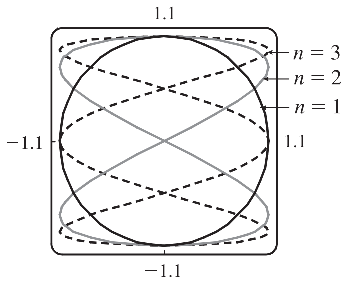
(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \;\Rightarrow\; \cos t = \frac{6}{5} > 1$.

47. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.

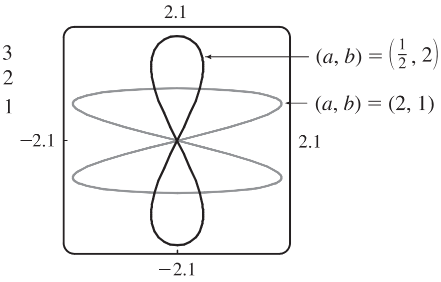




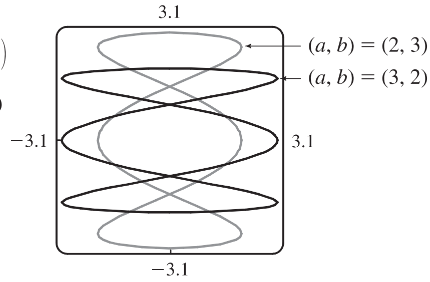
49. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a = b = 1$



$n = 2$



$n = 3$

10.2 Calculus with Parametric Curves

1. $x = t\sin t,\ \ y = t^2 + t \quad\Rightarrow\quad \dfrac{dy}{dt} = 2t + 1,\ \dfrac{dx}{dt} = t\cos t + \sin t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{t\cos t + \sin t}$.

3. $x = t^4 + 1,\ \ y = t^3 + t;\ \ t = -1.\quad \dfrac{dy}{dt} = 3t^2 + 1,\ \dfrac{dx}{dt} = 4t^3$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2+1}{4t^3}$. When $t = -1$, $(x, y) = (2, -2)$ and $dy/dx = \frac{4}{-4} = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - (-2) = (-1)(x - 2)$, or $y = -x$.

5. $x = e^{\sqrt{t}},\ \ y = t - \ln t^2;\ \ t = 1.\quad \dfrac{dy}{dt} = 1 - \dfrac{2t}{t^2} = 1 - \dfrac{2}{t},\ \dfrac{dx}{dt} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2/t}{e^{\sqrt{t}}/(2\sqrt{t})} \cdot \dfrac{2t}{2t} = \dfrac{2t-4}{\sqrt{t}e^{\sqrt{t}}}$.

When $t = 1$, $(x, y) = (e, 1)$ and $\dfrac{dy}{dx} = -\dfrac{2}{e}$, so an equation of the tangent line is $y - 1 = -\dfrac{2}{e}(x - e)$, or $y = -\dfrac{2}{e}x + 3$.

7. (a) $x = 1 + \ln t,\ \ y = t^2 + 2;\ \ (1, 3).\quad \dfrac{dy}{dt} = 2t,\ \dfrac{dx}{dt} = \dfrac{1}{t}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$.

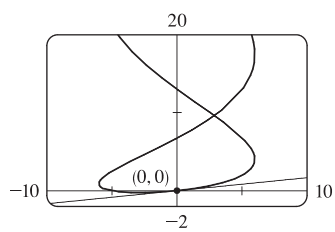
At $(1, 3)$, $x = 1 + \ln t = 1 \quad\Rightarrow\quad \ln t = 0 \quad\Rightarrow\quad t = 1$ and $\dfrac{dy}{dx} = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(b) $x = 1 + \ln t \quad\Rightarrow\quad x - 1 = \ln t \quad\Rightarrow\quad t = e^{x-1}$, so $y = (e^{x-1})^2 + 2 = e^{2x-2} + 2$ and $\dfrac{dy}{dx} = 2e^{2x-2}$.

When $x = 1$, $\dfrac{dy}{dx} = 2e^0 = 2$, so an equation of the tangent is $y = 2x + 1$, as in part (a).

9. $x = 6\sin t,\ \ y = t^2 + t;\ \ (0, 0)$.

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{6\cos t}$. The point $(0, 0)$ corresponds to $t = 0$, so the slope of the tangent at that point is $\frac{1}{6}$. An equation of the tangent is therefore $y - 0 = \frac{1}{6}(x - 0)$, or $y = \frac{1}{6}x$.



11. $x = 4 + t^2, \quad y = t^2 + t^3 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 3t^2}{2t} = 1 + \dfrac{3}{2}t \quad \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d(dy/dx)/dt}{dx/dt} = \frac{(d/dt)\left(1 + \frac{3}{2}t\right)}{2t} = \frac{3/2}{2t} = \frac{3}{4t}.$$

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > 0$.

13. $x = t - e^t, \quad y = t + e^{-t} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - e^{-t}}{1 - e^t} = \frac{1 - \dfrac{1}{e^t}}{1 - e^t} = \frac{\dfrac{e^t - 1}{e^t}}{1 - e^t} = -e^{-t} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{d}{dt}(-e^{-t})}{dx/dt} = \frac{e^{-t}}{1 - e^t}.$$

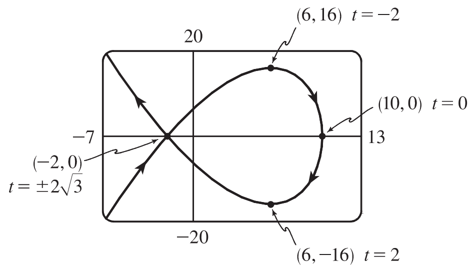
The curve is CU when $e^t < 1$ [since $e^{-t} > 0$] $\Rightarrow \quad t < 0$.

15. $x = 2\sin t, \; y = 3\cos t, \; 0 < t < 2\pi$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t, \text{ so } \frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t.$$

The curve is CU when $\sec^3 t < 0 \quad \Rightarrow \quad \sec t < 0 \quad \Rightarrow \quad \cos t < 0 \quad \Rightarrow \quad \frac{\pi}{2} < t < \frac{3\pi}{2}$.

17. $x = 10 - t^2, \; y = t^3 - 12t$.



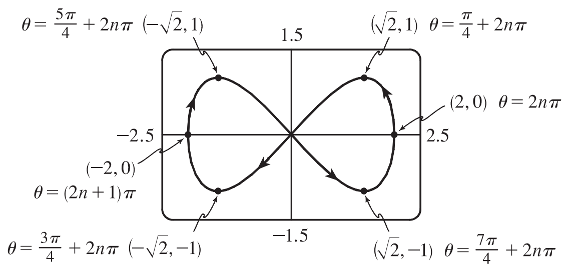
$\dfrac{dy}{dt} = 3t^2 - 12 = 3(t+2)(t-2)$, so $\dfrac{dy}{dt} = 0 \quad \Leftrightarrow$

$t = \pm 2 \quad \Leftrightarrow \quad (x, y) = (6, \mp 16)$.

$\dfrac{dx}{dt} = -2t$, so $\dfrac{dx}{dt} = 0 \quad \Leftrightarrow \quad t = 0 \quad \Leftrightarrow \quad (x, y) = (10, 0)$.

The curve has horizontal tangents at $(6, \pm 16)$ and a vertical tangent at $(10, 0)$.

19. $x = 2\cos\theta, \; y = \sin 2\theta$.



$\dfrac{dy}{d\theta} = 2\cos 2\theta$, so $\dfrac{dy}{d\theta} = 0 \quad \Leftrightarrow \quad 2\theta = \frac{\pi}{2} + n\pi$

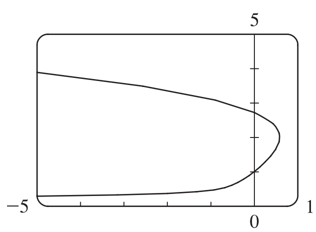
[n an integer] $\Leftrightarrow \quad \theta = \frac{\pi}{4} + \frac{\pi}{2}n \quad \Leftrightarrow$

$(x, y) = \left(\pm\sqrt{2}, \pm 1\right)$. Also, $\dfrac{dx}{d\theta} = -2\sin\theta$, so

$\dfrac{dx}{d\theta} = 0 \quad \Leftrightarrow \quad \theta = n\pi \quad \Leftrightarrow \quad (x, y) = (\pm 2, 0)$.

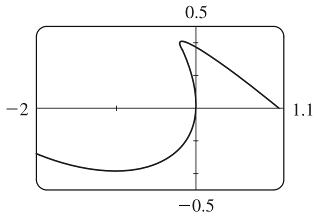
The curve has horizontal tangents at $\left(\pm\sqrt{2}, \pm 1\right)$ (four points), and vertical tangents at $(\pm 2, 0)$.

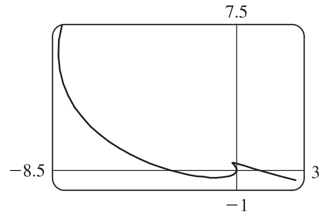
21. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \quad \Leftrightarrow \quad t = 1/\sqrt[5]{6}$. Hence, the rightmost point is



$$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$

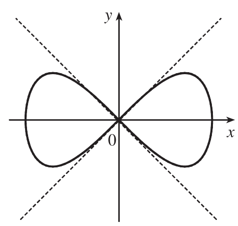
23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$.





We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad\Leftrightarrow\quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t(2t^2 - 3t - 2) = 0 \quad\Leftrightarrow\quad 2t(2t+1)(t-2) = 0 \quad\Leftrightarrow\quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

25. $x = \cos t$, $y = \sin t \cos t$. $\quad dx/dt = -\sin t$, $dy/dt = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \quad\Leftrightarrow\quad \cos t = 0 \quad\Leftrightarrow\quad t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



27. $x = r\theta - d\sin\theta,\ \ y = r - d\cos\theta$.

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

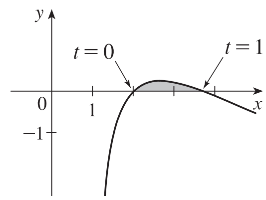
29. $x = 2t^3,\ \ y = 1 + 4t - t^2 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4 - 2t}{6t^2}$. Now solve $\dfrac{dy}{dx} = 1 \quad\Leftrightarrow\quad \dfrac{4 - 2t}{6t^2} = 1 \quad\Leftrightarrow\quad 6t^2 + 2t - 4 = 0 \quad\Leftrightarrow\quad 2(3t-2)(t+1) = 0 \quad\Leftrightarrow\quad t = \frac{2}{3}$ or $t = -1$. If $t = \frac{2}{3}$, the point is $\left(\frac{16}{27}, \frac{29}{9}\right)$, and if $t = -1$, the point is $(-2, -4)$.

31. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

33. The curve $x = 1 + e^t$, $y = t - t^2 = t(1-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$.



The shaded area is given by

$$\int_{x=2}^{x=1+e} (y_T - y_B)\,dx = \int_{t=0}^{t=1} [y(t) - 0]\,x'(t)\,dt = \int_0^1 (t - t^2)e^t\,dt$$

$$= \int_0^1 te^t\,dt - \int_0^1 t^2e^t\,dt = \int_0^1 te^t\,dt - \left[t^2e^t\right]_0^1 + 2\int_0^1 te^t\,dt \qquad \text{[Formula 97 or parts]}$$

$$= 3\int_0^1 te^t\,dt - (e - 0) = 3\left[(t-1)e^t\right]_0^1 - e \qquad \text{[Formula 96 or parts]}$$

$$= 3[0 - (-1)] - e = 3 - e$$

35. $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$.

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta$$

$$= \left[r^2\theta - 2dr\sin\theta + \tfrac{1}{2}d^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$$

37. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1 - 2t$ and $dy/dt = 2t^{1/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (1-2t)^2 + (2t^{1/2})^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$.

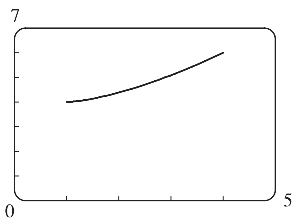
Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^2 \sqrt{1 + 4t^2}\,dt \approx 3.1678$.

39. $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$. $dx/dt = 1 - \sin t$ and $dy/dt = 1 - \cos t$, so

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - \sin t)^2 + (1 - \cos t)^2 = (1 - 2\sin t + \sin^2 t) + (1 - 2\cos t + \cos^2 t) = 3 - 2\sin t - 2\cos t$.

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi}\sqrt{3 - 2\sin t - 2\cos t}\,dt \approx 10.0367$.

41.



$x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

$dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$.

Thus, $L = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1 + t^2}\,dt$

$$= 6\int_1^2 \sqrt{u}\left(\tfrac{1}{2}du\right) \quad [u = 1 + t^2,\ du = 2t\,dt]$$

$$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2\left(2\sqrt{2} - 1\right)$$

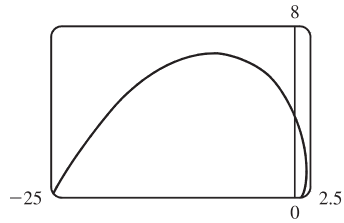
43. $x = \dfrac{t}{1+t}$, $y = \ln(1+t)$, $0 \le t \le 2$. $\dfrac{dx}{dt} = \dfrac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{1}{1+t}$,

so $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \dfrac{1}{(1+t)^4} + \dfrac{1}{(1+t)^2} = \dfrac{1}{(1+t)^4}\left[1 + (1+t)^2\right] = \dfrac{t^2 + 2t + 2}{(1+t)^4}$. Thus,

$$L = \int_0^2 \frac{\sqrt{t^2 + 2t + 2}}{(1+t)^2}\,dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du \quad \begin{bmatrix} u = t+1, \\ du = dt \end{bmatrix} \quad \overset{24}{=} \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\right]_1^3$$

$$= -\tfrac{\sqrt{10}}{3} + \ln\left(3 + \sqrt{10}\right) + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)$$

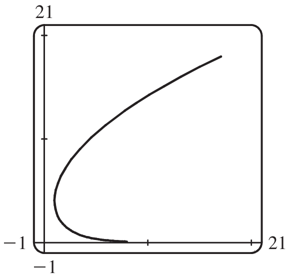
45. $x = e^t \cos t,\ y = e^t \sin t,\ 0 \le t \le \pi.$



$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2 \\ &= (e^t)^2(\cos^2 t - 2\cos t\ \sin t + \sin^2 t) \\ &\qquad + (e^t)^2(\sin^2 t + 2\sin t\ \cos t + \cos^2 t \\ &= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}\end{aligned}$$

Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\, dt = \int_0^\pi \sqrt{2}\, e^t\, dt = \sqrt{2}\, \left[\, e^t \,\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

47.



$x = e^t - t,\ y = 4e^{t/2},\ -8 \le t \le 3$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t \\ &= e^{2t} + 2e^t + 1 = (e^t + 1)^2\end{aligned}$$

$$\begin{aligned}L &= \int_{-8}^{3} \sqrt{(e^t + 1)^2}\, dt = \int_{-8}^{3} (e^t + 1)\, dt = \left[e^t + t\right]_{-8}^{3t} \\ &= (e^3 + 3) - (e^{-8} - 8) = e^3 - e^{-8} + 11\end{aligned}$$

49. $x = t - e^t,\ y = t + e^t,\ -6 \le t \le 6.$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}$, so $L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}\, dt.$

Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-(-6)}{6} = 2$, we get

$L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053.$

51. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 3\pi.$

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\ \cos t)^2 + (-2\cos t \sin t)^2 = 8\sin^2 t \cos^2 t = 2\sin^2 2t \quad \Rightarrow$

Distance $= \int_0^{3\pi} \sqrt{2}\,|\sin 2t|\ dt = 6\sqrt{2}\int_0^{\pi/2} \sin 2t\, dt$ [by symmetry] $= -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2} = -3\sqrt{2}\,(-1 - 1) = 6\sqrt{2}.$

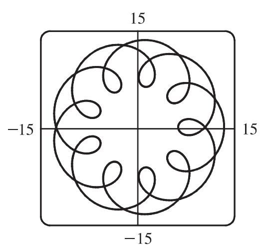
The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2} \sin 2t\, dt = \sqrt{2}$, as above.

53. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1 - \sin^2\theta) + b^2\sin^2\theta \\ &= a^2 - (a^2 - b^2)\ \sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2} \sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\, d\theta$ [by symmetry] $= 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta.$

55. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2).$



Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.

(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Formula 1 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ as $88E\big(2\sqrt{2}\,i\big)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi}\sqrt{-4\cos t\,\cos\!\left(\frac{11t}{2}\right) - 4\sin t\,\sin\!\left(\frac{11t}{2}\right) + 5}\,dt$.

57. $x = 1 + te^t,\ y = (t^2+1)e^t,\ 0 \le t \le 1.$

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (te^t + e^t)^2 + [(t^2+1)e^t + e^t(2t)]^2 = [e^t(t+1)]^2 + [e^t(t^2+2t+1)]^2\\ &= e^{2t}(t+1)^2 + e^{2t}(t+1)^4 = e^{2t}(t+1)^2[1+(t+1)^2], \quad \text{so}\end{aligned}$$

$$S = \textstyle\int 2\pi y\,ds = \int_0^1 2\pi(t^2+1)e^t\sqrt{e^{2t}(t+1)^2(t^2+2t+2)}\,dt = \int_0^1 2\pi(t^2+1)e^{2t}(t+1)\sqrt{t^2+2t+2}\,dt \approx 103.5999$$

59. $x = t^3,\ y = t^2,\ 0 \le t \le 1.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2.$

$$\begin{aligned}S &= \int_0^1 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi t^2\sqrt{9t^4+4t^2}\,dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt\\ &= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right)\quad \left[\begin{matrix}u = 9t^2+4,\ t^2 = (u-4)/9,\\ du = 18t\,dt,\ \text{so}\ t\,dt = \frac{1}{18}\,du\end{matrix}\right] \quad = \frac{2\pi}{9\cdot 18}\int_4^{13}\left(u^{3/2} - 4u^{1/2}\right)du\\ &= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13}\\ &= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right] = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)\end{aligned}$$

61. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}.\quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2 = 9a^2\sin^2\theta\,\cos^2\theta.$

$$S = \textstyle\int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\,\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\,\cos\theta\,d\theta = \frac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \frac{6}{5}\pi a^2$$

63. $x = t + t^3,\ y = t - \dfrac{1}{t^2},\ 1 \le t \le 2.\quad \frac{dx}{dt} = 1 + 3t^2$ and $\frac{dy}{dt} = 1 + \dfrac{2}{t^3}$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1+3t^2)^2 + \left(1 + \dfrac{2}{t^3}\right)^2$

and $S = \displaystyle\int 2\pi y\,ds = \int_1^2 2\pi\left(t - \frac{1}{t^2}\right)\sqrt{(1+3t^2)^2 + \left(1+\frac{2}{t^3}\right)^2}\,dt \approx 59.101.$

65. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5\quad\Rightarrow\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1+t^2)\quad\Rightarrow$

$$\begin{aligned}S &= \textstyle\int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1+t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1+t^2}\,2t\,dt\\ &= 18\pi\textstyle\int_1^{26}(u-1)\sqrt{u}\,du\quad\left[\begin{matrix}u = 1+t^2,\\ du = 2t\,dt\end{matrix}\right]\quad = 18\pi\int_1^{26}\left(u^{3/2} - u^{1/2}\right)du = 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26}\\ &= 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right] = \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right)\end{aligned}$$

67. If f' is continuous and $f'(t) \neq 0$ for $a \leq t \leq b$, then either $f'(t) > 0$ for all t in $[a,b]$ or $f'(t) < 0$ for all t in $[a,b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a,b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \;\Rightarrow\; f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

69. (a) $\phi = \tan^{-1}\left(\dfrac{dy}{dx}\right) \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{d}{dt}\tan^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{1+(dy/dx)^2}\left[\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right]$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}} \;\Rightarrow$

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right) = \dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \;\Rightarrow\; \dfrac{d\phi}{dt} = \dfrac{1}{1+(\dot{y}/\dot{x})^2}\left(\dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$. Using the Chain Rule, and the

fact that $s = \displaystyle\int_0^t \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt \;\Rightarrow\; \tfrac{ds}{dt} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\dfrac{d\phi}{ds} = \dfrac{d\phi/dt}{ds/dt} = \left(\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\dfrac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So $\kappa = \left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

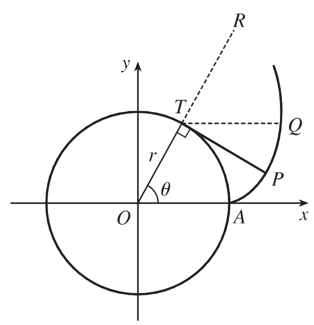
(b) $x = x$ and $y = f(x) \;\Rightarrow\; \dot{x} = 1,\, \ddot{x} = 0$ and $\dot{y} = \dfrac{dy}{dx},\, \ddot{y} = \dfrac{d^2y}{dx^2}$.

So $\kappa = \dfrac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}}$.

71. $x = \theta - \sin\theta \;\Rightarrow\; \dot{x} = 1 - \cos\theta \;\Rightarrow\; \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \;\Rightarrow\; \dot{y} = \sin\theta \;\Rightarrow\; \ddot{y} = \cos\theta$. Therefore,

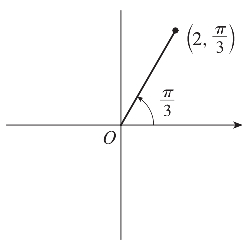
$\kappa = \dfrac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{[(1-\cos\theta)^2 + \sin^2\theta]^{3/2}} = \dfrac{\left|\cos\theta - (\cos^2\theta + \sin^2\theta)\right|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ: $\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1-1|}{[2 - 2(-1)]^{3/2}} = \dfrac{1}{4}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



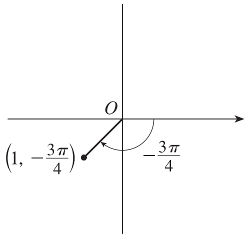
10.3 Polar Coordinates

1. (a) $\left(2, \frac{\pi}{3}\right)$



By adding 2π to $\frac{\pi}{3}$, we obtain the point $\left(2, \frac{7\pi}{3}\right)$. The direction opposite $\frac{\pi}{3}$ is $\frac{4\pi}{3}$, so $\left(-2, \frac{4\pi}{3}\right)$ is a point that satisfies the $r < 0$ requirement.

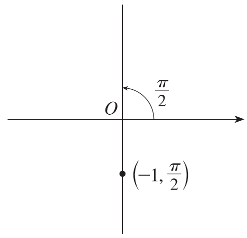
(b) $\left(1, -\frac{3\pi}{4}\right)$



$\boldsymbol{r > 0}$**:** $\left(1, -\frac{3\pi}{4} + 2\pi\right) = \left(1, \frac{5\pi}{4}\right)$

$\boldsymbol{r < 0}$**:** $\left(-1, -\frac{3\pi}{4} + \pi\right) = \left(-1, \frac{\pi}{4}\right)$

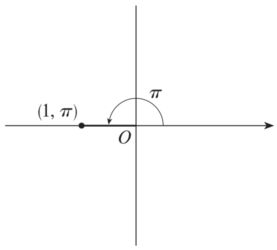
(c) $\left(-1, \frac{\pi}{2}\right)$



$\boldsymbol{r > 0}$**:** $\left(-(-1), \frac{\pi}{2} + \pi\right) = \left(1, \frac{3\pi}{2}\right)$

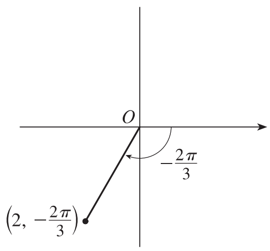
$\boldsymbol{r < 0}$**:** $\left(-1, \frac{\pi}{2} + 2\pi\right) = \left(-1, \frac{5\pi}{2}\right)$

3. (a)



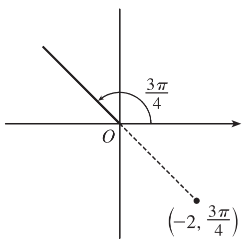
$x = 1\cos\pi = 1(-1) = -1$ and $y = 1\sin\pi = 1(0) = 0$ give us the Cartesian coordinates $(-1, 0)$.

(b)



$x = 2\cos\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$ and $y = 2\sin\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$ give us $\left(-1, -\sqrt{3}\right)$.

(c)

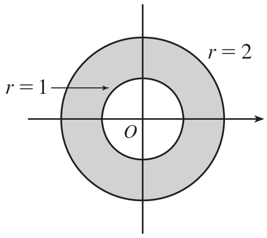


$x = -2\cos\frac{3\pi}{4} = -2\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$ and $y = -2\sin\frac{3\pi}{4} = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$ gives us $\left(\sqrt{2}, -\sqrt{2}\right)$.

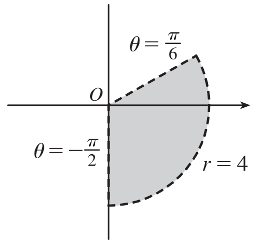
5. (a) $x = 2$ and $y = -2 \quad\Rightarrow\quad r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$. Since $(2, -2)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2\sqrt{2}, \frac{7\pi}{4}\right)$ and (ii) $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$.

(b) $x = -1$ and $y = \sqrt{3} \quad\Rightarrow\quad r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$ and $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3}$. Since $\left(-1, \sqrt{3}\right)$ is in the second quadrant, the polar coordinates are (i) $\left(2, \frac{2\pi}{3}\right)$ and (ii) $\left(-2, \frac{5\pi}{3}\right)$.

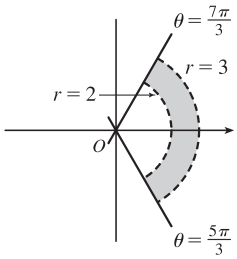
7. The curves $r = 1$ and $r = 2$ represent circles with center O and radii 1 and 2. The region in the plane satisfying $1 \le r \le 2$ consists of both circles and the shaded region between them in the figure.



9. The region satisfying $0 \le r < 4$ and $-\pi/2 \le \theta < \pi/6$ does not include the circle $r = 4$ nor the line $\theta = \frac{\pi}{6}$.



11. $2 < r < 3$, $\frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



13. Converting the polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$ to Cartesian coordinates gives us $\left(2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\right) = \left(1, \sqrt{3}\right)$ and $\left(4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3}\right) = \left(-2, 2\sqrt{3}\right)$. Now use the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-1)^2 + \left(2\sqrt{3} - \sqrt{3}\right)^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

15. $r = 2 \quad\Leftrightarrow\quad \sqrt{x^2 + y^2} = 2 \quad\Leftrightarrow\quad x^2 + y^2 = 4$, a circle of radius 2 centered at the origin.

17. $r = 3\sin\theta \quad\Rightarrow\quad r^2 = 3r\sin\theta \quad\Leftrightarrow\quad x^2 + y^2 = 3y \quad\Leftrightarrow\quad x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2 = 3r\sin\theta \quad\Rightarrow\quad r(r - 3\sin\theta) = 0 \quad\Rightarrow\quad r = 0$ or $r = 3\sin\theta$. But $r = 3\sin\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the single equation $r = 3\sin\theta$ is equivalent to the compound condition ($r = 0$ or $r = 3\sin\theta$).

19. $r = \csc\theta \quad\Leftrightarrow\quad r = \dfrac{1}{\sin\theta} \quad\Leftrightarrow\quad r\sin\theta = 1 \quad\Leftrightarrow\quad y = 1$, a horizontal line 1 unit above the x-axis.

21. $x = 3 \quad\Leftrightarrow\quad r\cos\theta = 3 \quad\Leftrightarrow\quad r = 3/\cos\theta \quad\Leftrightarrow\quad r = 3\sec\theta$.

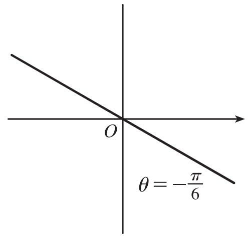
23. $x = -y^2 \quad\Leftrightarrow\quad r\cos\theta = -r^2\sin^2\theta \quad\Leftrightarrow\quad \cos\theta = -r\sin^2\theta \quad\Leftrightarrow\quad r = -\dfrac{\cos\theta}{\sin^2\theta} = -\cot\theta\,\csc\theta$.

25. $x^2 + y^2 = 2cx \quad\Leftrightarrow\quad r^2 = 2cr\cos\theta \quad\Leftrightarrow\quad r^2 - 2cr\cos\theta = 0 \quad\Leftrightarrow\quad r(r - 2c\cos\theta) = 0 \quad\Leftrightarrow\quad r = 0$ or $r = 2c\cos\theta$. $r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

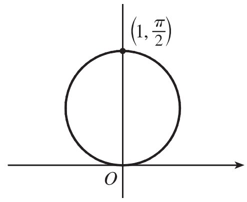
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}\,x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

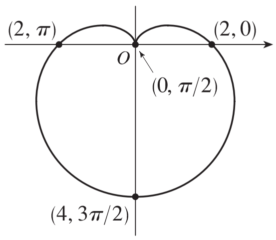
29. $\theta = -\pi/6$



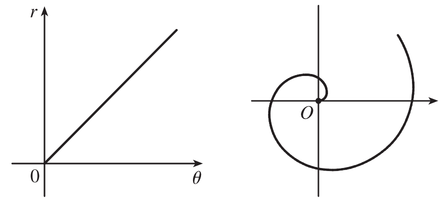
31. $r = \sin\theta \quad\Leftrightarrow\quad r^2 = r\sin\theta \quad\Leftrightarrow\quad x^2 + y^2 = y \quad\Leftrightarrow$ $x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 17. This is a circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$.



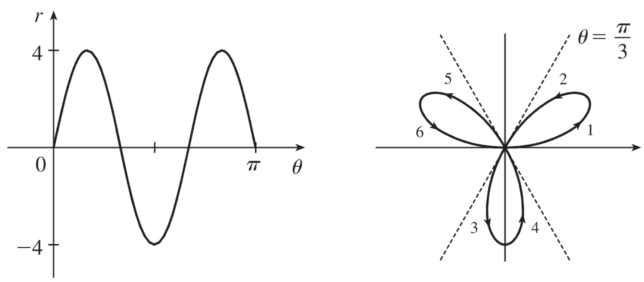
33. $r = 2(1 - \sin\theta)$. This curve is a cardioid.



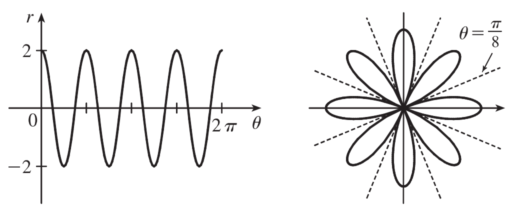
35. $r = \theta, \quad \theta \geq 0$



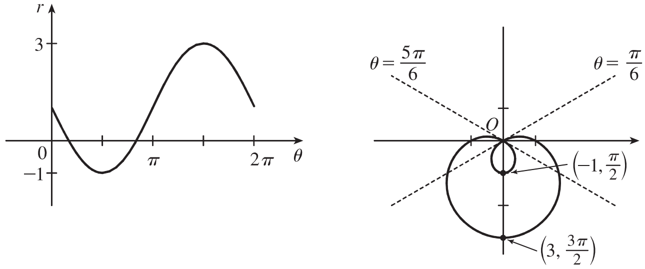
37. $r = 4\sin 3\theta$



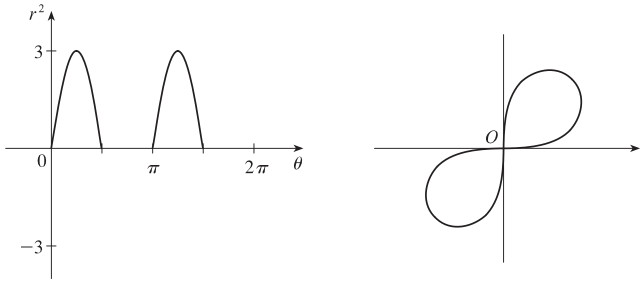
39. $r = 2\cos 4\theta$



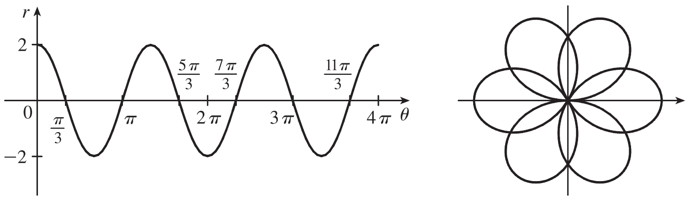
41. $r = 1 - 2\sin\theta$



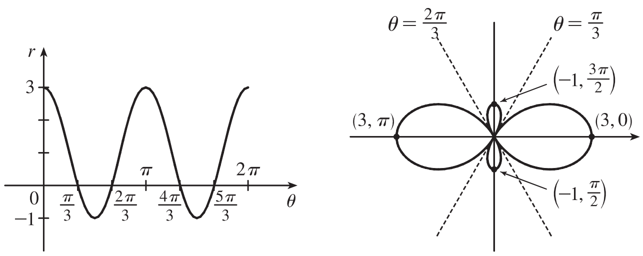
43. $r^2 = 9\sin 2\theta$



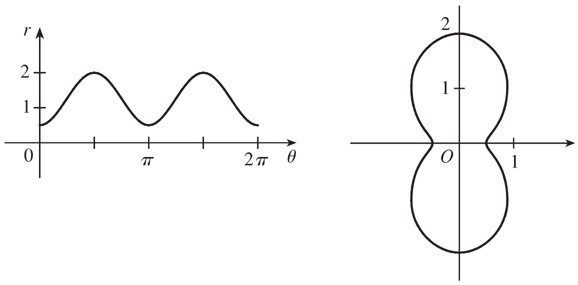
45. $r = 2\cos\left(\frac{3}{2}\theta\right)$



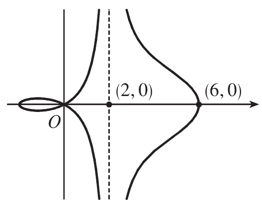
47. $r = 1 + 2\cos 2\theta$



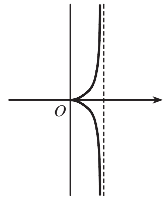
49. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.



51. $x = (r)\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \;\Rightarrow\; (4 + 2\sec\theta) \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^- \text{ or } \theta \to \left(\frac{3\pi}{2}\right)^+$ [since we need only consider $0 \le \theta < 2\pi$], so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \;\Rightarrow\; (4 + 2\sec\theta) \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^+ \text{ or } \theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \;\Rightarrow\; x = 2$ is a vertical asymptote.



53. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r \to \pm\infty} x = 1$.



$x = (r)\cos\theta = (\sin\theta\,\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \quad \Rightarrow \quad \sin\theta\,\tan\theta \to \infty \quad \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \quad \Rightarrow \quad \sin\theta\,\tan\theta \to -\infty \quad \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \quad \Rightarrow \quad x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \geq 0$ for all θ, and $x = \sin^2\theta \leq 1$ for all θ. And $x \neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \leq x < 1$.

55. (a) We see that the curve $r = 1 + c\sin\theta$ crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \quad \Leftrightarrow \quad c\sin\theta = -1 \quad \Leftrightarrow \quad \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum:

$y = r\sin\theta = \sin\theta + c\sin^2\theta \quad \Rightarrow \quad \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\,\cos\theta = \cos\theta + c\sin 2\theta \quad \Rightarrow \quad \dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta$.

At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

57. $r = 2\sin\theta \quad \Rightarrow \quad x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$, $y = r\sin\theta = 2\sin^2\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2\sin\theta\,\cos\theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

When $\theta = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = \tan\left(2 \cdot \dfrac{\pi}{6}\right) = \tan\dfrac{\pi}{3} = \sqrt{3}$. [*Another method:* Use Equation 3.]

59. $r = 1/\theta \quad \Rightarrow \quad x = r\cos\theta = (\cos\theta)/\theta$, $y = r\sin\theta = (\sin\theta)/\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \dfrac{-\pi}{1} = -\pi$.

61. $r = \cos 2\theta \quad \Rightarrow \quad x = r\cos\theta = \cos 2\theta\,\cos\theta$, $y = r\sin\theta = \cos 2\theta\,\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta\,\cos\theta + \sin\theta\,(-2\sin 2\theta)}{\cos 2\theta\,(-\sin\theta) + \cos\theta\,(-2\sin 2\theta)}$$

When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{0\big(\sqrt{2}/2\big) + \big(\sqrt{2}/2\big)(-2)}{0\big(-\sqrt{2}/2\big) + \big(\sqrt{2}/2\big)(-2)} = \dfrac{-\sqrt{2}}{-\sqrt{2}} = 1$.

63. $r = 3\cos\theta \quad \Rightarrow \quad x = r\cos\theta = 3\cos\theta\,\cos\theta,\ y = r\sin\theta = 3\cos\theta\,\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$.

$\frac{dx}{d\theta} = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \quad \Rightarrow \quad 2\theta = 0$ or $\pi \quad \Leftrightarrow \quad \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

65. $r = 1 + \cos\theta \quad \Rightarrow \quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad \Rightarrow$

$\frac{dy}{d\theta} = (1 + \cos\theta)\,\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$ or $-1 \quad \Rightarrow$

$\theta = \frac{\pi}{3}, \pi,$ or $\frac{5\pi}{3} \quad \Rightarrow \quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

$\frac{dx}{d\theta} = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad \Rightarrow \quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad \Rightarrow$

$\theta = 0, \pi, \frac{2\pi}{3},$ or $\frac{4\pi}{3} \quad \Rightarrow \quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\displaystyle\lim_{\theta\to\pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

67. $r = 2 + \sin\theta \quad \Rightarrow \quad x = r\cos\theta = (2 + \sin\theta)\,\cos\theta,\ y = r\sin\theta = (2 + \sin\theta)\,\sin\theta \quad \Rightarrow$

$\frac{dy}{d\theta} = (2 + \sin\theta)\,\cos\theta + \sin\theta\,\cos\theta = \cos\theta \cdot 2(1 + \sin\theta) = 0 \quad \Rightarrow \quad \cos\theta = 0$ or $\sin\theta = -1 \quad \Rightarrow$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad \Rightarrow \quad$ horizontal tangent at $\left(3, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.

$\frac{dx}{d\theta} = (2 + \sin\theta)(-\sin\theta) + \cos\theta\,\cos\theta = -2\sin\theta - \sin^2\theta + 1 - \sin^2\theta = -2\sin^2\theta - 2\sin\theta + 1 \quad \Rightarrow$

$$\sin\theta = \frac{2 \pm \sqrt{4+8}}{-4} = \frac{2 \pm 2\sqrt{3}}{-4} = \frac{1 - \sqrt{3}}{-2} \quad \left[\frac{1+\sqrt{3}}{-2} < -1\right] \quad \Rightarrow$$

$\theta_1 = \sin^{-1}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)$ and $\theta_2 = \pi - \theta_1 \quad \Rightarrow \quad$ vertical tangent at $\left(\frac{3}{2} + \frac{1}{2}\sqrt{3}, \theta_1\right)$ and $\left(\frac{3}{2} + \frac{1}{2}\sqrt{3}, \theta_2\right)$.

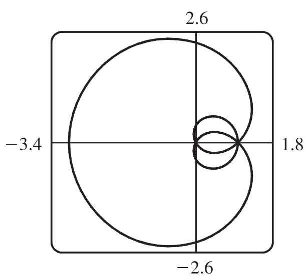
Note that $r(\theta_1) = 2 + \sin\left[\sin^{-1}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}\right)\right] = 2 - \frac{1}{2} + \frac{1}{2}\sqrt{3} = \frac{3}{2} + \frac{1}{2}\sqrt{3}$.

69. $r = a\sin\theta + b\cos\theta \quad \Rightarrow \quad r^2 = ar\sin\theta + br\cos\theta \quad \Rightarrow \quad x^2 + y^2 = ay + bx \quad \Rightarrow$

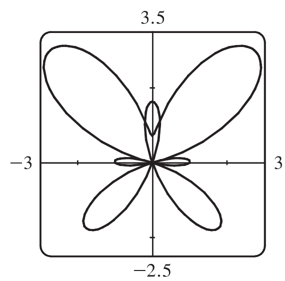
$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \quad \Rightarrow \quad \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

Note for Exercises 71–76: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under `Options State`. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 71, $x = r\cos\theta = [1 + 2\sin(\theta/2)]\cos\theta$, $y = r\sin\theta = [1 + 2\sin(\theta/2)]\sin\theta$.

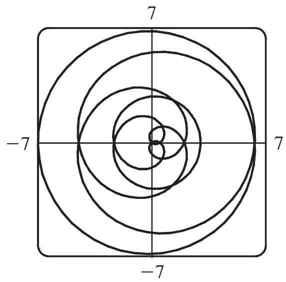
71. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



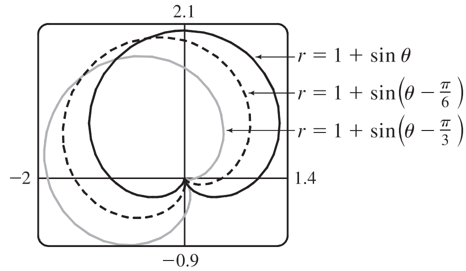
73. $r = e^{\sin\theta} - 2\cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



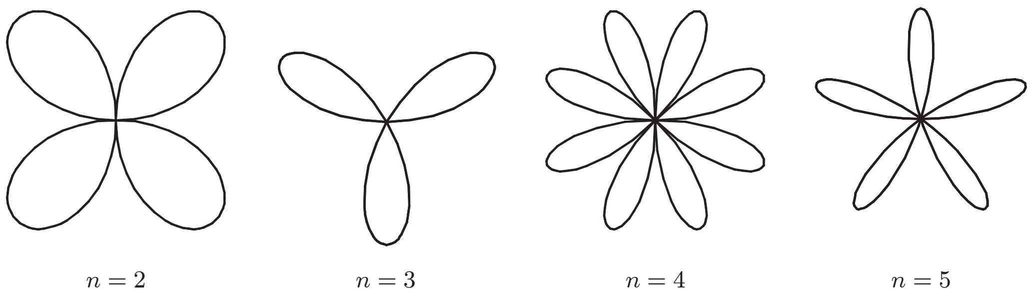
75. $r = 2 - 5\sin(\theta/6)$. The parameter interval is $[-6\pi, 6\pi]$.



77. It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.



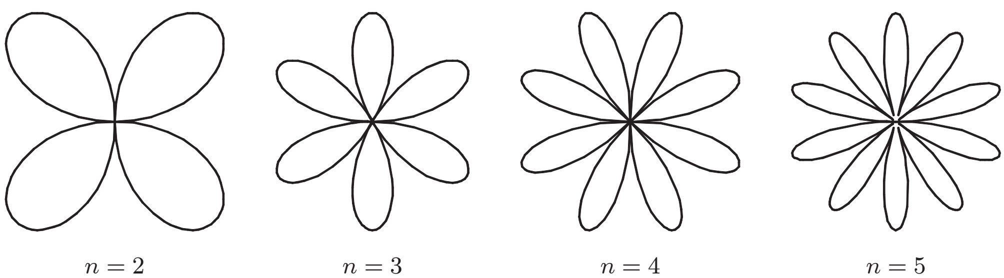
79. (a) $r = \sin n\theta$.



From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

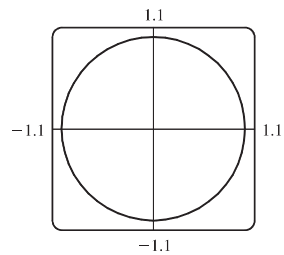
$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \, \cos n\pi + \cos n\theta \, \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.

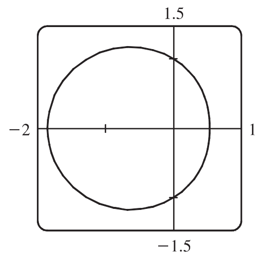


81. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

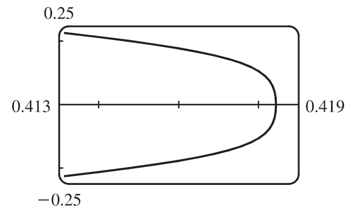
As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ [the actual value is $\sqrt{2} - 1$]. As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow \quad \theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ [actually, $\sqrt{2} + 1$]. As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 80.



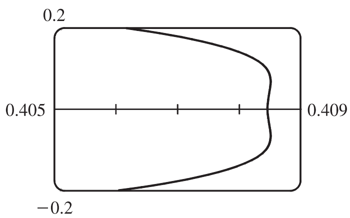
$a = 0$



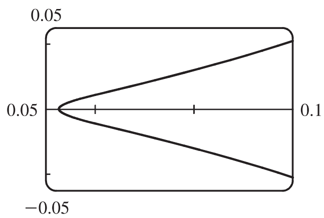
$a = 0.3$



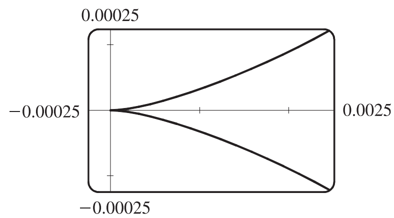
$a = 0.41, |\theta| \leq 0.5$



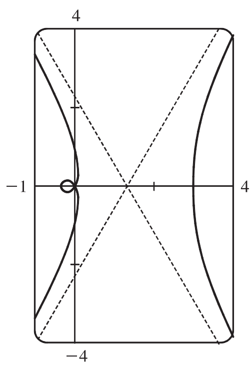
$a = 0.42, |\theta| \leq 0.5$



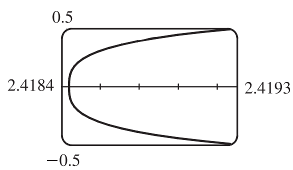
$a = 0.9, |\theta| \leq 0.5$



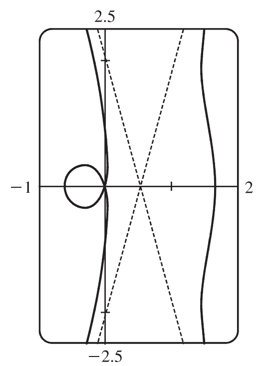
$a = 1, |\theta| \leq 0.1$



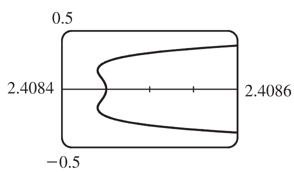
$a = 2$



$a = 2.41, |\theta - \pi| \leq 0.2$



$a = 4$



$a = 2.42, |\theta - \pi| \leq 0.2$

83. $\tan\psi = \tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \dfrac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)} = \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

10.4 Areas and Lengths in Polar Coordinates

1. $r = \theta^2,\ 0 \le \theta \le \frac{\pi}{4}.\quad A = \int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}(\theta^2)^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}\theta^4\,d\theta = \left[\frac{1}{10}\theta^5\right]_0^{\pi/4} = \frac{1}{10}\left(\frac{\pi}{4}\right)^5 = \frac{1}{10{,}240}\pi^5$

3. $r = \sin\theta,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}.$

$$A = \int_{\pi/3}^{2\pi/3} \tfrac{1}{2}\sin^2\theta\,d\theta = \tfrac{1}{4}\int_{\pi/3}^{2\pi/3}(1 - \cos 2\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3} = \tfrac{1}{4}\left[\tfrac{2\pi}{3} - \tfrac{1}{2}\sin\tfrac{4\pi}{3} - \tfrac{\pi}{3} + \tfrac{1}{2}\sin\tfrac{2\pi}{3}\right]$$

$$= \tfrac{1}{4}\left[\tfrac{2\pi}{3} - \tfrac{1}{2}\left(-\tfrac{\sqrt{3}}{2}\right) - \tfrac{\pi}{3} + \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}\right) = \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{8}$$

5. $r = \sqrt{\theta},\ 0 \le \theta \le 2\pi.\quad A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}\left(\sqrt{\theta}\right)^2\,d\theta = \int_0^{2\pi} \frac{1}{2}\theta\,d\theta = \left[\frac{1}{4}\theta^2\right]_0^{2\pi} = \pi^2$

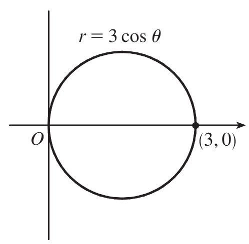
7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

$$A = \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}((4 + 3\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}(16 + 24\sin\theta + 9\sin^2\theta)\,d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}(16 + 9\sin^2\theta)\,d\theta \quad \text{[by Theorem 5.5.7(b)]}$$

$$= \tfrac{1}{2}\cdot 2\int_0^{\pi/2}\left[16 + 9\cdot\tfrac{1}{2}(1 - \cos 2\theta)\right]d\theta \qquad \text{[by Theorem 5.5.7(a)]}$$

$$= \int_0^{\pi/2}\left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right)d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4}$$

9. The area above the polar axis is bounded by $r = 3\cos\theta$ for $\theta = 0$ to $\theta = \pi/2$ [*not* π]. By symmetry,

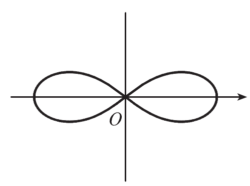


$$A = 2\int_0^{\pi/2}\tfrac{1}{2}r^2\,d\theta = \int_0^{\pi/2}(3\cos\theta)^2\,d\theta = 3^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 9\int_0^{\pi/2}\tfrac{1}{2}(1 + \cos 2\theta)\,d\theta = \tfrac{9}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{9}{2}\left[\left(\tfrac{\pi}{2} + 0\right) - (0 + 0)\right] = \tfrac{9\pi}{4}$$

Also, note that this is a circle with radius $\frac{3}{2}$, so its area is $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$.

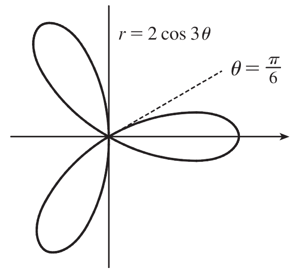
11. The curve goes through the pole when $\theta = \pi/4$, so we'll find the area for $0 \le \theta \le \pi/4$ and multiply it by 4.



$$A = 4\int_0^{\pi/4}\tfrac{1}{2}r^2\,d\theta = 2\int_0^{\pi/4}(4\cos 2\theta)\,d\theta$$

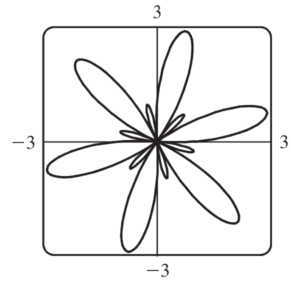
$$= 8\int_0^{\pi/4}\cos 2\theta\,d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4$$

13. One-sixth of the area lies above the polar axis and is bounded by the curve $r = 2\cos 3\theta$ for $\theta = 0$ to $\theta = \pi/6$.

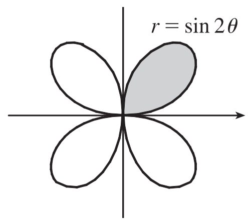


$$\begin{aligned} A &= 6\int_0^{\pi/6} \tfrac{1}{2}(2\cos 3\theta)^2\,d\theta = 12\int_0^{\pi/6} \cos^2 3\theta\,d\theta \\ &= \tfrac{12}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta \\ &= 6\left[\theta + \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 6\left(\tfrac{\pi}{6}\right) = \pi \end{aligned}$$

15. $$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}(1+2\sin 6\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi}(1+4\sin 6\theta + 4\sin^2 6\theta)\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[1+4\sin 6\theta + 4\cdot\tfrac{1}{2}(1-\cos 12\theta)\right]d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}(3+4\sin 6\theta - 2\cos 12\theta)\,d\theta \\ &= \tfrac{1}{2}\left[3\theta - \tfrac{2}{3}\cos 6\theta - \tfrac{1}{6}\sin 12\theta\right]_0^{2\pi} \\ &= \tfrac{1}{2}\left[\left(6\pi - \tfrac{2}{3} - 0\right) - \left(0 - \tfrac{2}{3} - 0\right)\right] = 3\pi \end{aligned}$$



17. The shaded loop is traced out from $\theta = 0$ to $\theta = \pi/2$.

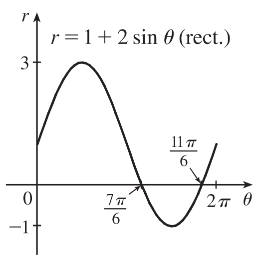


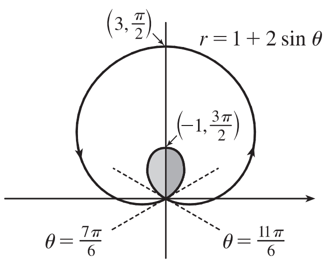
$$\begin{aligned} A &= \int_0^{\pi/2} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta \\ &= \tfrac{1}{2}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} \\ &= \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8} \end{aligned}$$

19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{10}$.

$$A = \int_{-\pi/10}^{\pi/10}\tfrac{1}{2}(3\cos 5\theta)^2\,d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\,d\theta = \tfrac{9}{2}\int_0^{\pi/10}(1+\cos 10\theta)\,d\theta = \tfrac{9}{2}\left[\theta + \tfrac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \tfrac{9\pi}{20}$$

21.

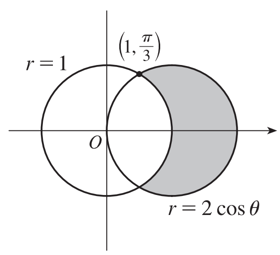




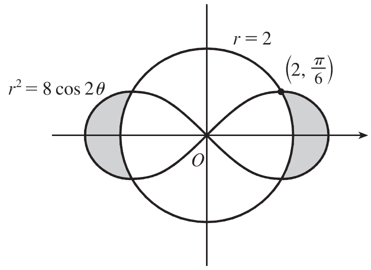
This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$\begin{aligned} A &= 2\int_{7\pi/6}^{3\pi/2}\tfrac{1}{2}(1+2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta = \int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta \\ &= \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2} \end{aligned}$$

23. $2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

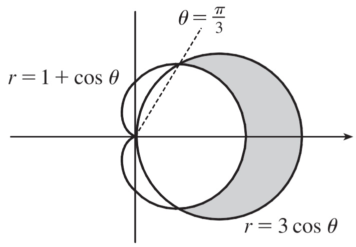


$$\begin{aligned} A &= 2\int_0^{\pi/3}\tfrac{1}{2}\left[(2\cos\theta)^2 - 1^2\right]d\theta = \int_0^{\pi/3}(4\cos^2\theta - 1)\,d\theta \\ &= \int_0^{\pi/3}\left\{4\left[\tfrac{1}{2}(1+\cos 2\theta)\right] - 1\right\}d\theta = \int_0^{\pi/3}(1+2\cos 2\theta)\,d\theta \\ &= \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2} \end{aligned}$$



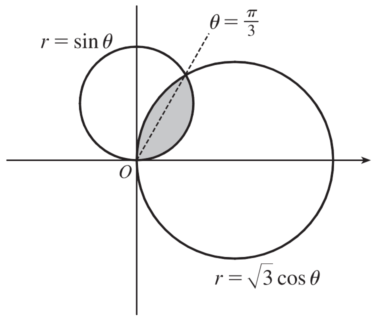
25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \;\Leftrightarrow\; 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \;\Leftrightarrow\; \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

$$A = 4\int_0^{\pi/6}\left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6}(2\cos 2\theta - 1)\,d\theta$$
$$= 8\Big[\sin 2\theta - \theta\Big]_0^{\pi/6} = 8\big(\sqrt{3}/2 - \pi/6\big) = 4\sqrt{3} - 4\pi/3$$



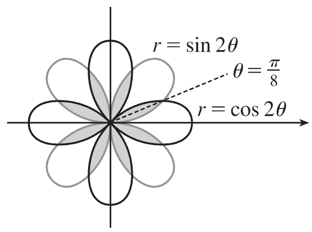
27. $3\cos\theta = 1 + \cos\theta \;\Leftrightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.

$$A = 2\int_0^{\pi/3}\tfrac{1}{2}[(3\cos\theta)^2 - (1+\cos\theta)^2]\,d\theta$$
$$= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta = \int_0^{\pi/3}[4(1+\cos 2\theta) - 2\cos\theta - 1]\,d\theta$$
$$= \int_0^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta = \big[3\theta + 2\sin 2\theta - 2\sin\theta\big]_0^{\pi/3}$$
$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$



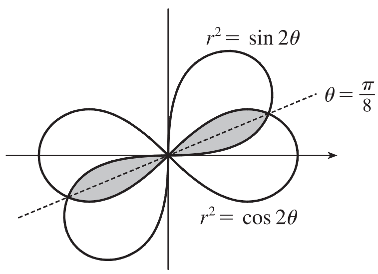
29. $\sqrt{3}\cos\theta = \sin\theta \;\Rightarrow\; \sqrt{3} = \dfrac{\sin\theta}{\cos\theta} \;\Rightarrow\; \tan\theta = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$.

$$A = \int_0^{\pi/3}\tfrac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2}\tfrac{1}{2}\left(\sqrt{3}\cos\theta\right)^2\,d\theta$$
$$= \int_0^{\pi/3}\tfrac{1}{2}\cdot\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta + \int_{\pi/3}^{\pi/2}\tfrac{1}{2}\cdot 3\cdot\tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$$
$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{3}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$= \tfrac{1}{4}\left[\left(\tfrac{\pi}{3} - \tfrac{\sqrt{3}}{4}\right) - 0\right] + \tfrac{3}{4}\left[\left(\tfrac{\pi}{2} + 0\right) - \left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right)\right]$$
$$= \tfrac{\pi}{12} - \tfrac{\sqrt{3}}{16} + \tfrac{\pi}{8} - \tfrac{3\sqrt{3}}{16} = \tfrac{5\pi}{24} - \tfrac{\sqrt{3}}{4}$$



31. $\sin 2\theta = \cos 2\theta \;\Rightarrow\; \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \;\Rightarrow\; \tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4} \;\Rightarrow\; \theta = \frac{\pi}{8} \;\Rightarrow$

$$A = 8\cdot 2\int_0^{\pi/8}\tfrac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8}\tfrac{1}{2}(1 - \cos 4\theta)\,d\theta$$
$$= 4\big[\theta - \tfrac{1}{4}\sin 4\theta\big]_0^{\pi/8} = 4\big(\tfrac{\pi}{8} - \tfrac{1}{4}\cdot 1\big) = \tfrac{\pi}{2} - 1$$

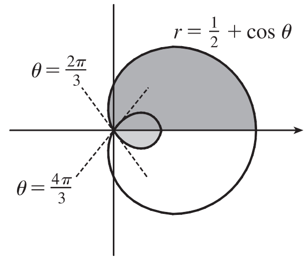


33. $\sin 2\theta = \cos 2\theta \;\Rightarrow\; \tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4} \;\Rightarrow\; \theta = \frac{\pi}{8}$

$$A = 4\int_0^{\pi/8}\tfrac{1}{2}\sin 2\theta\,d\theta \quad [\text{since } r^2 = \sin 2\theta]$$
$$= \int_0^{\pi/8}2\sin 2\theta\,d\theta = \big[-\cos 2\theta\big]_0^{\pi/8}$$
$$= -\tfrac{1}{2}\sqrt{2} - (-1) = 1 - \tfrac{1}{2}\sqrt{2}$$

35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.

$$
\begin{aligned}
A &= 2\left[\int_0^{2\pi/3} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2 d\theta\right] \\
&= \int_0^{2\pi/3} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta - \int_{2\pi/3}^{\pi} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta \\
&= \int_0^{2\pi/3} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&\qquad - \int_{2\pi/3}^{\pi} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi} \\
&= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) \\
&= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)
\end{aligned}
$$

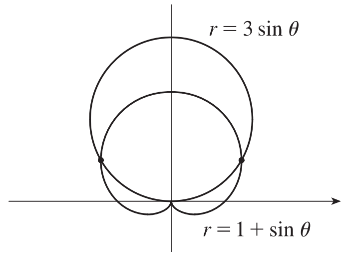


37. The pole is a point of intersection.

$1 + \sin\theta = 3\sin\theta \quad\Rightarrow\quad 1 = 2\sin\theta \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

The other two points of intersection are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.



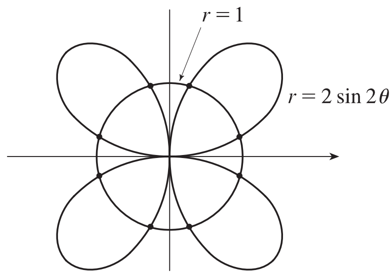
39. $2\sin 2\theta = 1 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

[There are many ways to describe these points.]

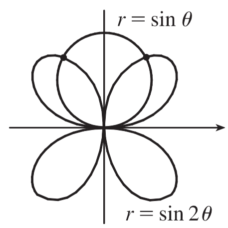


41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\,\cos\theta \quad\Leftrightarrow$

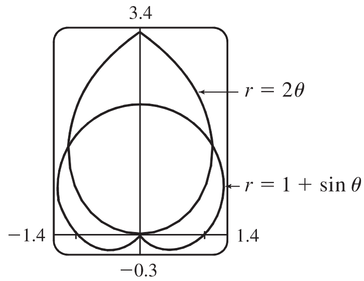
$\sin\theta\,(1 - 2\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \quad\Rightarrow$

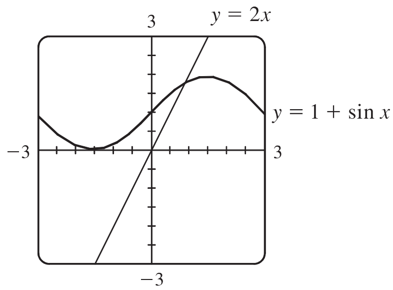
$\theta = 0, \pi, \frac{\pi}{3}$, or $-\frac{\pi}{3} \quad\Rightarrow\quad$ the other intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$

and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ [by symmetry].



43.





From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^{\alpha} \tfrac{1}{2}(2\theta)^2\, d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\, d\theta = \int_0^{\alpha} 4\theta^2\, d\theta + \int_{\alpha}^{\pi/2} \left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2} + \tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

45. $$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi/3} \sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = \int_0^{\pi/3} \sqrt{9(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= 3\int_0^{\pi/3} d\theta = 3\left[\theta\right]_0^{\pi/3} = 3\left(\tfrac{\pi}{3}\right) = \pi.$$

As a check, note that the circumference of a circle with radius $\frac{3}{2}$ is $2\pi\left(\frac{3}{2}\right) = 3\pi$, and since $\theta = 0$ to $\pi = \frac{\pi}{3}$ traces out $\frac{1}{3}$ of the circle (from $\theta = 0$ to $\theta = \pi$), $\frac{1}{3}(3\pi) = \pi$.

47. $$L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

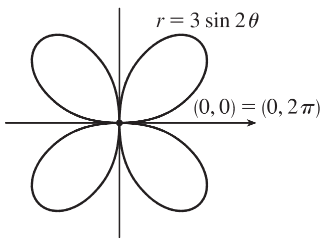
$$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^{4\pi^2+4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}\left[4^{3/2}(\pi^2+1)^{3/2} - 4^{3/2}\right] = \tfrac{8}{3}\left[(\pi^2+1)^{3/2} - 1\right]$$

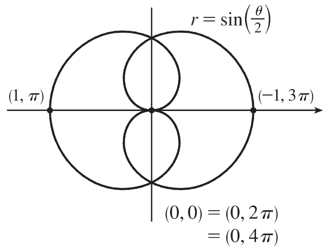
49. The curve $r = 3\sin 2\theta$ is completely traced with $0 \le \theta \le 2\pi$. $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (3\sin 2\theta)^2 + (6\cos 2\theta)^2 \Rightarrow$

$L = \int_0^{2\pi} \sqrt{9\sin^2 2\theta + 36\cos^2 2\theta}\, d\theta \approx 29.0653$



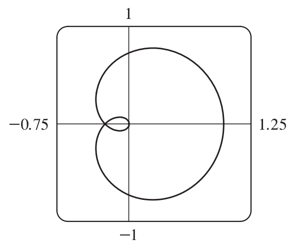
51. The curve $r = \sin\left(\frac{\theta}{2}\right)$ is completely traced with $0 \le \theta \le 4\pi$. $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2\left(\frac{\theta}{2}\right) + \left[\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right]^2 \Rightarrow$

$$L = \int_0^{4\pi} \sqrt{\sin^2\left(\tfrac{\theta}{2}\right) + \tfrac{1}{4}\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta \approx 9.6884$$



53. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \le \theta \le 4\pi$.

$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^4(\theta/4)]^2 + \left[4\cos^3(\theta/4)\cdot(-\sin(\theta/4))\cdot\tfrac{1}{4}\right]^2 \\ &= \cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4) \\ &= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4) \end{aligned}$$

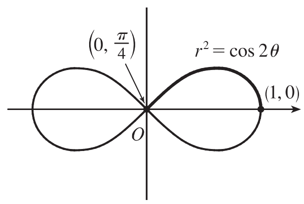


$$\begin{aligned} L &= \textstyle\int_0^{4\pi} \sqrt{\cos^6(\theta/4)}\, d\theta = \int_0^{4\pi} \left|\cos^3(\theta/4)\right|\, d\theta \\ &= 2\textstyle\int_0^{2\pi} \cos^3(\theta/4)\, d\theta \quad [\text{since } \cos^3(\theta/4) \ge 0 \text{ for } 0 \le \theta \le 2\pi] \quad = 8\int_0^{\pi/2} \cos^3 u\, du \quad \left[u = \tfrac{1}{4}\theta\right] \\ &\overset{68}{=} 8\left[\tfrac{1}{3}(2+\cos^2 u)\sin u\right]_0^{\pi/2} = \tfrac{8}{3}[(2\cdot 1) - (3\cdot 0)] = \tfrac{16}{3} \end{aligned}$$

55. (a) From (10.2.7),

$$\begin{aligned} S &= \textstyle\int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta \\ &= \textstyle\int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\, d\theta \qquad [\text{from the derivation of Equation 10.4.5}] \\ &= \textstyle\int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\, d\theta \end{aligned}$$

(b) The curve $r^2 = \cos 2\theta$ goes through the pole when $\cos 2\theta = 0 \Rightarrow$ $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$. We'll rotate the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface area generated.

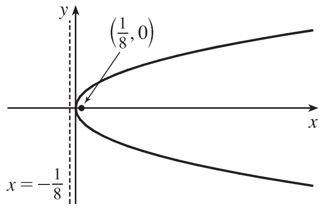


$$r^2 = \cos 2\theta \Rightarrow 2r\,\frac{dr}{d\theta} = -2\sin 2\theta \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2} = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

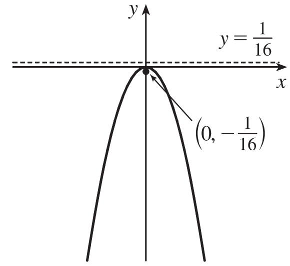
$$\begin{aligned} S &= 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\,\sin\theta\sqrt{\cos 2\theta + \left(\sin^2 2\theta\right)/\cos 2\theta}\, d\theta = 4\pi\int_0^{\pi/4} \sqrt{\cos 2\theta}\,\sin\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta \\ &= 4\pi\int_0^{\pi/4} \sqrt{\cos 2\theta}\,\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4} \sin\theta\, d\theta = 4\pi\left[-\cos\theta\right]_0^{\pi/4} = -4\pi\left(\tfrac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right) \end{aligned}$$

10.5 Conic Sections

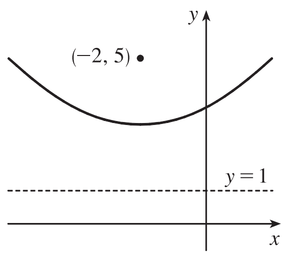
1. $x = 2y^2 \quad \Rightarrow \quad y^2 = \frac{1}{2}x$. $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(0, 0)$, the focus is $\left(\frac{1}{8}, 0\right)$, and the directrix is $x = -\frac{1}{8}$.



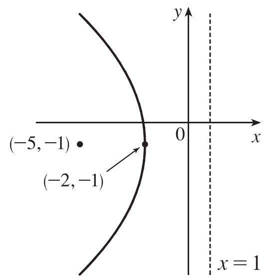
3. $4x^2 = -y \quad \Rightarrow \quad x^2 = -\frac{1}{4}y$. $4p = -\frac{1}{4}$, so $p = -\frac{1}{16}$. The vertex is $(0, 0)$, the focus is $\left(0, -\frac{1}{16}\right)$, and the directrix is $y = \frac{1}{16}$.



5. $(x + 2)^2 = 8\,(y - 3)$. $4p = 8$, so $p = 2$. The vertex is $(-2, 3)$, the focus is $(-2, 5)$, and the directrix is $y = 1$.

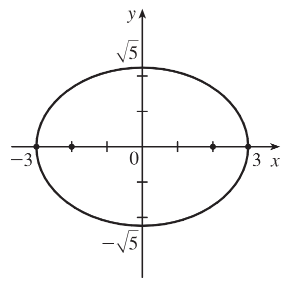


7. $y^2 + 2y + 12x + 25 = 0 \quad \Rightarrow$
$y^2 + 2y + 1 = -12x - 24 \quad \Rightarrow$
$(y + 1)^2 = -12(x + 2)$. $4p = -12$, so $p = -3$.
The vertex is $(-2, -1)$, the focus is $(-5, -1)$, and the directrix is $x = 1$.

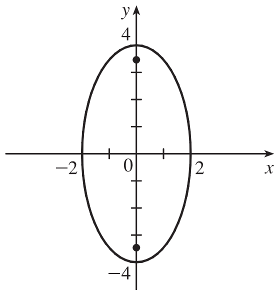


9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

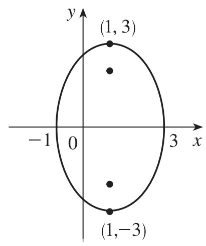
11. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1 \quad \Rightarrow \quad a = \sqrt{9} = 3$, $b = \sqrt{5}$, $c = \sqrt{a^2 - b^2} = \sqrt{9 - 5} = 2$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm 3, 0)$. The foci are $(\pm 2, 0)$.



13. $4x^2 + y^2 = 16 \;\Rightarrow\; \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1 \;\Rightarrow$
$a = \sqrt{16} = 4$, $b = \sqrt{4} = 2$,
$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = 2\sqrt{3}$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 4)$. The foci are $\left(0, \pm 2\sqrt{3}\right)$.



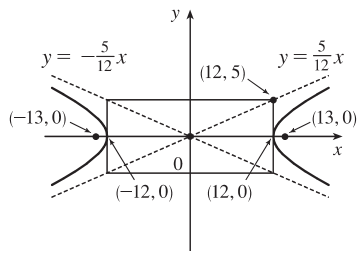
15. $9x^2 - 18x + 4y^2 = 27 \;\Leftrightarrow$
$9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \;\Leftrightarrow$
$9(x - 1)^2 + 4y^2 = 36 \;\Leftrightarrow\; \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1 \;\Rightarrow$
$a = 3$, $b = 2$, $c = \sqrt{5} \;\Rightarrow$ center $(1, 0)$,
vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$



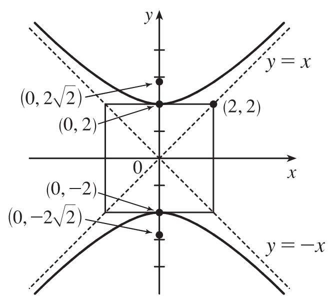
17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1 \;\Rightarrow\; a = 12, b = 5, c = \sqrt{144 + 25} = 13 \;\Rightarrow$
center $(0, 0)$, vertices $(\pm 12, 0)$, foci $(\pm 13, 0)$, asymptotes $y = \pm\frac{5}{12}x$.

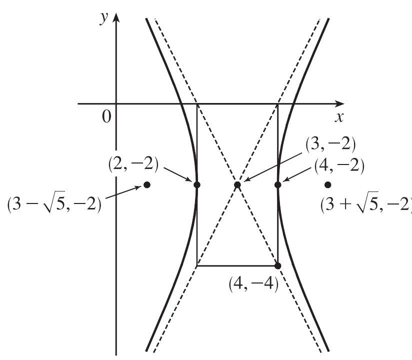
Note: It is helpful to draw a $2a$-by-$2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



21. $y^2 - x^2 = 4 \;\Leftrightarrow\; \dfrac{y^2}{4} - \dfrac{x^2}{4} = 1 \;\Rightarrow\; a = \sqrt{4} = 2 = b$,
$c = \sqrt{4 + 4} = 2\sqrt{2} \;\Rightarrow$ center $(0, 0)$, vertices $(0, \pm 2)$,
foci $\left(0, \pm 2\sqrt{2}\right)$, asymptotes $y = \pm x$



23. $4x^2 - y^2 - 24x - 4y + 28 = 0 \;\Leftrightarrow$
$4(x^2 - 6x + 9) - (y^2 + 4y + 4) = -28 + 36 - 4 \;\Leftrightarrow$
$4(x - 3)^2 - (y + 2)^2 = 4 \;\Leftrightarrow\; \dfrac{(x-3)^2}{1} - \dfrac{(y+2)^2}{4} = 1 \;\Rightarrow$
$a = \sqrt{1} = 1$, $b = \sqrt{4} = 2$, $c = \sqrt{1 + 4} = \sqrt{5} \;\Rightarrow$
center $(3, -2)$, vertices $(4, -2)$ and $(2, -2)$, foci $\left(3 \pm \sqrt{5}, -2\right)$,
asymptotes $y + 2 = \pm 2(x - 3)$.



25. $x^2 = y + 1 \quad \Leftrightarrow \quad x^2 = 1(y+1)$. This is an equation of a *parabola* with $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(0, -1)$ and the focus is $\left(0, -\frac{3}{4}\right)$.

27. $x^2 = 4y - 2y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 - 4y = 0 \quad \Leftrightarrow \quad x^2 + 2(y^2 - 2y + 1) = 2 \quad \Leftrightarrow \quad x^2 + 2(y-1)^2 = 2 \quad \Leftrightarrow$ $\dfrac{x^2}{2} + \dfrac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

29. $y^2 + 2y = 4x^2 + 3 \quad \Leftrightarrow \quad y^2 + 2y + 1 = 4x^2 + 4 \quad \Leftrightarrow \quad (y+1)^2 - 4x^2 = 4 \quad \Leftrightarrow \quad \dfrac{(y+1)^2}{4} - x^2 = 1$. This is an equation of a *hyperbola* with vertices $(0, -1 \pm 2) = (0, 1)$ and $(0, -3)$. The foci are at $\left(0, -1 \pm \sqrt{4+1}\right) = \left(0, -1 \pm \sqrt{5}\right)$.

31. The parabola with vertex $(0, 0)$ and focus $(0, -2)$ opens downward and has $p = -2$, so its equation is $x^2 = 4py = -8y$.

33. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x+1) \quad \Rightarrow$ $y^2 = -12(x+1)$.

35. A parabola with vertical axis and vertex $(2, 3)$ has equation $y - 3 = a(x-2)^2$. Since it passes through $(1, 5)$, we have $5 - 3 = a(1-2)^2 \quad \Rightarrow \quad a = 2$, so an equation is $y - 3 = 2(x-2)^2$.

37. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 2$, so $b^2 = a^2 - c^2 = 25 - 4 = 21$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$.

39. Since the vertices are $(0, 0)$ and $(0, 8)$, the ellipse has center $(0, 4)$ with a vertical axis and $a = 4$. The foci at $(0, 2)$ and $(0, 6)$ are 2 units from the center, so $c = 2$ and $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$. An equation is $\dfrac{(x-0)^2}{b^2} + \dfrac{(y-4)^2}{a^2} = 1 \quad \Rightarrow$ $\dfrac{x^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

41. An equation of an ellipse with center $(-1, 4)$ and vertex $(-1, 0)$ is $\dfrac{(x+1)^2}{b^2} + \dfrac{(y-4)^2}{4^2} = 1$. The focus $(-1, 6)$ is 2 units from the center, so $c = 2$. Thus, $b^2 + 2^2 = 4^2 \quad \Rightarrow \quad b^2 = 12$, and the equation is $\dfrac{(x+1)^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

43. An equation of a hyperbola with vertices $(\pm 3, 0)$ is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$. Foci $(\pm 5, 0) \quad \Rightarrow \quad c = 5$ and $3^2 + b^2 = 5^2 \quad \Rightarrow$ $b^2 = 25 - 9 = 16$, so the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

45. The center of a hyperbola with vertices $(-3, -4)$ and $(-3, 6)$ is $(-3, 1)$, so $a = 5$ and an equation is $\dfrac{(y-1)^2}{5^2} - \dfrac{(x+3)^2}{b^2} = 1$. Foci $(-3, -7)$ and $(-3, 9) \quad \Rightarrow \quad c = 8$, so $5^2 + b^2 = 8^2 \quad \Rightarrow \quad b^2 = 64 - 25 = 39$ and the equation is $\dfrac{(y-1)^2}{25} - \dfrac{(x+3)^2}{39} = 1$.

47. The center of a hyperbola with vertices $(\pm 3, 0)$ is $(0, 0)$, so $a = 3$ and an equation is $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$.

Asymptotes $y = \pm 2x \;\Rightarrow\; \dfrac{b}{a} = 2 \;\Rightarrow\; b = 2(3) = 6$ and the equation is $\dfrac{x^2}{9} - \dfrac{y^2}{36} = 1$.

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit, $(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314)$, or $a = 1940$; and $(a + c) - (a - c) = 2c = 314 - 110$, or $c = 102$. Thus, $b^2 = a^2 - c^2 = 3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$.

51. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.

$|PA| - |PB| = (1200)(980) = 1{,}176{,}000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \;\Rightarrow\; a = \frac{1225}{11}$, and $c = 200$ so

$b^2 = c^2 - a^2 = \dfrac{3{,}339{,}375}{121} \;\Rightarrow\; \dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$.

(b) Due north of $B \;\Rightarrow\; x = 200 \;\Rightarrow\; \dfrac{(121)(200)^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1 \;\Rightarrow\; y = \dfrac{133{,}575}{539} \approx 248$ mi

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is

$y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2 + x^2}$, so $y' = \dfrac{a}{b}x(b^2 + x^2)^{-1/2}$ and

$y'' = \dfrac{a}{b}\left[(b^2 + x^2)^{-1/2} - x^2(b^2 + x^2)^{-3/2}\right] = ab(b^2 + x^2)^{-3/2} > 0$ for all x, and so f is concave upward.

55. (a) If $k > 16$, then $k - 16 > 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k - 16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0 < k < 16$, then $k - 16 < 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k - 16} = 1$ is a *hyperbola* since it is the difference of two squares on the left side.

(c) If $k < 0$, then $k - 16 < 0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

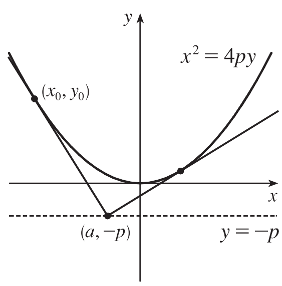
(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

57. $x^2 = 4py \;\Rightarrow\; 2x = 4py' \;\Rightarrow\; y' = \dfrac{x}{2p}$, so the tangent line at (x_0, y_0) is

$y - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(x - x_0)$. This line passes through the point $(a, -p)$ on the

directrix, so $-p - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(a - x_0) \;\Rightarrow\; -4p^2 - x_0^2 = 2ax_0 - 2x_0^2 \;\Leftrightarrow$

$x_0^2 - 2ax_0 - 4p^2 = 0 \;\Leftrightarrow\; x_0^2 - 2ax_0 + a^2 = a^2 + 4p^2 \;\Leftrightarrow$



$(x_0 - a)^2 = a^2 + 4p^2 \quad \Leftrightarrow \quad x_0 = a \pm \sqrt{a^2 + 4p^2}$. The slopes of the tangent lines at $x = a \pm \sqrt{a^2 + 4p^2}$ are $\dfrac{a \pm \sqrt{a^2 + 4p^2}}{2p}$, so the product of the two slopes is

$$\frac{a + \sqrt{a^2 + 4p^2}}{2p} \cdot \frac{a - \sqrt{a^2 + 4p^2}}{2p} = \frac{a^2 - (a^2 + 4p^2)}{4p^2} = \frac{-4p^2}{4p^2} = -1,$$

showing that the tangent lines are perpendicular.

59. For $x^2 + 4y^2 = 4$, or $x^2/4 + y^2 = 1$, use the parametrization $x = 2\cos t,\ y = \sin t,\ 0 \le t \le 2\pi$ to get

$$L = 4\int_0^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = 4\int_0^{\pi/2} \sqrt{4\sin^2 t + \cos^2 t}\,dt = 4\int_0^{\pi/2} \sqrt{3\sin^2 t + 1}\,dt$$

Using Simpson's Rule with $n = 10$, $\Delta t = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(t) = \sqrt{3\sin^2 t + 1}$, we get

$$L \approx \tfrac{4}{3}\left(\tfrac{\pi}{20}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 2f\left(\tfrac{8\pi}{20}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 9.69$$

61. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{y^2}{b^2} = \dfrac{x^2 - a^2}{a^2} \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$.

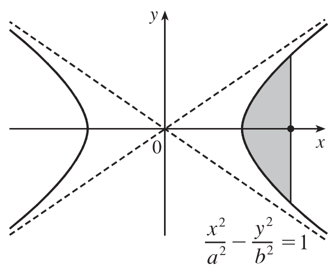
$$A = 2\int_a^c \frac{b}{a}\sqrt{x^2 - a^2}\,dx \overset{39}{=} \frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right|\right]_a^c$$

$$= \frac{b}{a}\left[c\sqrt{c^2 - a^2} - a^2\ln\left|c + \sqrt{c^2 - a^2}\right| + a^2\ln|a|\right]$$

Since $a^2 + b^2 = c^2$, $c^2 - a^2 = b^2$, and $\sqrt{c^2 - a^2} = b$.

$$= \frac{b}{a}\left[cb - a^2\ln(c + b) + a^2\ln a\right] = \frac{b}{a}\left[cb + a^2(\ln a - \ln(b + c))\right]$$

$$= b^2c/a + ab\ln[a/(b + c)], \text{ where } c^2 = a^2 + b^2.$$



63. Differentiating implicitly, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad \Rightarrow \quad y' = -\dfrac{b^2x}{a^2y}$ $[y \ne 0]$. Thus, the slope of the tangent line at P is $-\dfrac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1 + c}$ and of F_2P is $\dfrac{y_1}{x_1 - c}$. By the formula in Problem 17 on text page 268, we have

$$\tan\alpha = \frac{\dfrac{y_1}{x_1 + c} + \dfrac{b^2x_1}{a^2y_1}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1 + c)}} = \frac{a^2y_1^2 + b^2x_1(x_1 + c)}{a^2y_1(x_1 + c) - b^2x_1y_1} = \frac{a^2b^2 + b^2cx_1}{c^2x_1y_1 + a^2cy_1} \quad \left[\begin{array}{c}\text{using } b^2x_1^2 + a^2y_1^2 = a^2b^2, \\ \text{and } a^2 - b^2 = c^2\end{array}\right]$$

$$= \frac{b^2(cx_1 + a^2)}{cy_1(cx_1 + a^2)} = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1 - c}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1 - c)}} = \frac{-a^2y_1^2 - b^2x_1(x_1 - c)}{a^2y_1(x_1 - c) - b^2x_1y_1} = \frac{-a^2b^2 + b^2cx_1}{c^2x_1y_1 - a^2cy_1} = \frac{b^2(cx_1 - a^2)}{cy_1(cx_1 - a^2)} = \frac{b^2}{cy_1}$$

Thus, $\alpha = \beta$.

10.6 Conic Sections in Polar Coordinates

1. The directrix $y = 6$ is above the focus at the origin, so we use the form with "$+\, e\sin\theta$" in the denominator. [See Theorem 6 and Figure 2(c).] $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{\frac{7}{4}\cdot 6}{1 + \frac{7}{4}\sin\theta} = \dfrac{42}{4 + 7\sin\theta}$

3. The directrix $x = -5$ is to the left of the focus at the origin, so we use the form with "$-\, e\cos\theta$" in the denominator.

$$r = \frac{ed}{1 - e\cos\theta} = \frac{\frac{3}{4}\cdot 5}{1 - \frac{3}{4}\cos\theta} = \frac{15}{4 - 3\cos\theta}$$

5. The vertex $(4, 3\pi/2)$ is 4 units below the focus at the origin, so the directrix is 8 units below the focus ($d = 8$), and we use the form with "$-e\sin\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1 - e\sin\theta} = \dfrac{1(8)}{1 - 1\sin\theta} = \dfrac{8}{1 - \sin\theta}$.

7. The directrix $r = 4\sec\theta$ (equivalent to $r\cos\theta = 4$ or $x = 4$) is to the right of the focus at the origin, so we will use the form with "$+e\cos\theta$" in the denominator. The distance from the focus to the directrix is $d = 4$, so an equation is

$$r = \frac{ed}{1 + e\cos\theta} = \frac{\frac{1}{2}(4)}{1 + \frac{1}{2}\cos\theta}\cdot\frac{2}{2} = \frac{4}{2 + \cos\theta}.$$

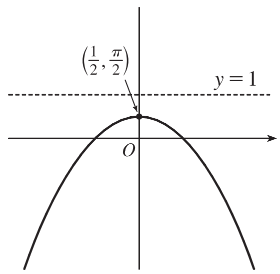
9. $r = \dfrac{1}{1 + \sin\theta} = \dfrac{ed}{1 + e\sin\theta}$, where $d = e = 1$.

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$+e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = 1$, so an equation of the directrix is $y = 1$.

(d) The vertex is at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$, midway between the focus and the directrix.



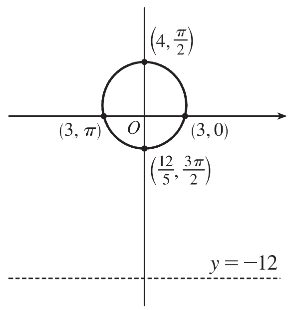
11. $r = \dfrac{12}{4 - \sin\theta}\cdot\dfrac{1/4}{1/4} = \dfrac{3}{1 - \frac{1}{4}\sin\theta}$, where $e = \frac{1}{4}$ and $ed = 3 \;\Rightarrow\; d = 12$.

(a) Eccentricity $= e = \frac{1}{4}$

(b) Since $e = \frac{1}{4} < 1$, the conic is an ellipse.

(c) Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = 12$, so an equation of the directrix is $y = -12$.

(d) The vertices are $\left(4, \frac{\pi}{2}\right)$ and $\left(\frac{12}{5}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{4}{5}, \frac{\pi}{2}\right)$.



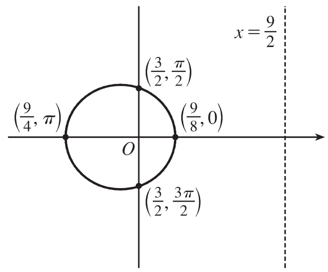
13. $r = \dfrac{9}{6 + 2\cos\theta} \cdot \dfrac{1/6}{1/6} = \dfrac{3/2}{1 + \frac{1}{3}\cos\theta}$, where $e = \frac{1}{3}$ and $ed = \frac{3}{2} \quad\Rightarrow\quad d = \frac{9}{2}$.

(a) Eccentricity $= e = \frac{1}{3}$

(b) Since $e = \frac{1}{3} < 1$, the conic is an ellipse.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{9}{2}$, so an equation of the directrix is $x = \frac{9}{2}$.

(d) The vertices are $\left(\frac{9}{8}, 0\right)$ and $\left(\frac{9}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{9}{16}, \pi\right)$.



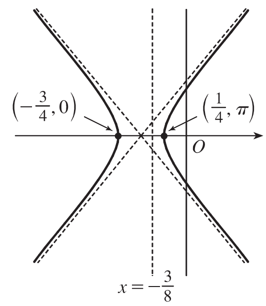
15. $r = \dfrac{3}{4 - 8\cos\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{3/4}{1 - 2\cos\theta}$, where $e = 2$ and $ed = \frac{3}{4} \quad\Rightarrow\quad d = \frac{3}{8}$.

(a) Eccentricity $= e = 2$

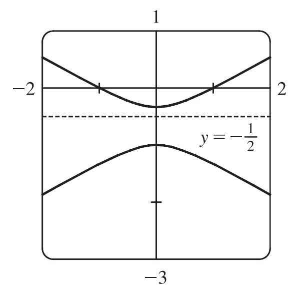
(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{3}{8}$, so an equation of the directrix is $x = -\frac{3}{8}$.

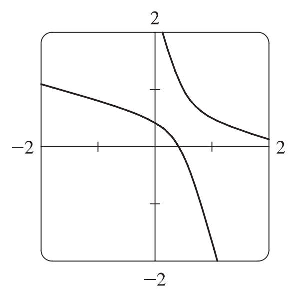
(d) The vertices are $\left(-\frac{3}{4}, 0\right)$ and $\left(\frac{1}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2}, \pi\right)$.



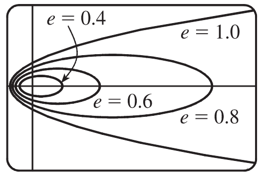
17. (a) $r = \dfrac{1}{1 - 2\sin\theta}$, where $e = 2$ and $ed = 1 \quad\Rightarrow\quad d = \frac{1}{2}$. The eccentricity $e = 2 > 1$, so the conic is a hyperbola. Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = \frac{1}{2}$, so an equation of the directrix is $y = -\frac{1}{2}$. The vertices are $\left(-1, \frac{\pi}{2}\right)$ and $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{2}{3}, \frac{3\pi}{2}\right)$.



(b) By the discussion that precedes Example 4, the equation is $r = \dfrac{1}{1 - 2\sin\left(\theta - \frac{3\pi}{4}\right)}$.

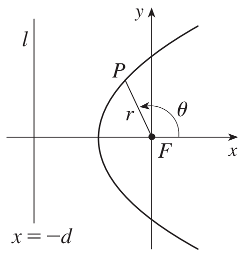


19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



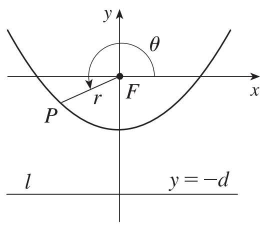
21. $|PF| = e\,|Pl| \quad\Rightarrow\quad r = e[d - r\cos(\pi - \theta)] = e(d + r\cos\theta) \quad\Rightarrow$

$r(1 - e\cos\theta) = ed \quad\Rightarrow\quad r = \dfrac{ed}{1 - e\cos\theta}$



23. $|PF| = e\,|Pl| \quad\Rightarrow\quad r = e[d - r\sin(\theta - \pi)] = e(d + r\sin\theta) \quad\Rightarrow$

$r(1 - e\sin\theta) = ed \quad\Rightarrow\quad r = \dfrac{ed}{1 - e\sin\theta}$



25. We are given $e = 0.093$ and $a = 2.28 \times 10^8$. By (7), we have

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{2.28 \times 10^8[1 - (0.093)^2]}{1 + 0.093\cos\theta} \approx \frac{2.26 \times 10^8}{1 + 0.093\cos\theta}$$

27. Here $2a$ = length of major axis = 36.18 AU $\quad\Rightarrow\quad a = 18.09$ AU and $e = 0.97$. By (7), the equation of the orbit is

$r = \dfrac{18.09\left[1 - (0.97)^2\right]}{1 - 0.97\cos\theta} \approx \dfrac{1.07}{1 - 0.97\cos\theta}$. By (8), the maximum distance from the comet to the sun is

$18.09(1 + 0.97) \approx 35.64$ AU or about 3.314 billion miles.

29. The minimum distance is at perihelion, where $4.6 \times 10^7 = r = a(1 - e) = a(1 - 0.206) = a(0.794) \quad\Rightarrow$

$a = 4.6 \times 10^7/0.794$. So the maximum distance, which is at aphelion, is

$r = a(1 + e) = \left(4.6 \times 10^7/0.794\right)(1.206) \approx 7.0 \times 10^7$ km.

31. From Exercise 29, we have $e = 0.206$ and $a(1 - e) = 4.6 \times 10^7$ km. Thus, $a = 4.6 \times 10^7/0.794$. From (7), we can write the equation of Mercury's orbit as $r = a\dfrac{1 - e^2}{1 - e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{-a(1 - e^2)e\sin\theta}{(1 - e\cos\theta)^2} \quad\Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1 - e^2)^2}{(1 - e\cos\theta)^2} + \frac{a^2(1 - e^2)^2\,e^2\sin^2\theta}{(1 - e\cos\theta)^4} = \frac{a^2(1 - e^2)^2}{(1 - e\cos\theta)^4}\,(1 - 2e\cos\theta + e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = a(1 - e^2) \int_0^{2\pi} \frac{\sqrt{1 + e^2 - 2e\cos\theta}}{(1 - e\cos\theta)^2}\, d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6 \times 10^8$ km.

10 Review

CONCEPT CHECK

1. (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t.

(b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) You can find $\dfrac{dy}{dx}$ as a function of t by calculating $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ [if $dx/dt \neq 0$].

(b) Calculate the area as $\int_a^b y\, dx = \int_\alpha^\beta g(t)\, f'(t) dt$ [or $\int_\beta^\alpha g(t)\, f'(t) dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. (a) $L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

(b) $S = \int_\alpha^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_\alpha^\beta 2\pi g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

4. (a) See Figure 5 in Section 10.3.

(b) $x = r\cos\theta,\ y = r\sin\theta$

(c) To find a polar representation (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\cos\theta = x/r$ and $\sin\theta = y/r$.

5. (a) Calculate $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \dfrac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \dfrac{\left(\frac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\frac{dr}{d\theta}\right)\cos\theta - r\sin\theta}$, where $r = f(\theta)$.

(b) Calculate $A = \int_a^b \frac{1}{2} r^2\, d\theta = \int_a^b \frac{1}{2} [f(\theta)]^2\, d\theta$

(c) $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$

6. (a) A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line l (the directrix) are equal.

(b) $x^2 = 4py$; $y^2 = 4px$

7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$

(c) $y = \pm\dfrac{\sqrt{c^2 - a^2}}{a}x$

9. (a) If a conic section has focus F and corresponding directrix l, then the eccentricity e is the fixed ratio $|PF| \,/\, |Pl|$ for points P of the conic section.

(b) $e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola.

(c) $x = d$: $r = \dfrac{ed}{1 + e\cos\theta}$. $x = -d$: $r = \dfrac{ed}{1 - e\cos\theta}$. $y = d$: $r = \dfrac{ed}{1 + e\sin\theta}$. $y = -d$: $r = \dfrac{ed}{1 - e\sin\theta}$.

TRUE-FALSE QUIZ

1. False. Consider the curve defined by $x = f(t) = (t-1)^3$ and $y = g(t) = (t-1)^2$. Then $g'(t) = 2(t-1)$, so $g'(1) = 0$, but its graph has a *vertical* tangent when $t = 1$. *Note:* The statement is true if $f'(1) \neq 0$ when $g'(1) = 0$.

3. False. For example, if $f(t) = \cos t$ and $g(t) = \sin t$ for $0 \le t \le 4\pi$, then the curve is a circle of radius 1, hence its length is 2π, but $\int_0^{4\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{4\pi} 1\,dt = 4\pi$, since as t increases from 0 to 4π, the circle is traversed twice.

5. True. The curve $r = 1 - \sin 2\theta$ is unchanged if we rotate it through 180° about O because $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$. So it's unchanged if we replace r by $-r$. (See the discussion after Example 8 in Section 10.3.) In other words, it's the same curve as $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$.

7. False. The first pair of equations gives the portion of the parabola $y = x^2$ with $x \ge 0$, whereas the second pair of equations traces out the whole parabola $y = x^2$.

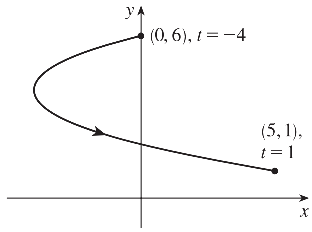
9. True. By rotating and translating the parabola, we can assume it has an equation of the form $y = cx^2$, where $c > 0$. The tangent at the point $\left(a, ca^2\right)$ is the line $y - ca^2 = 2ca(x - a)$; i.e., $y = 2cax - ca^2$. This tangent meets the parabola at the points $\left(x, cx^2\right)$ where $cx^2 = 2cax - ca^2$. This equation is equivalent to $x^2 = 2ax - a^2$ [since $c > 0$]. But $x^2 = 2ax - a^2 \;\Leftrightarrow\; x^2 - 2ax + a^2 = 0 \;\Leftrightarrow\; (x-a)^2 = 0 \;\Leftrightarrow\; x = a \;\Leftrightarrow\; \left(x, cx^2\right) = \left(a, ca^2\right)$. This shows that each tangent meets the parabola at exactly one point.

EXERCISES

1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \le t \le 1$. $t = 2 - y$, so

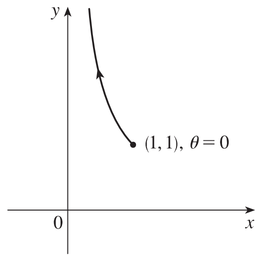
$x = (2 - y)^2 + 4(2 - y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \quad \Leftrightarrow$

$x + 4 = y^2 - 8y + 16 = (y - 4)^2$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



3. $y = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{x}$. Since $0 \le \theta \le \pi/2$, $0 < x \le 1$ and $y \ge 1$.

This is part of the hyperbola $y = 1/x$.



5. Three different sets of parametric equations for the curve $y = \sqrt{x}$ are

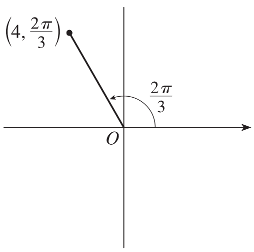
(i) $x = t, \quad y = \sqrt{t}$

(ii) $x = t^4, \quad y = t^2$

(iii) $x = \tan^2 t, \quad y = \tan t, \quad 0 \le t < \pi/2$

There are many other sets of equations that also give this curve.

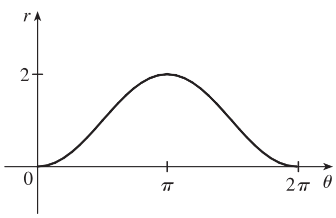
7. (a)

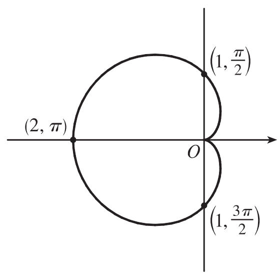


The Cartesian coordinates are $x = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ and $y = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$, that is, the point $\left(-2, 2\sqrt{3}\right)$.

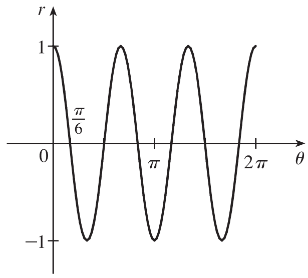
(b) Given $x = -3$ and $y = 3$, we have $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Also, $\tan\theta = \dfrac{y}{x} \quad\Rightarrow\quad \tan\theta = \dfrac{3}{-3}$, and since $(-3, 3)$ is in the second quadrant, $\theta = \frac{3\pi}{4}$. Thus, one set of polar coordinates for $(-3, 3)$ is $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$, and two others are $\left(3\sqrt{2}, \frac{11\pi}{4}\right)$ and $\left(-3\sqrt{2}, \frac{7\pi}{4}\right)$.

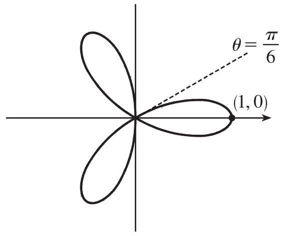
9. $r = 1 - \cos\theta$. This cardioid is symmetric about the polar axis.



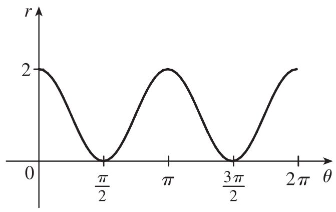


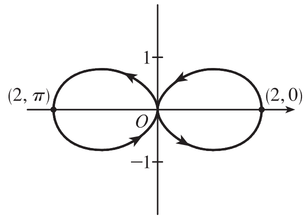
11. $r = \cos 3\theta$. This is a three-leaved rose. The curve is traced twice.



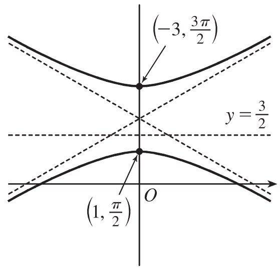


13. $r = 1 + \cos 2\theta$. The curve is symmetric about the pole and both the horizontal and vertical axes.



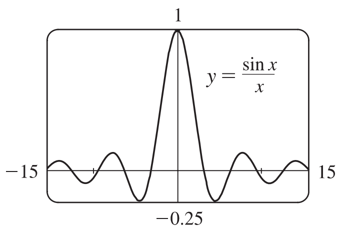


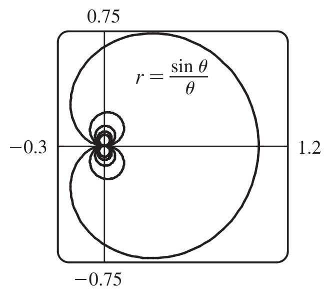
15. $r = \dfrac{3}{1 + 2\sin\theta} \quad \Rightarrow \quad e = 2 > 1$, so the conic is a hyperbola. $de = 3 \quad \Rightarrow$ $d = \frac{3}{2}$ and the form "$+2\sin\theta$" imply that the directrix is above the focus at the origin and has equation $y = \frac{3}{2}$. The vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(-3, \frac{3\pi}{2}\right)$.



17. $x + y = 2 \quad \Leftrightarrow \quad r\cos\theta + r\sin\theta = 2 \quad \Leftrightarrow \quad r(\cos\theta + \sin\theta) = 2 \quad \Leftrightarrow \quad r = \dfrac{2}{\cos\theta + \sin\theta}$

19. $r = (\sin\theta)/\theta$. As $\theta \to \pm\infty$, $r \to 0$. As $\theta \to 0$, $r \to 1$. In the first figure, there are an infinite number of x-intercepts at $x = \pi n$, n a nonzero integer. These correspond to pole points in the second figure.





21. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\dfrac{dy}{dt} = 2t$ and $\dfrac{dx}{dt} = \dfrac{1}{t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$.

When $t = 1$, $(x, y) = (0, 2)$ and $dy/dx = 2$.

23. $r = e^{-\theta} \quad \Rightarrow \quad y = r\sin\theta = e^{-\theta}\sin\theta$ and $x = r\cos\theta = e^{-\theta}\cos\theta \quad \Rightarrow$

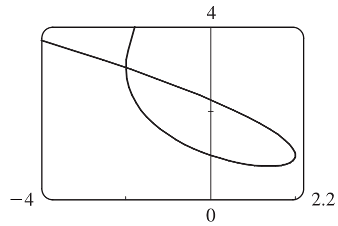
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta} \cdot \frac{-e^{\theta}}{-e^{\theta}} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}.$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{0 - (-1)}{-1 + 0} = \dfrac{1}{-1} = -1$.

25. $x = t + \sin t,\ y = t - \cos t \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \sin t}{1 + \cos t} \quad \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \frac{\dfrac{(1+\cos t)\,\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}}{1+\cos t} = \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^3} = \frac{1 + \cos t + \sin t}{(1+\cos t)^3}$$

27. We graph the curve $x = t^3 - 3t$, $y = t^2 + t + 1$ for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \quad \Leftrightarrow \quad t = -\frac{1}{2} \quad \Leftrightarrow \quad (x, y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



29. $x = 2a\cos t - a\cos 2t \quad \Rightarrow \quad \dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t(2\cos t - 1) = 0 \quad \Leftrightarrow$

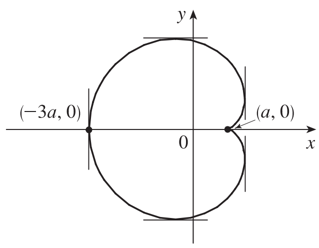
$\sin t = 0$ or $\cos t = \frac{1}{2} \quad \Rightarrow \quad t = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

$y = 2a\sin t - a\sin 2t \quad \Rightarrow \quad \dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\big(1 + \cos t - 2\cos^2 t\big) = 2a(1 - \cos t)(1 + 2\cos t) = 0 \quad \Rightarrow$

$t = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}$, π and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\lim\limits_{t \to 0} \dfrac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

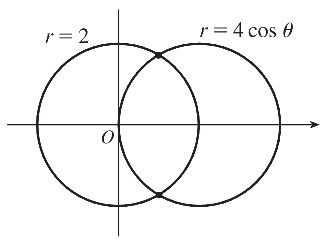
t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



31. The curve $r^2 = 9\cos 5\theta$ has 10 "petals." For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

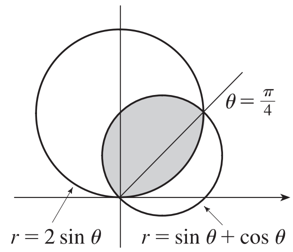
$$A = 10\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta\,d\theta = 5 \cdot 9 \cdot 2\int_0^{\pi/10} \cos 5\theta\,d\theta = 18\big[\sin 5\theta\big]_0^{\pi/10} = 18$$

33. The curves intersect when $4\cos\theta = 2 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{3}$ for $-\pi \le \theta \le \pi$. The points of intersection are $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



35. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \quad\Rightarrow$ $\sin\theta = \cos\theta \quad\Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).

$$\begin{aligned} A &= \int_0^{\pi/4} \tfrac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{3\pi/4} \tfrac{1}{2}(\sin\theta+\cos\theta)^2\,d\theta \\ &= \int_0^{\pi/4} (1-\cos 2\theta)\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4} (1+\sin 2\theta)\,d\theta \\ &= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}(\pi - 1) \end{aligned}$$



37. $x = 3t^2,\ y = 2t^3$.

$$\begin{aligned} L &= \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\,dt = \int_0^2 \sqrt{36t^2 + 36t^4}\,dt = \int_0^2 \sqrt{36t^2}\,\sqrt{1+t^2}\,dt \\ &= \int_0^2 6\,|t|\,\sqrt{1+t^2}\,dt = 6\int_0^2 t\,\sqrt{1+t^2}\,dt = 6\int_1^5 u^{1/2}\left(\tfrac{1}{2}du\right) \qquad \left[u = 1+t^2,\ du = 2t\,dt\right] \\ &= 6\cdot\tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_1^5 = 2(5^{3/2}-1) = 2\left(5\sqrt{5}-1\right) \end{aligned}$$

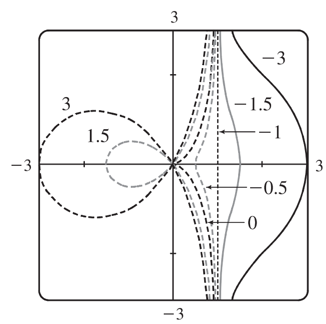
39. $L = \int_\pi^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_\pi^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2}\,d\theta = \displaystyle\int_\pi^{2\pi} \frac{\sqrt{\theta^2+1}}{\theta^2}\,d\theta$

$$\begin{aligned} &\overset{24}{=} \left[-\frac{\sqrt{\theta^2+1}}{\theta} + \ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_\pi^{2\pi} = \frac{\sqrt{\pi^2+1}}{\pi} - \frac{\sqrt{4\pi^2+1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right) \\ &= \frac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right) \end{aligned}$$

41. $x = 4\sqrt{t},\ y = \dfrac{t^3}{3} + \dfrac{1}{2t^2},\ 1 \le t \le 4 \quad\Rightarrow$

$$\begin{aligned} S &= \int_1^4 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4 2\pi\left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + (t^2 - t^{-3})^2}\,dt \\ &= 2\pi\int_1^4 \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^{-2}\right)\sqrt{(t^2+t^{-3})^2}\,dt = 2\pi\int_1^4 \left(\tfrac{1}{3}t^5 + \tfrac{5}{6} + \tfrac{1}{2}t^{-5}\right)dt = 2\pi\left[\tfrac{1}{18}t^6 + \tfrac{5}{6}t - \tfrac{1}{8}t^{-4}\right]_1^4 = \tfrac{471{,}295}{1024}\pi \end{aligned}$$

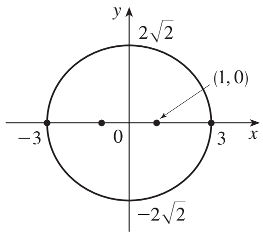
43. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x-intercept of the curve is always $-c$.



45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$ is an ellipse with center $(0, 0)$.

$a = 3$, $b = 2\sqrt{2}$, $c = 1$ $\Rightarrow$

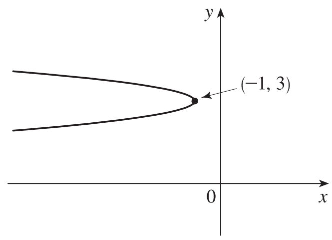
foci $(\pm 1, 0)$, vertices $(\pm 3, 0)$.



47. $6y^2 + x - 36y + 55 = 0 \quad \Leftrightarrow$

$6(y^2 - 6y + 9) = -(x + 1) \quad \Leftrightarrow$

$(y - 3)^2 = -\frac{1}{6}(x + 1)$, a parabola with vertex $(-1, 3)$, opening to the left, $p = -\frac{1}{24} \quad \Rightarrow$ focus $\left(-\frac{25}{24}, 3\right)$ and directrix $x = -\frac{23}{24}$.



49. The ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 4$, so $b^2 = a^2 - c^2 = 5^2 - 4^2 = 9$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

51. The center of a hyperbola with foci $(0, \pm 4)$ is $(0, 0)$, so $c = 4$ and an equation is $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$. The asymptote $y = 3x$ has slope 3, so $\dfrac{a}{b} = \dfrac{3}{1} \quad \Rightarrow \quad a = 3b$ and $a^2 + b^2 = c^2 \quad \Rightarrow \quad (3b)^2 + b^2 = 4^2 \quad \Rightarrow$ $10b^2 = 16 \quad \Rightarrow \quad b^2 = \frac{8}{5}$ and so $a^2 = 16 - \frac{8}{5} = \frac{72}{5}$. Thus, an equation is $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$, or $\dfrac{5y^2}{72} - \dfrac{5x^2}{8} = 1$.

53. $x^2 = -(y - 100)$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y - 100) = -(y - 100) \quad \Rightarrow \quad 4p = -1 \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \dfrac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$, or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$.

55. Directrix $x = 4 \quad \Rightarrow \quad d = 4$, so $e = \frac{1}{3} \quad \Rightarrow \quad r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{4}{3 + \cos\theta}$.

57. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\,\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\,\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

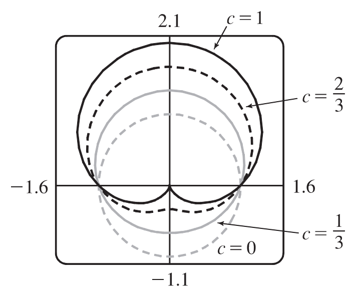
PROBLEMS PLUS

1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \quad\Leftrightarrow\quad \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = \big[\ln t\big]_1^{\pi/2} = \ln \tfrac{\pi}{2}$$

3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \le \sin\theta \le 1 \quad\Rightarrow\quad -1 \le \sin\theta + c\sin^2\theta \le 1 + c \le 2$, so $-1 \le y \le 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \le y \le 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$ and $\sin 2\theta \le 0$ for $-\frac{\pi}{2} \le \theta \le 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \quad\Rightarrow$ $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \quad\Rightarrow\quad \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ [but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$]. If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\frac{3t_1}{1 + t_1^3} = a$ and $\frac{3t_1^2}{1 + t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \ne 0$, then the point corresponding to $t = \frac{1}{t_1}$ is given by

$x = \frac{3(1/t_1)}{1 + (1/t_1)^3} = \frac{3t_1^2}{t_1^3 + 1} = b$, $y = \frac{3(1/t_1)^2}{1 + (1/t_1)^3} = \frac{3t_1}{t_1^3 + 1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\frac{3t}{1 + t^3} = \frac{3t^2}{1 + t^3} \quad\Rightarrow$ $t = t^2 \quad\Rightarrow\quad t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

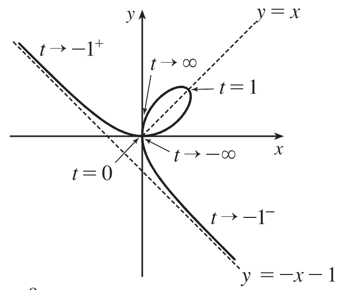
(b) $\dfrac{dy}{dt} = \dfrac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t-3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t(2-t^3) = 0 \;\Rightarrow\; t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0,0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x-1) = y + x + 1 = \dfrac{3t + 3t^2 + (1+t^3)}{1+t^3} = \dfrac{(t+1)^3}{1+t^3} = \dfrac{(t+1)^2}{t^2-t+1} \to 0$ as $t \to -1$. So $y = -x-1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \dfrac{3-6t^3}{(1+t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t-3t^4}{(1+t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t(2-t^3)}{1-2t^3}$.

Also $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \;\Leftrightarrow\; t < \dfrac{1}{\sqrt[3]{2}}$.

So the curve is concave upward there and has a minimum point at $(0,0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3 + y^3 = \left(\dfrac{3t}{1+t^3}\right)^3 + \left(\dfrac{3t^2}{1+t^3}\right)^3 = \dfrac{27t^3 + 27t^6}{(1+t^3)^3} = \dfrac{27t^3(1+t^3)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$ and

$3xy = 3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right) = \dfrac{27t^3}{(1+t^3)^2}$, so $x^3 + y^3 = 3xy$.

(f) We start with the equation from part (e) and substitute $x = r\cos\theta$, $y = r\sin\theta$. Then $x^3 + y^3 = 3xy \;\Rightarrow$ $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\,\sin\theta$. For $r \neq 0$, this gives $r = \dfrac{3\cos\theta\,\sin\theta}{\cos^3\theta + \sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r = \dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^3\theta}{\cos^3\theta}} = \dfrac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta \in \left(0, \frac{\pi}{2}\right)$, so its area is

$$A = \int_0^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2 d\theta = \frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\,\tan^2\theta}{(1+\tan^3\theta)^2}\,d\theta = \frac{9}{2}\int_0^{\infty}\frac{u^2\,du}{(1+u^3)^2} \quad [\text{let } u = \tan\theta]$$

$$= \lim_{b\to\infty} \tfrac{9}{2}\left[-\tfrac{1}{3}(1+u^3)^{-1}\right]_0^b = \tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x-1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y = -x-1 \;\Rightarrow$ $r\sin\theta = -r\cos\theta - 1 \;\Rightarrow\; r = -\dfrac{1}{\sin\theta + \cos\theta}$, the area in the fourth quadrant is

$$\frac{1}{2}\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2 - \left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2\right]d\theta \overset{\text{CAS}}{=} \frac{1}{2}.$$ Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.

11 □ INFINITE SEQUENCES AND SERIES

11.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \ldots\}$.

5. $a_n = \dfrac{3\,(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \ldots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \ldots\right\}$.

7. $a_1 = 3$, $a_{n+1} = 2a_n - 1$. Each term is defined in terms of the preceding term.

$a_2 = 2a_1 - 1 = 2(3) - 1 = 5$. $a_3 = 2a_2 - 1 = 2(5) - 1 = 9$. $a_4 = 2a_3 - 1 = 2(9) - 1 = 17$.

$a_5 = 2a_4 - 1 = 2(17) - 1 = 33$. The sequence is $\{3, 5, 9, 17, 33, \ldots\}$.

9. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$. The denominator of the nth term is the nth positive odd integer, so $a_n = \dfrac{1}{2n-1}$.

11. $\{2, 7, 12, 17, \ldots\}$. Each term is larger than the preceding one by 5, so $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

13. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

15. The first six terms of $a_n = \dfrac{n}{2n+1}$ are $\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}, \dfrac{5}{11}, \dfrac{6}{13}$. It appears that the sequence is approaching $\dfrac{1}{2}$.

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2}$$

17. $a_n = 1 - (0.2)^n$, so $\lim\limits_{n\to\infty} a_n = 1 - 0 = 1$ by (9). Converges

19. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

21. Because the natural exponential function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty}(1/n)} = e^0 = 1. \quad \text{Converges}$$

23. If $b_n = \dfrac{2n\pi}{1+8n}$, then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{(2n\pi)/n}{(1+8n)/n} = \lim\limits_{n\to\infty} \dfrac{2\pi}{1/n+8} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$. Since tan is continuous at $\frac{\pi}{4}$, by

Theorem 7, $\lim\limits_{n\to\infty} \tan\left(\dfrac{2n\pi}{1+8n}\right) = \tan\left(\lim\limits_{n\to\infty} \dfrac{2n\pi}{1+8n}\right) = \tan\dfrac{\pi}{4} = 1$. Converges

25. $a_n = \dfrac{(-1)^{n-1}\,n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 6. Converges

27. $a_n = \cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n \to \infty$. The terms take on values between -1 and 1.

29. $a_n = \dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \dfrac{1}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Converges

31. $a_n = \dfrac{e^n + e^{-n}}{e^{2n}-1} \cdot \dfrac{e^{-n}}{e^{-n}} = \dfrac{1+e^{-2n}}{e^n - e^{-n}} \to 0$ as $n \to \infty$ because $1 + e^{-2n} \to 1$ and $e^n - e^{-n} \to \infty$. Converges

33. $a_n = n^2 e^{-n} = \dfrac{n^2}{e^n}$. Since $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$, it follows from Theorem 3 that $\displaystyle\lim_{n\to\infty} a_n = 0$. Converges

35. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\displaystyle\lim_{n\to\infty} \frac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

37. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\displaystyle\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{t\to 0^+} \frac{\sin t}{t}$ [where $t = 1/x$] $= 1$, it follows from Theorem 3 that $\{a_n\}$ converges to 1.

39. $y = \left(1 + \dfrac{2}{x}\right)^x \quad\Rightarrow\quad \ln y = x\ln\left(1+\dfrac{2}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(\dfrac{1}{1+2/x}\right)\left(-\dfrac{2}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2 \quad\Rightarrow$$

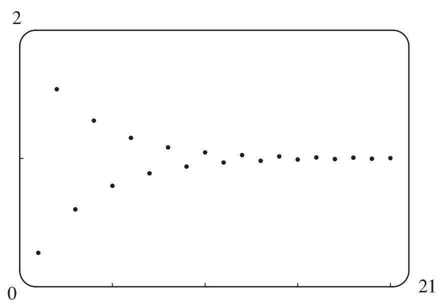
$\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2$, so by Theorem 3, $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$. Convergent

41. $a_n = \ln(2n^2+1) - \ln(n^2+1) = \ln\left(\dfrac{2n^2+1}{n^2+1}\right) = \ln\left(\dfrac{2+1/n^2}{1+1/n^2}\right) \to \ln 2$ as $n \to \infty$. Convergent

43. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

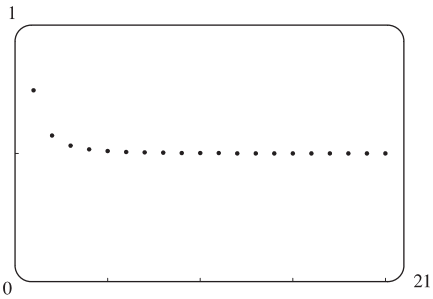
45. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \ge \dfrac{1}{2}\cdot\dfrac{n}{2}$ [for $n > 1$] $= \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

47.



From the graph, it appears that the sequence converges to 1. $\{(-2/e)^n\}$ converges to 0 by (9), and hence $\{1 + (-2/e)^n\}$ converges to $1 + 0 = 1$.

49.



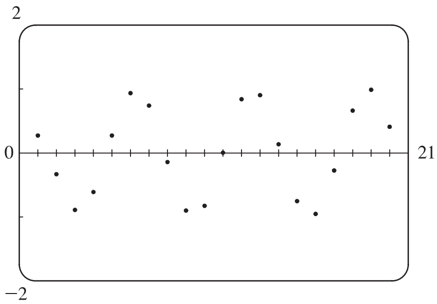
From the graph, it appears that the sequence converges to $\frac{1}{2}$.

As $n \to \infty$,

$$a_n = \sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\frac{3/n^2+2}{8+1/n}} \quad \Rightarrow \quad \sqrt{\frac{0+2}{8+0}} = \sqrt{\frac{1}{4}} = \frac{1}{2},$$

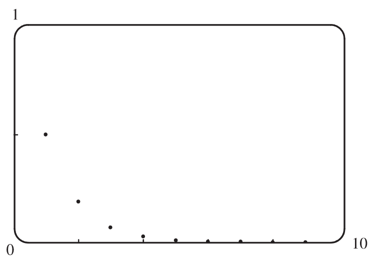
so $\lim\limits_{n\to\infty} a_n = \frac{1}{2}$.

51.



From the graph, it appears that the sequence $\{a_n\} = \left\{\dfrac{n^2 \cos n}{1+n^2}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately). To prove this, suppose that $\{a_n\}$ converges to L. If $b_n = \dfrac{n^2}{1+n^2}$, then $\{b_n\}$ converges to 1, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{1} = L$. But $\dfrac{a_n}{b_n} = \cos n$, so $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist. This contradiction shows that $\{a_n\}$ diverges.

53.



From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0.

55. (a) $a_n = 1000(1.06)^n \quad \Rightarrow \quad a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (9) with $r = 1.06 > 1$.

57. If $|r| \geq 1$, then $\{r^n\}$ diverges by (9), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim\limits_{x\to\infty} xr^x = \lim\limits_{x\to\infty} \dfrac{x}{r^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{(-\ln r)\,r^{-x}} = \lim\limits_{x\to\infty} \dfrac{r^x}{-\ln r} = 0$, so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

59. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

61. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

63. The terms of $a_n = n(-1)^n$ alternate in sign, so the sequence is not monotonic. The first five terms are -1, 2, -3, 4, and -5. Since $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} n = \infty$, the sequence is not bounded.

65. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$, $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$. The sequence is bounded since $0 < a_n \le \frac{1}{2}$ for all $n \ge 1$.

67. For $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$, $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, …, so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

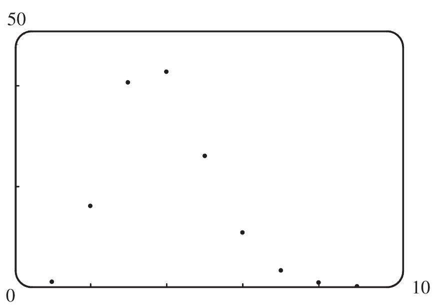
Alternate solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \;\Rightarrow\; L^2 = 2L \;\Rightarrow\; L(L-2) = 0$. $L \ne 0$ since the sequence increases, so $L = 2$.

69. $a_1 = 1$, $a_{n+1} = 3 - \dfrac{1}{a_n}$. We show by induction that $\{a_n\}$ is increasing and bounded above by 3. Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1} > a_n \;\Rightarrow\; \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \;\Rightarrow\; -\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$. Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \;\Leftrightarrow\; P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \;\Rightarrow\; L^2 - 3L + 1 = 0 \;\Rightarrow\; L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3+\sqrt{5}}{2}$.

71. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

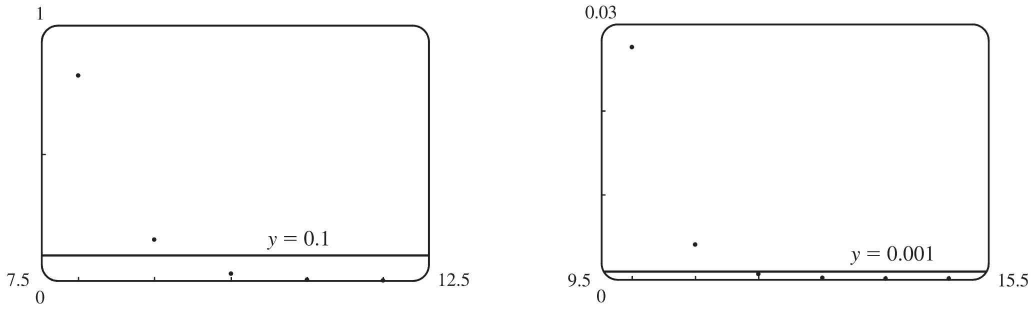
(b) $a_n = \dfrac{f_{n+1}}{f_n} \;\Rightarrow\; a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1}+f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \;\Rightarrow\; L^2 - L - 1 = 0 \;\Rightarrow\; L = \frac{1+\sqrt{5}}{2}$ [since L must be positive].

73. (a)



From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$ converges to 0, that is, $\lim\limits_{n\to\infty} \dfrac{n^5}{n!} = 0$.

(b)



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11 since $n^5/n! < 0.001$ whenever $n \geq 12$.

75. Theorem 6: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

77. To Prove: If $\lim\limits_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim\limits_{n\to\infty} (a_n b_n) = 0$.

Proof: Since $\{b_n\}$ is bounded, there is a positive number M such that $|b_n| \leq M$ and hence, $|a_n|\,|b_n| \leq |a_n|\,M$ for all $n \geq 1$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = 0$, there is an integer N such that $|a_n - 0| < \dfrac{\varepsilon}{M}$ if $n > N$. Then $|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \leq |a_n|\,M = |a_n - 0|\,M < \dfrac{\varepsilon}{M} \cdot M = \varepsilon$ for all $n > N$. Since ε was arbitrary, $\lim\limits_{n\to\infty} (a_n b_n) = 0$.

79. (a) First we show that $a > a_1 > b_1 > b$.

$$a_1 - b_1 = \tfrac{a+b}{2} - \sqrt{ab} = \tfrac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \tfrac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0 \quad [\text{since } a > b] \quad \Rightarrow \quad a_1 > b_1. \text{ Also}$$

$a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$. In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right) = \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0,$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \tfrac{1}{2}(a_{k+1} + b_{k+1}) = \tfrac{1}{2}(a_{k+1} - b_{k+1}) > 0, \text{ and}$$

$$b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0 \quad \Rightarrow \quad a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1},$$

so the assertion is true for $n = k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2} \quad \Rightarrow \quad \alpha = \dfrac{\alpha + \beta}{2} \quad \Rightarrow$ $2\alpha = \alpha + \beta \quad \Rightarrow \quad \alpha = \beta$.

81. (a) Suppose $\{p_n\}$ converges to p. Then $p_{n+1} = \dfrac{bp_n}{a+p_n} \quad \Rightarrow \quad \lim\limits_{n\to\infty} p_{n+1} = \dfrac{b \lim\limits_{n\to\infty} p_n}{a + \lim\limits_{n\to\infty} p_n} \quad \Rightarrow \quad p = \dfrac{bp}{a+p} \quad \Rightarrow$

$p^2 + ap = bp \quad \Rightarrow \quad p(p+a-b) = 0 \quad \Rightarrow \quad p = 0$ or $p = b - a$.

(b) $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{\left(\dfrac{b}{a}\right)p_n}{1+\dfrac{p_n}{a}} < \left(\dfrac{b}{a}\right)p_n$ since $1 + \dfrac{p_n}{a} > 1$.

(c) By part (b), $p_1 < \left(\dfrac{b}{a}\right)p_0$, $p_2 < \left(\dfrac{b}{a}\right)p_1 < \left(\dfrac{b}{a}\right)^2 p_0$, $p_3 < \left(\dfrac{b}{a}\right)p_2 < \left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n < \left(\dfrac{b}{a}\right)^n p_0$,

so $\lim\limits_{n\to\infty} p_n \le \lim\limits_{n\to\infty} \left(\dfrac{b}{a}\right)^n \cdot p_0 = 0$ since $b < a$. $\left[\text{By result 9, } \lim\limits_{n\to\infty} r^n = 0 \text{ if } -1 < r < 1. \text{ Here } r = \dfrac{b}{a} \in (0,1).\right]$

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

For $n = 0$, we have $p_1 - p_0 = \dfrac{bp_0}{a+p_0} - p_0 = \dfrac{p_0(b-a-p_0)}{a+p_0} > 0$ since $p_0 < b - a$. So $p_1 > p_0$.

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$b - a - p_{k+1} = b - a - \dfrac{bp_k}{a+p_k} = \dfrac{a(b-a) + bp_k - ap_k - bp_k}{a+p_k} = \dfrac{a(b-a-p_k)}{a+p_k} > 0$ because $p_k < b - a$. So

$p_{k+1} < b - a$. And $p_{k+2} - p_{k+1} = \dfrac{bp_{k+1}}{a+p_{k+1}} - p_{k+1} = \dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}} > 0$ since $p_{k+1} < b - a$. Therefore,

$p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k + 1$. It is therefore true for all n by mathematical induction.

A similar proof by induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim\limits_{n\to\infty} p_n = b - a$.

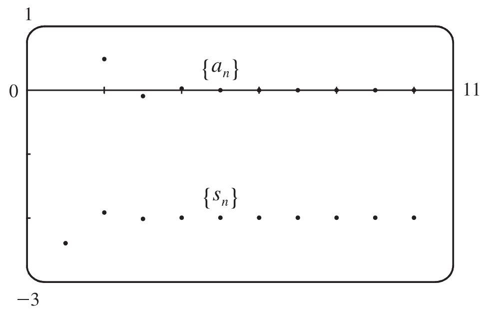
11.2 Series

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

3.

n	s_n
1	-2.40000
2	-1.92000
3	-2.01600
4	-1.99680
5	-2.00064
6	-1.99987
7	-2.00003
8	-1.99999
9	-2.00000
10	-2.00000



From the graph and the table, it seems that the series converges to -2. In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is $\sum\limits_{n=1}^{\infty} \dfrac{12}{(-5)^n} = \dfrac{-2.4}{1-\left(-\frac{1}{5}\right)} = \dfrac{-2.4}{1.2} = -2$.

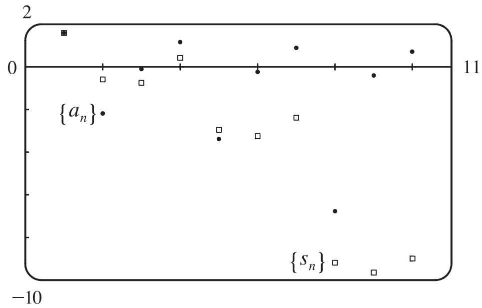
Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under

GRAPH, MORE, FORMT (F3).) Now under `E(t)=` make the assignments: `xt1=t, yt1=12/(-5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1).` (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use `1,10,1,0,10,1,-3,1,1` to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

5.

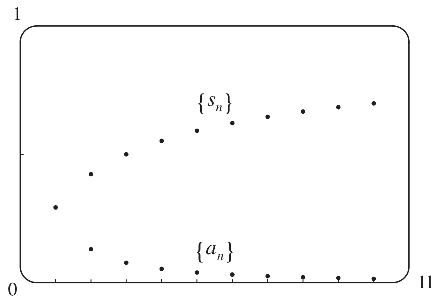
n	s_n
1	1.55741
2	−0.62763
3	−0.77018
4	0.38764
5	−2.99287
6	−3.28388
7	−2.41243
8	−9.21214
9	−9.66446
10	−9.01610



The series $\sum\limits_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

7.

n	s_n
1	0.29289
2	0.42265
3	0.50000
4	0.55279
5	0.59175
6	0.62204
7	0.64645
8	0.66667
9	0.68377
10	0.69849



From the graph and the table, it seems that the series converges.

$$\sum_{n=1}^{k}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\cdots+\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$$
$$=1-\frac{1}{\sqrt{k+1}},$$

so $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=\lim_{k\to\infty}\left(1-\frac{1}{\sqrt{k+1}}\right)=1.$

9. (a) $\displaystyle\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{2n}{3n+1}=\frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (11.1.1).

(b) Since $\lim\limits_{n\to\infty} a_n=\frac{2}{3}\neq 0$, the *series* $\sum\limits_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

11. $3+2+\frac{4}{3}+\frac{8}{9}+\cdots$ is a geometric series with first term $a=3$ and common ratio $r=\frac{2}{3}$. Since $|r|=\frac{2}{3}<1$, the series converges to $\frac{a}{1-r}=\frac{3}{1-2/3}=\frac{3}{1/3}=9$.

13. $3-4+\frac{16}{3}-\frac{64}{9}+\cdots$ is a geometric series with ratio $r=-\frac{4}{3}$. Since $|r|=\frac{4}{3}>1$, the series diverges.

15. $\sum_{n=1}^{\infty} 6(0.9)^{n-1}$ is a geometric series with first term $a = 6$ and ratio $r = 0.9$. Since $|r| = 0.9 < 1$, the series converges to

$$\frac{a}{1-r} = \frac{6}{1-0.9} = \frac{6}{0.1} = 60.$$

17. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

19. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \frac{\pi}{3}$. Since $|r| > 1$, the series diverges.

21. $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is $\frac{1}{2}$ times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. $\left[\text{If } \sum_{n=1}^{\infty} \frac{1}{2n} \text{ were to converge, then } \sum_{n=1}^{\infty} \frac{1}{n} \text{ would also have to converge by Theorem 8(i).}\right]$ In general, constant multiples of divergent series are divergent.

23. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

25. Converges.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} &= \sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{2^n}{3^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right] \qquad \text{[sum of two convergent geometric series]} \\ &= \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = \frac{1}{2} + 2 = \frac{5}{2}\end{aligned}$$

27. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

29. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverges by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n^2+1}{2n^2+1}\right) = \ln \tfrac{1}{2} \neq 0.$$

31. $\sum_{n=1}^{\infty} \arctan n$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$.

33. $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with first term $a = \frac{1}{e}$ and ratio $r = \frac{1}{e}$. Since $|r| = \frac{1}{e} < 1$, the series converges to $\frac{1/e}{1-1/e} = \frac{1/e}{1-1/e} \cdot \frac{e}{e} = \frac{1}{e-1}$. By Example 6, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Thus, by Theorem 8(ii),

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \frac{1}{e-1} + \frac{e-1}{e-1} = \frac{e}{e-1}.$$

35. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right)$$
$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

37. For the series $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$, $s_n = \sum_{i=1}^{n} \frac{3}{i(i+3)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+3}\right)$ [using partial fractions]. The latter sum is

$$\left(1 - \tfrac{1}{4}\right) + \left(\tfrac{1}{2} - \tfrac{1}{5}\right) + \left(\tfrac{1}{3} - \tfrac{1}{6}\right) + \left(\tfrac{1}{4} - \tfrac{1}{7}\right) + \cdots + \left(\tfrac{1}{n-3} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n-2} - \tfrac{1}{n+1}\right) + \left(\tfrac{1}{n-1} - \tfrac{1}{n+2}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+3}\right)$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{n+1} - \tfrac{1}{n+2} - \tfrac{1}{n+3} \quad \text{[telescoping series]}$$

Thus, $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right) = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$. Converges

39. For the series $\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$,

$$s_n = \sum_{i=1}^{n} \left(e^{1/i} - e^{1/(i+1)}\right) = (e^1 - e^{1/2}) + (e^{1/2} - e^{1/3}) + \cdots + \left(e^{1/n} - e^{1/(n+1)}\right) = e - e^{1/(n+1)}$$

[telescoping series]

Thus, $\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(e - e^{1/(n+1)}\right) = e - e^0 = e - 1$. Converges

41. $0.\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \cdots$ is a geometric series with $a = \frac{2}{10}$ and $r = \frac{1}{10}$. It converges to $\frac{a}{1-r} = \frac{2/10}{1-1/10} = \frac{2}{9}$.

43. $3.\overline{417} = 3 + \frac{417}{10^3} + \frac{417}{10^6} + \cdots$. Now $\frac{417}{10^3} + \frac{417}{10^6} + \cdots$ is a geometric series with $a = \frac{417}{10^3}$ and $r = \frac{1}{10^3}$.

It converges to $\frac{a}{1-r} = \frac{417/10^3}{1-1/10^3} = \frac{417/10^3}{999/10^3} = \frac{417}{999}$. Thus, $3.\overline{417} = 3 + \frac{417}{999} = \frac{3414}{999} = \frac{1138}{333}$.

45. $1.53\overline{42} = 1.53 + \frac{42}{10^4} + \frac{42}{10^6} + \cdots$. Now $\frac{42}{10^4} + \frac{42}{10^6} + \cdots$ is a geometric series with $a = \frac{42}{10^4}$ and $r = \frac{1}{10^2}$.

It converges to $\frac{a}{1-r} = \frac{42/10^4}{1-1/10^2} = \frac{42/10^4}{99/10^2} = \frac{42}{9900}$.

Thus, $1.53\overline{42} = 1.53 + \frac{42}{9900} = \frac{153}{100} + \frac{42}{9900} = \frac{15{,}147}{9900} + \frac{42}{9900} = \frac{15{,}189}{9900}$ or $\frac{5063}{3300}$.

47. $\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with $r = \frac{x}{3}$, so the series converges $\Leftrightarrow |r| < 1 \Leftrightarrow \frac{|x|}{3} < 1 \Leftrightarrow |x| < 3$;

that is, $-3 < x < 3$. In that case, the sum of the series is $\frac{a}{1-r} = \frac{x/3}{1-x/3} = \frac{x/3}{1-x/3} \cdot \frac{3}{3} = \frac{x}{3-x}$.

49. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

51. $\sum_{n=0}^{\infty} \dfrac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \dfrac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \dfrac{|\cos x|}{2} \leq \dfrac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is $\dfrac{1}{1-(\cos x)/2} = \dfrac{2}{2-\cos x}$.

53. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{3n^2+3n+1}{(n^2+n)^3} = \dfrac{1}{n^3} - \dfrac{1}{(n+1)^3}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k^3} - \frac{1}{(k+1)^3} \right) = \left(1 - \frac{1}{2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{3^3}\right) + \cdots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) = 1 - \frac{1}{(n+1)^3}$$

The series converges to $\lim_{n\to\infty} s_n = 1$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

55. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n - (n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$.

57. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \frac{D(1-c^n)}{1-c} \text{ by (3).}$$

(b) $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n\to\infty} (1-c^n) = \frac{D}{1-c} \quad \left[\text{since } 0 < c < 1 \quad \Rightarrow \quad \lim_{n\to\infty} c^n = 0\right]$$
$$= \frac{D}{s} \quad [\text{since } c + s = 1] = kD \quad [\text{since } k = 1/s]$$

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

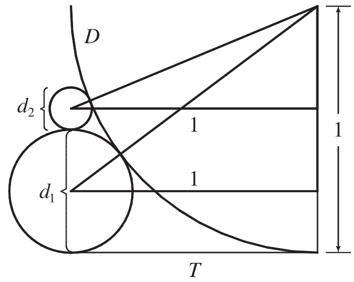
59. $\sum_{n=2}^{\infty} (1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1$ $\Leftrightarrow$ $|1+c| > 1$ $\Leftrightarrow$ $1+c > 1$ or $1+c < -1$ $\Leftrightarrow$ $c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2$ $\Leftrightarrow$ $\left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right)$ $\Leftrightarrow$ $1 = 2(1+c)^2 - 2(1+c)$ $\Leftrightarrow$ $2c^2 + 2c - 1 = 0$ $\Leftrightarrow$ $c = \frac{-2 \pm \sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

61. $e^{s_n} = e^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}} = e^1 e^{1/2} e^{1/3} \cdots e^{1/n} > (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) \qquad [e^x > 1+x]$

$= \frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\frac{n+1}{n} = n+1$

Thus, $e^{s_n} > n+1$ and $\lim\limits_{n\to\infty} e^{s_n} = \infty$. Since $\{s_n\}$ is increasing, $\lim\limits_{n\to\infty} s_n = \infty$, implying that the harmonic series is divergent.

63. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write



$1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \quad \Leftrightarrow$

$1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ [difference of squares] $\Rightarrow \quad d_1 = \frac{1}{2}$.

Similarly,

$1 = \left(1 + \frac{1}{2}d_2\right)^2 - \left(1 - d_1 - \frac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$

$= (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$

$d_2 = \dfrac{1}{2 - d_1} - d_1 = \dfrac{(1-d_1)^2}{2 - d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \quad \Leftrightarrow \quad d_3 = \dfrac{[1 - (d_1 + d_2)]^2}{2 - (d_1 + d_2)}$, and in general,

$d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^n d_i\right)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\dfrac{1}{6} = \dfrac{1}{2\cdot 3}$ and $\dfrac{1}{12} = \dfrac{1}{3\cdot 4}$ respectively, so we suspect that in general, $d_n = \dfrac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k \le n$, $d_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then $\sum\limits_{i=1}^n d_i = 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$ [telescoping sum]. Substituting this into our formula for d_{n+1}, we get $d_{n+1} = \dfrac{\left[1 - \dfrac{n}{n+1}\right]^2}{2 - \left(\dfrac{n}{n+1}\right)} = \dfrac{\dfrac{1}{(n+1)^2}}{\dfrac{n+2}{n+1}} = \dfrac{1}{(n+1)(n+2)}$, and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^n d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is,

$\sum\limits_{n=1}^\infty a_n = \sum\limits_{n=1}^\infty \dfrac{1}{n(n+1)} = 1$, which is what we wanted to prove.

65. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

67. $\sum_{n=1}^\infty ca_n = \lim\limits_{n\to\infty} \sum_{i=1}^n ca_i = \lim\limits_{n\to\infty} c\sum_{i=1}^n a_i = c \lim\limits_{n\to\infty} \sum_{i=1}^n a_i = c\sum_{n=1}^\infty a_n$, which exists by hypothesis.

69. Suppose on the contrary that $\sum(a_n + b_n)$ converges. Then $\sum(a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum[(a_n + b_n) - a_n]$ would also be convergent. But $\sum[(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

71. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \le 1000$ for all n. So by the Monotonic Sequence Theorem, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

73. (a) At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2 \cdot \left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum\limits_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = 1$ $\left[\text{geometric series with } a = \frac{1}{3} \text{ and } r = \frac{2}{3}\right]$. Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1 - \left(\frac{2}{3}\right)^n, 1 - \left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1 - 8/9} = 1.$$

75. (a) For $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)!}$, $s_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{5}{6}$, $s_3 = \frac{5}{6} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{23}{24}$,

$s_4 = \frac{23}{24} + \frac{4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \frac{(n+1)! - 1}{(n+1)!}$.

(b) For $n = 1$, $s_1 = \frac{1}{2} = \frac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \frac{(k+1)! - 1}{(k+1)!}$. Then

$$\begin{aligned} s_{k+1} &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!(k+2)} = \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} \\ &= \frac{(k+2)! - 1}{(k+2)!} \end{aligned}$$

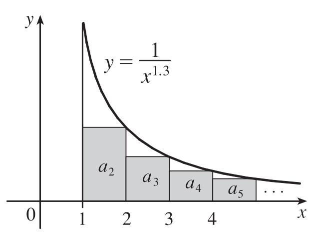
Thus, the formula is true for $n = k + 1$. So by induction, the guess is correct.

(c) $\lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} \frac{(n+1)! - 1}{(n+1)!} = \lim\limits_{n \to \infty} \left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

11.3 The Integral Test and Estimates of Sums

1. The picture shows that $a_2 = \frac{1}{2^{1.3}} < \int_1^2 \frac{1}{x^{1.3}}\, dx$,

$a_3 = \frac{1}{3^{1.3}} < \int_2^3 \frac{1}{x^{1.3}}\, dx$, and so on, so $\sum\limits_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\, dx$. The integral converges by (7.8.2) with $p = 1.3 > 1$, so the series converges.



3. The function $f(x) = 1/\sqrt[5]{x} = x^{-1/5}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-1/5}\, dx = \lim\limits_{t \to \infty} \int_1^t x^{-1/5}\, dx = \lim\limits_{t \to \infty} \left[\frac{5}{4}x^{4/5}\right]_1^t = \lim\limits_{t \to \infty} \left(\frac{5}{4}t^{4/5} - \frac{5}{4}\right) = \infty$, so $\sum\limits_{n=1}^{\infty} 1/\sqrt[5]{n}$ diverges.

5. The function $f(x) = \dfrac{1}{(2x+1)^3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4}\,\frac{1}{(2x+1)^2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{4(2t+1)^2}+\frac{1}{36}\right) = \frac{1}{36}.$$

Since this improper integral is convergent, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n+1)^3}$ is also convergent by the Integral Test.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^b \quad \text{[by parts]} \;= \lim_{b\to\infty}\left[-be^{-b}-e^{-b}+e^{-1}+e^{-1}\right] = 2/e$$

since $\lim\limits_{b\to\infty} be^{-b} = \lim\limits_{b\to\infty}(b/e^b) \overset{\text{H}}{=} \lim\limits_{b\to\infty}(1/e^b) = 0$ and $\lim\limits_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges.

9. The series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\sum\limits_{n=1}^{\infty}\dfrac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{0.85}}$ would have to converge [by Theorem 8(i) in Section 11.2].

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

13. $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \cdots = \sum\limits_{n=1}^{\infty}\dfrac{1}{2n-1}$. The function $f(x) = \dfrac{1}{2x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{2x-1}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln|2x-1|\right]_1^t = \tfrac{1}{2}\lim_{t\to\infty}(\ln(2t-1)-0) = \infty,$$ so the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{2n-1}$ diverges.

15. $\sum\limits_{n=1}^{\infty}\dfrac{5-2\sqrt{n}}{n^3} = 5\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3} - 2\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{5/2}}$ by Theorem 11.2.8, since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$ and $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{5/2}}$ both converge by (1) $\left[\text{with } p = 3 > 1 \text{ and } p = \tfrac{5}{2} > 1\right]$. Thus, $\sum\limits_{n=1}^{\infty}\dfrac{5-2\sqrt{n}}{n^3}$ converges.

17. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+4}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{2}\right)-\tan^{-1}\left(\frac{1}{2}\right)\right]\\ &= \frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{2}\right)\right]\end{aligned}$$

Therefore, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+4}$ converges.

19. $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^3} = \sum\limits_{n=2}^{\infty}\dfrac{\ln n}{n^3}$ since $\dfrac{\ln 1}{1} = 0$. The function $f(x) = \dfrac{\ln x}{x^3}$ is continuous and positive on $[2, \infty)$.

$$f'(x) = \frac{x^3(1/x)-(\ln x)(3x^2)}{(x^3)^2} = \frac{x^2-3x^2\ln x}{x^6} = \frac{1-3\ln x}{x^4} < 0 \quad\Leftrightarrow\quad 1-3\ln x < 0 \quad\Leftrightarrow\quad \ln x > \tfrac{1}{3} \quad\Leftrightarrow$$

$x > e^{1/3} \approx 1.4$, so f is decreasing on $[2, \infty)$, and the Integral Test applies.

$\displaystyle\int_2^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_2^t \frac{\ln x}{x^3}\,dx \overset{(\star)}{=} \lim_{t\to\infty}\left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{4t^2}(2\ln t + 1) + \frac{1}{4}\right] \overset{(\star\star)}{=} \frac{1}{4}$, so the series $\displaystyle\sum_{n=2}^\infty \frac{\ln n}{n^3}$ converges.

$(\star)$: $u = \ln x$, $dv = x^{-3}\,dx \quad\Rightarrow\quad du = (1/x)\,dx$, $v = -\frac{1}{2}x^{-2}$, so

$$\int \frac{\ln x}{x^3}\,dx = -\tfrac{1}{2}x^{-2}\ln x - \int -\tfrac{1}{2}x^{-2}(1/x)\,dx = -\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{2}\int x^{-3}\,dx = -\tfrac{1}{2}x^{-2}\ln x - \tfrac{1}{4}x^{-2} + C.$$

$(\star\star)$: $\displaystyle\lim_{t\to\infty}\left(-\frac{2\ln t + 1}{4t^2}\right) \overset{\text{H}}{=} -\lim_{t\to\infty}\frac{2/t}{8t} = -\tfrac{1}{4}\lim_{t\to\infty}\frac{1}{t^2} = 0.$

21. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1 + \ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln n}$ diverges.

23. The function $f(x) = e^{1/x}/x^2$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$[g(x) = e^{1/x}$ is decreasing and dividing by x^2 doesn't change that fact.$]$

$\displaystyle\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to\infty}\left[-e^{1/x}\right]_1^t = -\lim_{t\to\infty}(e^{1/t} - e) = -(1 - e) = e - 1$, so the series $\displaystyle\sum_{n=1}^\infty \frac{e^{1/n}}{n^2}$ converges.

25. The function $f(x) = \dfrac{1}{x^3 + x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\begin{aligned}\int_1^\infty \frac{1}{x^3 + x}\,dx &= \lim_{t\to\infty}\int_1^t \left[\frac{1}{x} - \frac{x}{1 + x^2}\right]dx = \lim_{t\to\infty}\left[\ln x - \frac{1}{2}\ln(1 + x^2)\right]_1^t = \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1 + x^2}}\right]_1^t \\ &= \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1 + t^2}} - \ln\frac{1}{\sqrt{2}}\right) = \lim_{t\to\infty}\left(\ln\frac{1}{\sqrt{1 + 1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2\end{aligned}$$

so the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3 + n}$ converges.

27. We have already shown (in Exercise 21) that when $p = 1$ the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$.

$f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\dfrac{p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1 - p}\right]_2^t \quad [\text{for } p \neq 1] = \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1 - p} - \frac{(\ln 2)^{1-p}}{1 - p}\right]$$

This limit exists whenever $1 - p < 0 \quad\Leftrightarrow\quad p > 1$, so the series converges for $p > 1$.

29. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim\limits_{n\to\infty} n(1+n^2)^p \neq 0$. So assume $p < -\frac{1}{2}$. Then

$f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1,\infty)$, and we can use the Integral Test.

$$\int_1^\infty x(1+x^2)^p\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \frac{1}{2(p+1)}\lim_{t\to\infty}\left[(1+t^2)^{p+1} - 2^{p+1}\right].$$

This limit exists and is finite $\Leftrightarrow$ $p+1<0$ $\Leftrightarrow$ $p<-1$, so the series converges whenever $p<-1$.

31. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of real numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of real numbers x such that the series is convergent.

33. (a) $f(x) = \dfrac{1}{x^2}$ is positive and continuous and $f'(x) = -\dfrac{2}{x^3}$ is negative for $x > 0$, and so the Integral Test applies.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \approx s_{10} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2} \approx 1.549768.$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{10}\right) = \frac{1}{10},$$ so the error is at most 0.1.

(b) $s_{10} + \displaystyle\int_{11}^\infty \frac{1}{x^2}\,dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^2}\,dx \Rightarrow s_{10} + \frac{1}{11} \leq s \leq s_{10} + \frac{1}{10} \Rightarrow$

$1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) $R_n \leq \displaystyle\int_n^\infty \frac{1}{x^2}\,dx = \frac{1}{n}$. So $R_n < 0.001$ if $\dfrac{1}{n} < \dfrac{1}{1000} \Leftrightarrow n > 1000$.

35. $f(x) = 1/(2x+1)^6$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. Using (2),

$R_n \leq \displaystyle\int_n^\infty (2x+1)^{-6}\,dx = \lim_{t\to\infty}\left[\frac{-1}{10(2x+1)^5}\right]_n^t = \frac{1}{10(2n+1)^5}$. To be correct to five decimal places, we want

$\dfrac{1}{10(2n+1)^5} \leq \dfrac{5}{10^6} \Leftrightarrow (2n+1)^5 \geq 20{,}000 \Leftrightarrow n \geq \frac{1}{2}\left(\sqrt[5]{20{,}000}-1\right) \approx 3.12$, so use $n = 4$.

$$s_4 = \sum_{n=1}^{4}\frac{1}{(2n+1)^6} = \frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6} \approx 0.001\,446 \approx 0.00145.$$

37. $\displaystyle\sum_{n=1}^\infty n^{-1.001} = \sum_{n=1}^\infty \frac{1}{n^{1.001}}$ is a convergent p-series with $p = 1.001 > 1$. Using (2), we get

$$R_n \leq \int_n^\infty x^{-1.001}\,dx = \lim_{t\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_n^t = -1000\lim_{t\to\infty}\left[\frac{1}{x^{0.001}}\right]_n^t = -1000\left(-\frac{1}{n^{0.001}}\right) = \frac{1000}{n^{0.001}}.$$

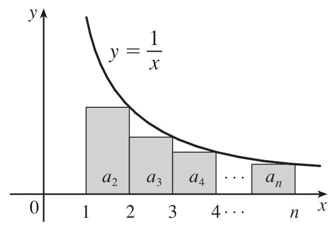
We want $R_n < 0.000\,000\,005 \Leftrightarrow \dfrac{1000}{n^{0.001}} < 5\times 10^{-9} \Leftrightarrow n^{0.001} > \dfrac{1000}{5\times 10^{-9}} \Leftrightarrow$

$n > \left(2\times 10^{11}\right)^{1000} = 2^{1000}\times 10^{11{,}000} \approx 1.07\times 10^{301}\times 10^{11{,}000} = 1.07\times 10^{11{,}301}$.

39. (a) From the figure, $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$, so with

$$f(x) = \frac{1}{x},\ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le \int_1^n \frac{1}{x}\,dx = \ln n.$$

Thus, $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n$.



(b) By part (a), $s_{10^6} \le 1 + \ln 10^6 \approx 14.82 < 15$ and

$s_{10^9} \le 1 + \ln 10^9 \approx 21.72 < 22$.

41. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. This is a p-series, which converges for all b such that $-\ln b > 1 \quad\Leftrightarrow$

$\ln b < -1 \quad\Leftrightarrow\quad b < e^{-1} \quad\Leftrightarrow\quad b < 1/e$ [with $b > 0$].

11.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

3. $\dfrac{n}{2n^3+1} < \dfrac{n}{2n^3} = \dfrac{1}{2n^2} < \dfrac{1}{n^2}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

5. $\dfrac{n+1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$.

7. $\dfrac{9^n}{3+10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \ge 1$. $\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ converges by the Comparison Test.

9. $\dfrac{\cos^2 n}{n^2+1} \le \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ converges by comparison with the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $[p = 2 > 1]$.

11. $\dfrac{n-1}{n\,4^n}$ is positive for $n > 1$ and $\dfrac{n-1}{n\,4^n} < \dfrac{n}{n\,4^n} = \dfrac{1}{4^n} = \left(\dfrac{1}{4}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n\,4^n}$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$.

13. $\dfrac{\arctan n}{n^{1.2}} < \dfrac{\pi/2}{n^{1.2}}$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$ converges by comparison with $\dfrac{\pi}{2}\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$, which converges because it is a constant times a p-series with $p = 1.2 > 1$.

15. $\dfrac{2+(-1)^n}{n\sqrt{n}} \le \dfrac{3}{n\sqrt{n}}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $\left[p = \frac{3}{2} > 1\right]$, so the given series converges by the Comparison Test.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

19. Use the Limit Comparison Test with $a_n = \dfrac{1+4^n}{1+3^n}$ and $b_n = \dfrac{4^n}{3^n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1+4^n}{1+3^n}}{\dfrac{4^n}{3^n}} = \lim_{n\to\infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{1+4^n}{4^n} \cdot \frac{3^n}{1+3^n} = \lim_{n\to\infty} \left(\frac{1}{4^n}+1\right) \cdot \frac{1}{\dfrac{1}{3^n}+1} = 1 > 0$$

Since the geometric series $\sum b_n = \sum \left(\frac{4}{3}\right)^n$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$. Alternatively, use the Comparison Test with $\dfrac{1+4^n}{1+3^n} > \dfrac{1+4^n}{3^n+3^n} > \dfrac{4^n}{2(3^n)} = \dfrac{1}{2}\left(\dfrac{4}{3}\right)^n$ or use the Test for Divergence.

21. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+2}}{2n^2+n+1}$ and $b_n = \dfrac{1}{n^{3/2}}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+2}}{2n^2+n+1} = \lim_{n\to\infty} \frac{(n^{3/2}\sqrt{n+2})/(n^{3/2}\sqrt{n})}{(2n^2+n+1)/n^2} = \lim_{n\to\infty} \frac{\sqrt{1+2/n}}{2+1/n+1/n^2} = \frac{\sqrt{1}}{2} = \frac{1}{2} > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series $\left[p = \frac{3}{2} > 1\right]$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ also converges.

23. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty} \frac{5n^3+2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series $[p = 3 > 1]$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty} \frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series $\left[|r| = \frac{1}{e} < 1\right]$, the series $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ also converges.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 \cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent geometric series $\left[|r| = \frac{1}{2} < 1\right]$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series,

$\sum\limits_{n=1}^{\infty} \sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$\lim\limits_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1.$]

33. $\sum\limits_{n=1}^{10} \frac{1}{\sqrt{n^4+1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{17}} + \frac{1}{\sqrt{82}} + \cdots + \frac{1}{\sqrt{10{,}001}} \approx 1.24856$. Now $\frac{1}{\sqrt{n^4+1}} < \frac{1}{\sqrt{n^4}} = \frac{1}{n^2}$, so the error is

$R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{x}\right]_{10}^{t} = \lim\limits_{t\to\infty}\left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10} = 0.1.$

35. $\sum\limits_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is

$R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2}$ [geometric series] ≈ 0.00098.

37. Since $\frac{d_n}{10^n} \le \frac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series $\left(|r| = \frac{1}{10} < 1\right)$, $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \le a_n < 1$ for all $n > N \Rightarrow 0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

41. (a) Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, there is an integer N such that $\frac{a_n}{b_n} > 1$ whenever $n > N$. (Take $M = 1$ in Definition 11.1.5.)

Then $a_n > b_n$ whenever $n > N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$ for $n \ge 2$, then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n}{\ln n} = \lim\limits_{x\to\infty} \frac{x}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \frac{1}{1/x} = \lim\limits_{x\to\infty} x = \infty$,

so by part (a), $\sum\limits_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \ln n = \lim\limits_{x\to\infty} \ln x = \infty$,

so $\sum\limits_{n=1}^{\infty} a_n$ diverges by part (a).

43. $\lim\limits_{n\to\infty} na_n = \lim\limits_{n\to\infty} \frac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n = \frac{1}{n}$. Since $\lim\limits_{n\to\infty} na_n > 0$ we know that either both series converge or both series diverge, and we also know that $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges [p-series with $p = 1$]. Therefore, $\sum a_n$ must be divergent.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 11.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.3.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

11.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim\limits_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

3. $\dfrac{4}{7} - \dfrac{4}{8} + \dfrac{4}{9} - \dfrac{4}{10} + \dfrac{4}{11} - \cdots = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{4}{n+6}$. Now $b_n = \dfrac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

5. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{2n+1} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$. Now $b_n = \dfrac{1}{2n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

7. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{3n-1}{2n+1} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{3-1/n}{2+1/n} = \dfrac{3}{2} \ne 0$. Since $\lim\limits_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

9. $b_n = \dfrac{n}{10^n} > 0$ for $n \ge 1$. $\{b_n\}$ is decreasing for $n \ge 1$ since

$$\left(\frac{x}{10^x}\right)' = \frac{10^x(1) - x \cdot 10^x \ln 10}{(10^x)^2} = \frac{10^x(1 - x\ln 10)}{(10^x)^2} = \frac{1 - x\ln 10}{10^x} < 0 \text{ for } 1 - x\ln 10 < 0 \;\Rightarrow\; x\ln 10 > 1 \;\Rightarrow$$

$x > \dfrac{1}{\ln 10} \approx 0.4$. Also, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n}{10^n} = \lim\limits_{x\to\infty} \dfrac{x}{10^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{x}{10^x \ln 10} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{10^n}$ converges by the Alternating Series Test.

11. $b_n = \dfrac{n^2}{n^3+4} > 0$ for $n \ge 1$. $\{b_n\}$ is decreasing for $n \ge 2$ since

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0 \text{ for } x > 2. \text{ Also,}$$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+4}$ converges by the Alternating Series Test.

13. $\sum\limits_{n=2}^{\infty} (-1)^n \dfrac{n}{\ln n}$. $\lim\limits_{n\to\infty} \dfrac{n}{\ln n} = \lim\limits_{x\to\infty} \dfrac{x}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

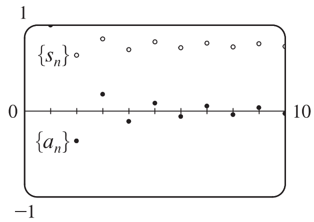
15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$. $b_n = \sin\left(\frac{\pi}{n}\right) > 0$ for $n \geq 2$ and $\sin\left(\frac{\pi}{n}\right) \geq \sin\left(\frac{\pi}{n+1}\right)$, and $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0$, so the series converges by the Alternating Series Test.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \;\Rightarrow\; \lim_{n\to\infty} \frac{n^n}{n!} = \infty \;\Rightarrow\; \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

21.

n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is $|s - s_{10}| \leq b_{11} = 1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

23. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^6} < \frac{1}{n^6}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^6} = 0$, so the series is convergent. Now $b_5 = \frac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \frac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

25. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n\, n!}$ satisfies (i) of the Alternating Series Test because $\frac{1}{10^{n+1}(n+1)!} < \frac{1}{10^n\, n!}$ and (ii) $\lim_{n\to\infty} \frac{1}{10^n\, n!} = 0$, so the series is convergent. Now $b_3 = \frac{1}{10^3\, 3!} \approx 0.000\,167 > 0.000\,005$ and $b_4 = \frac{1}{10^4\, 4!} = 0.000\,004 < 0.000\,005$, so by the Alternating Series Estimation Theorem, $n = 4$ (since the series starts with $n = 0$, not $n = 1$). (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

27. $b_7 = \frac{1}{7^5} = \frac{1}{16{,}807} \approx 0.000\,059\,5$, so

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.972\,080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.9721.

29. $b_7 = \dfrac{7^2}{10^7} = 0.000\,004\,9$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{10^n} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n-1}n^2}{10^n} = \tfrac{1}{10} - \tfrac{4}{100} + \tfrac{9}{1000} - \tfrac{16}{10{,}000} + \tfrac{25}{100{,}000} - \tfrac{36}{1{,}000{,}000} = 0.067\,614$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.0676.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

33. Clearly $b_n = \dfrac{1}{n+p}$ is decreasing and eventually positive and $\lim\limits_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \neq 0$ for $n \geq 1$, or p is not a negative integer.

35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p-series for $p = 2$). So suppose that $\sum (-1)^{n-1}\, b_n$ converges. Then by Theorem 11.2.8(ii), so does $\sum \left[(-1)^{n-1} b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum \dfrac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1}\, b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

11.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$. Using the Ratio Test, $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-10)^n}\right| = \lim_{n\to\infty} \left|\frac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ is a divergent p-series $\left(p = \frac{1}{4} \leq 1\right)$, so the given series is conditionally convergent.

7. $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left[\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right] = \lim_{k\to\infty}\frac{k+1}{k}\left(\frac{2}{3}\right)^1 = \frac{2}{3}\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = \frac{2}{3}(1) = \frac{2}{3} < 1$, so the series $\sum\limits_{n=1}^{\infty} k\left(\frac{2}{3}\right)^k$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

9. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(1.1)^{n+1}}{(n+1)^4}\cdot\frac{n^4}{(1.1)^n}\right] = \lim_{n\to\infty}\frac{(1.1)n^4}{(n+1)^4} = (1.1)\lim_{n\to\infty}\frac{1}{\frac{(n+1)^4}{n^4}} = (1.1)\lim_{n\to\infty}\frac{1}{(1+1/n)^4}$

$= (1.1)(1) = 1.1 > 1,$

so the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(1.1)^n}{n^4}$ diverges by the Ratio Test.

11. Since $0 \le \dfrac{e^{1/n}}{n^3} \le \dfrac{e}{n^3} = e\left(\dfrac{1}{n^3}\right)$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series [$p = 3 > 1$], $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^3}$ converges, and so $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

13. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+2)\,4^{2n+3}}\cdot\frac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim_{n\to\infty}\left(\frac{10}{4^2}\cdot\frac{n+1}{n+2}\right) = \frac{5}{8} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

15. $\left|\dfrac{(-1)^n\arctan n}{n^2}\right| < \dfrac{\pi/2}{n^2}$, so since $\displaystyle\sum_{n=1}^{\infty}\frac{\pi/2}{n^2} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges ($p = 2 > 1$), the given series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\arctan n}{n^2}$ converges absolutely by the Comparison Test.

17. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\displaystyle\lim_{n\to\infty}\frac{1}{\ln n} = 0$ and $\left\{\dfrac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ is the divergent (partial) harmonic series, $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ is conditionally convergent.

19. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (use the Ratio Test), so the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

21. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

23. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$ (by Equation 3.6.6), so the series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ diverges by the Root Test.

25. Use the Ratio Test with the series

$$1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n-1)!}.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)[2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{(-1)^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!} \right| = \lim_{n\to\infty} \frac{1}{2n} = 0 < 1,$$

so the given series is absolutely convergent and therefore convergent.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!} = \sum_{n=1}^{\infty} \frac{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \cdots \cdot (2 \cdot n)}{n!} = \sum_{n=1}^{\infty} \frac{2^n n!}{n!} = \sum_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\displaystyle\lim_{n\to\infty} 2^n = \infty$.

29. By the recursive definition, $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5n+1}{4n+3} \right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

31. (a) $\displaystyle\lim_{n\to\infty} \left| \frac{1/(n+1)^3}{1/n^3} \right| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} \frac{1}{(1+1/n)^3} = 1.$ Inconclusive

(b) $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n} \right) = \frac{1}{2}.$ Conclusive (convergent)

(c) $\displaystyle\lim_{n\to\infty} \left| \frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}} \right| = 3 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = 3.$ Conclusive (divergent)

(d) $\displaystyle\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right| = \lim_{n\to\infty} \left[\sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2} \right] = 1.$ Inconclusive

33. (a) $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$, so by the Ratio Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ by Theorem 11.2.6.

35. (a) $\displaystyle s_5 = \sum_{n=1}^{5} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios

$\displaystyle r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$ form an increasing sequence, since

$\displaystyle r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0$. So by Exercise 34(b), the error in using s_5 is $\displaystyle R_5 \le \frac{a_6}{1 - \lim_{n\to\infty} r_n} = \frac{1/(6 \cdot 2^6)}{1 - 1/2} = \frac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $\displaystyle R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)2^{n+1}}$. We want $R_n < 0.00005 \Leftrightarrow$ $\displaystyle\frac{1}{(n+1)2^n} < 0.00005 \Leftrightarrow (n+1)2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate $(11+1)2^{11} = 24{,}576$, so $\displaystyle s_{11} = \sum_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

37. (i) Following the hint, we get that $|a_n| < r^n$ for $n \geq N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $[0 < r < 1]$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \geq N$, so $|a_n| > 1$ for $n \geq N$. Thus, $\lim\limits_{n\to\infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

(iii) Consider $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ [diverges] and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ [converges]. For each sum, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, so the Root Test is inconclusive.

39. (a) Since $\sum a_n$ is absolutely convergent, and since $\left|a_n^+\right| \leq |a_n|$ and $\left|a_n^-\right| \leq |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent.
Or: Use Theorem 11.2.8.

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum\left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 11.2.8. But $\sum\left(a_n^+ - \frac{1}{2}a_n\right) = \sum\left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

11.7 Strategy for Testing Series

1. $\dfrac{1}{n+3^n} < \dfrac{1}{3^n} = \left(\dfrac{1}{3}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{3}\right)^n$ is a convergent geometric series $\left[|r| = \frac{1}{3} < 1\right]$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n+3^n}$ converges by the Comparison Test.

3. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{n}{n+2} = 1$, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n \dfrac{n}{n+2}$ does not exist. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{n+2}$ diverges by the Test for Divergence.

5. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)^2\, 2^n}{(-5)^{n+1}} \cdot \dfrac{(-5)^n}{n^2\, 2^{n-1}}\right| = \lim\limits_{n\to\infty} \dfrac{2(n+1)^2}{5n^2} = \dfrac{2}{5} \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 = \dfrac{2}{5}(1) = \dfrac{2}{5} < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2\, 2^{n-1}}{(-5)^n}$ converges by the Ratio Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \; \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e}\right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1$, so the series converges.

11. $b_n = \frac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2}\right| = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = 3 \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

15. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)[3(n+1)+2]} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$,

so the series $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$ converges by the Ratio Test.

17. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty} (-1)^n\, 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

19. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then $f'(x) = \frac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\frac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{1/\left(2\sqrt{n}\right)} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sec^2(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

25. Use the Ratio Test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \frac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

27. $\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ [using integration by parts] $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.

29. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{\cosh n} = \sum\limits_{n=1}^{\infty} (-1)^n b_n$. Now $b_n = \dfrac{1}{\cosh n} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

Or: Write $\dfrac{1}{\cosh n} = \dfrac{2}{e^n + e^{-n}} < \dfrac{2}{e^n}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{e^n}$ is a convergent geometric series, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{\cosh n}$ is convergent by the Comparison Test. So $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{\cosh n}$ is absolutely convergent and therefore convergent.

31. $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \dfrac{5^k}{3^k + 4^k} =$ [divide by 4^k] $\quad \lim\limits_{k\to\infty} \dfrac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim\limits_{k\to\infty} \left(\dfrac{3}{4}\right)^k = 0$ and $\lim\limits_{k\to\infty} \left(\dfrac{5}{4}\right)^k = \infty$.

Thus, $\sum\limits_{k=1}^{\infty} \dfrac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

33. Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = 1 > 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ $\quad [p = 3/2 > 1]$.

35. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty} \dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty} (1 + 1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

37. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} (2^{1/n} - 1) = 1 - 1 = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

11.8 Power Series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a, where a is a constant.

3. If $a_n = \dfrac{x^n}{\sqrt{n}}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{\sqrt{1+1/n}} = |x|$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} \le 1$. When $x = -1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

5. If $a_n = \dfrac{(-1)^{n-1} x^n}{n^3}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \dfrac{n^3}{(-1)^{n-1} x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)x n^3}{(n+1)^3}\right| = \lim\limits_{n\to\infty} \left[\left(\dfrac{n}{n+1}\right)^3 |x|\right] = 1^3 \cdot |x| = |x|$. By the

Ratio Test, the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^n}{n^3}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(-1)^n}{n^3} = -\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series $[p = 3 > 1]$. Thus, the interval of convergence is $I = [-1, 1]$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\dfrac{x}{n+1}\right| = |x| \lim_{n\to\infty} \dfrac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

9. If $a_n = (-1)^n \dfrac{n^2 x^n}{2^n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2\, x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n^2\, x^n}\right| = \lim_{n\to\infty} \left|\frac{x(n+1)^2}{2n^2}\right| = \lim_{n\to\infty} \left[\frac{|x|}{2}\left(1 + \frac{1}{n}\right)^2\right] = \frac{|x|}{2}(1)^2 = \tfrac{1}{2}\,|x|.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2 x^n}{2^n}$ converges when $\frac{1}{2}\,|x| < 1 \quad\Leftrightarrow\quad |x| < 2$, so the radius of convergence is $R = 2$. When $x = \pm 2$, both series $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2(\pm 2)^n}{2^n} = \sum_{n=1}^{\infty} (\mp 1)^n n^2$ diverge by the Test for Divergence since $\lim_{n\to\infty} \left|(\mp 1)^n\, n^2\right| = \infty$. Thus, the interval of convergence is $I = (-2, 2)$.

11. $a_n = \dfrac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{2^{n+1}\,|x|^{n+1}}{\sqrt[4]{n+1}} \cdot \dfrac{\sqrt[4]{n}}{2^n\,|x|^n} = \lim_{n\to\infty} 2\,|x| \sqrt[4]{\dfrac{n}{n+1}} = 2\,|x|$, so by the Ratio Test, the series converges when $2\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. When $x = -\frac{1}{2}$, we get the divergent p-series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[4]{n}}$ $\left[p = \frac{1}{4} \le 1\right]$. When $x = \frac{1}{2}$, we get the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt[4]{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{1}{2}, \frac{1}{2}\right]$.

13. If $a_n = (-1)^n \dfrac{x^n}{4^n \ln n}$, then $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \dfrac{4^n \ln n}{x^n}\right| = \dfrac{|x|}{4} \lim_{n\to\infty} \dfrac{\ln n}{\ln(n+1)} = \dfrac{|x|}{4} \cdot 1$ [by l'Hospital's Rule] $= \dfrac{|x|}{4}$. By the Ratio Test, the series converges when $\dfrac{|x|}{4} < 1 \quad\Leftrightarrow\quad |x| < 4$, so $R = 4$. When $x = -4$, $\sum_{n=2}^{\infty} (-1)^n \dfrac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \dfrac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \dfrac{1}{\ln n}$. Since $\ln n < n$ for $n \ge 2$, $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$, $\sum_{n=2}^{\infty} (-1)^n \dfrac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \dfrac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

15. If $a_n = \dfrac{(x-2)^n}{n^2+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\dfrac{n^2+1}{(x-2)^n}\right| = |x-2|\lim\limits_{n\to\infty}\dfrac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(x-2)^n}{n^2+1}$ converges when $|x-2|<1 \quad [R=1] \quad \Leftrightarrow \quad -1<x-2<1 \quad \Leftrightarrow \quad 1<x<3$. When $x=1$, the series $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{n^2+1}$ converges by the Alternating Series Test; when $x=3$, the series $\sum\limits_{n=0}^{\infty}\dfrac{1}{n^2+1}$ converges by comparison with the p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ $[p=2>1]$. Thus, the interval of convergence is $I=[1,3]$.

17. If $a_n = \dfrac{3^n(x+4)^n}{\sqrt{n}}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{3^n(x+4)^n}\right| = 3\,|x+4|\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{\sqrt{n+1}} = 3\,|x+4|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{3^n(x+4)^n}{\sqrt{n}}$ converges when $3\,|x+4|<1 \quad \Leftrightarrow \quad |x+4|<\frac{1}{3} \quad \left[R=\frac{1}{3}\right] \quad \Leftrightarrow$ $-\frac{1}{3}<x+4<\frac{1}{3} \quad \Leftrightarrow \quad -\frac{13}{3}<x<-\frac{11}{3}$. When $x=-\frac{13}{3}$, the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{\sqrt{n}}$ converges by the Alternating Series Test; when $x=-\frac{11}{3}$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ diverges $\left[p=\frac{1}{2}\le 1\right]$. Thus, the interval of convergence is $I=\left[-\frac{13}{3},-\frac{11}{3}\right)$.

19. If $a_n = \dfrac{(x-2)^n}{n^n}$, then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\dfrac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

21. $a_n = \dfrac{n}{b^n}(x-a)^n$, where $b>0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b}<1 \quad \Leftrightarrow \quad |x-a|<b \quad [\text{so } R=b] \quad \Leftrightarrow \quad -b<x-a<b \quad \Leftrightarrow$ $a-b<x<a+b$. When $|x-a|=b$, $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty} n = \infty$, so the series diverges. Thus, $I=(a-b,a+b)$.

23. If $a_n = n!\,(2x-1)^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim\limits_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$ for all $x\neq\frac{1}{2}$. Since the series diverges for all $x\neq\frac{1}{2}$, $R=0$ and $I=\left\{\frac{1}{2}\right\}$.

25. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|4x+1|^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{|4x+1|^n}\right] = \lim\limits_{n\to\infty}\dfrac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio Test, the series converges when $|4x+1|<1 \quad \Leftrightarrow \quad -1<4x+1<1 \quad \Leftrightarrow \quad -2<4x<0 \quad \Leftrightarrow \quad -\frac{1}{2}<x<0$, so $R=\frac{1}{4}$. When $x=-\frac{1}{2}$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x=0$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$, a convergent p-series $[p=2>1]$. $\quad I=\left[-\frac{1}{2},0\right]$.

27. If $a_n = \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2n+1} = 0 < 1.$$ Thus, by the Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \le 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

31. If $a_n = \dfrac{(n!)^k}{(kn)!}x^n$, then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{[(n+1)!]^k\,(kn)!}{(n!)^k\,[k(n+1)]!}|x| = \lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}|x| \\
&= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x| \\
&= \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x| \\
&= \left(\frac{1}{k}\right)^k|x| < 1 \quad\Leftrightarrow\quad |x| < k^k \text{ for convergence, and the radius of convergence is } R = k^k.
\end{aligned}$$

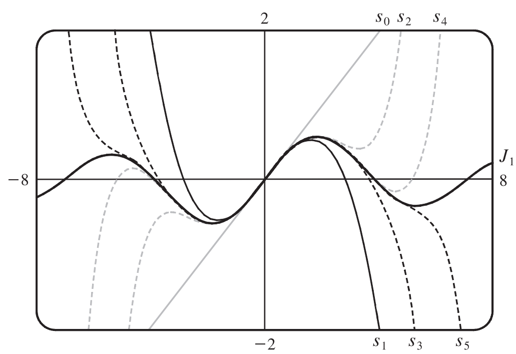
33. No. If a power series is centered at a, its interval of convergence is symmetric about a. If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty, \infty)$, not $[0, \infty)$.

35. (a) If $a_n = \dfrac{(-1)^n\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!\,(n+2)!\,2^{2n+3}} \cdot \frac{n!\,(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0 \text{ for all } x.$$

So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

(b), (c) The initial terms of $J_1(x)$ up to $n = 5$ are $a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$, $a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



37. $s_{2n-1} = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots + x^{2n-2} + 2x^{2n-1}$

$= 1(1+2x) + x^2(1+2x) + x^4(1+2x) + \cdots + x^{2n-2}(1+2x) = (1+2x)(1+x^2+x^4+\cdots+x^{2n-2})$

$= (1+2x)\dfrac{1-x^{2n}}{1-x^2}$ [by (11.2.3)] with $r = x^2$] $\to \dfrac{1+2x}{1-x^2}$ as $n \to \infty$ [by (11.2.4)], when $|x| < 1$.

Also $s_{2n} = s_{2n-1} + x^{2n} \to \dfrac{1+2x}{1-x^2}$ since $x^{2n} \to 0$ for $|x| < 1$. Therefore, $s_n \to \dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n \to \infty$. Thus, the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{1+2x}{1-x^2}$.

39. We use the Root Test on the series $\sum c_n x^n$. We need $\lim\limits_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

41. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 11.2.69, $\sum (c_n + d_n)\, x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)\, x^n$ is 2.

11.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \quad \Leftrightarrow \quad |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

5. $f(x) = \dfrac{2}{3-x} = \dfrac{2}{3}\left(\dfrac{1}{1-x/3}\right) = \dfrac{2}{3} \sum\limits_{n=0}^{\infty} \left(\dfrac{x}{3}\right)^n$ or, equivalently, $2 \sum\limits_{n=0}^{\infty} \dfrac{1}{3^{n+1}} x^n$. The series converges when $\left|\dfrac{x}{3}\right| < 1$, that is, when $|x| < 3$, so $R = 3$ and $I = (-3, 3)$.

7. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9} \sum\limits_{n=0}^{\infty} \left[-\left(\dfrac{x}{3}\right)^2\right]^n = \dfrac{x}{9} \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{9^n} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum\limits_{n=0}^{\infty} \left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when $\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \quad \Leftrightarrow \quad \dfrac{|x^2|}{9} < 1 \quad \Leftrightarrow \quad |x|^2 < 9 \quad \Leftrightarrow \quad |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

9. $f(x) = \dfrac{1+x}{1-x} = (1+x)\left(\dfrac{1}{1-x}\right) = (1+x) \sum\limits_{n=0}^{\infty} x^n = \sum\limits_{n=0}^{\infty} x^n + \sum\limits_{n=0}^{\infty} x^{n+1} = 1 + \sum\limits_{n=1}^{\infty} x^n + \sum\limits_{n=1}^{\infty} x^n = 1 + 2\sum\limits_{n=1}^{\infty} x^n$.

The series converges when $|x| < 1$, so $R = 1$ and $I = (-1, 1)$.

A second approach: $f(x) = \dfrac{1+x}{1-x} = \dfrac{-(1-x)+2}{1-x} = -1 + 2\left(\dfrac{1}{1-x}\right) = -1 + 2\sum\limits_{n=0}^{\infty} x^n = 1 + 2\sum\limits_{n=1}^{\infty} x^n$.

A third approach:

$f(x) = \dfrac{1+x}{1-x} = (1+x)\left(\dfrac{1}{1-x}\right) = (1+x)(1+x+x^2+x^3+\cdots)$

$= (1+x+x^2+x^3+\cdots) + (x+x^2+x^3+x^4+\cdots) = 1 + 2x + 2x^2 + 2x^3 + \cdots = 1 + 2\sum\limits_{n=1}^{\infty} x^n$.

11. $f(x) = \dfrac{3}{x^2 - x - 2} = \dfrac{3}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} \quad \Rightarrow \quad 3 = A(x+1) + B(x-2)$. Let $x = 2$ to get $A = 1$ and $x = -1$ to get $B = -1$. Thus

$$\frac{3}{x^2 - x - 2} = \frac{1}{x-2} - \frac{1}{x+1} = \frac{1}{-2}\left(\frac{1}{1 - (x/2)}\right) - \frac{1}{1-(-x)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n - 1(-1)^n\right]x^n = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-2, 2)$ and the second converges for $(-1, 1)$. Thus, the sum converges for $x \in (-1, 1) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} \quad \text{[from Theorem 2(i)]} \quad = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \text{ with } R = 1.$$

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$$= -\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n \text{ with } R = 1.$$

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R = 1$.

15. $f(x) = \ln(5 - x) = -\displaystyle\int \frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1 - x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$.

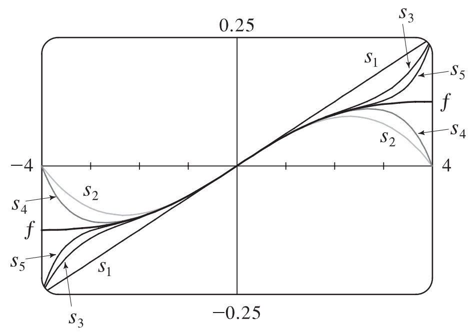
17. $\dfrac{1}{2-x} = \dfrac{1}{2(1 - x/2)} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$ for $\left|\dfrac{x}{2}\right| < 1 \quad \Leftrightarrow \quad |x| < 2$. Now

$$\frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n\right) = \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}x^{n-1} = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n. \text{ So}$$

$f(x) = \dfrac{x^3}{(x-2)^2} = x^3\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^n = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}x^{n+3}$ or $\displaystyle\sum_{n=3}^{\infty}\frac{n-2}{2^{n-1}}x^n$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

19. $f(x) = \dfrac{x}{x^2+16} = \dfrac{x}{16}\left(\dfrac{1}{1-(-x^2/16)}\right) = \dfrac{x}{16}\sum\limits_{n=0}^{\infty}\left(-\dfrac{x^2}{16}\right)^n = \dfrac{x}{16}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{16^n}x^{2n} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{16^{n+1}}x^{2n+1}$.

The series converges when $\left|-x^2/16\right| < 1 \iff x^2 < 16 \iff |x| < 4$, so $R = 4$. The partial sums are $s_1 = \dfrac{x}{16}$, $s_2 = s_1 - \dfrac{x^3}{16^2}$, $s_3 = s_2 + \dfrac{x^5}{16^3}$, $s_4 = s_3 - \dfrac{x^7}{16^4}$, $s_5 = s_4 + \dfrac{x^9}{16^5}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent.



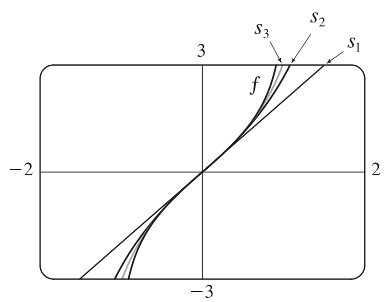
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-4, 4)$.

21. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int\frac{dx}{1+x} + \int\frac{dx}{1-x} = \int\frac{dx}{1-(-x)} + \int\frac{dx}{1-x}$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}x^n\right]dx = \int\left[(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)\right]dx$$

$$= \int(2 + 2x^2 + 2x^4 + \cdots)\,dx = \int\sum_{n=0}^{\infty}2x^{2n}\,dx = C + \sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln\frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \sum\limits_{n=0}^{\infty}\dfrac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2\sum\limits_{n=0}^{\infty}\dfrac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$. The partial sums are $s_1 = \dfrac{2x}{1}$, $s_2 = s_1 + \dfrac{2x^3}{3}$, $s_3 = s_2 + \dfrac{2x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

23. $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t\sum\limits_{n=0}^{\infty}(t^8)^n = \sum\limits_{n=0}^{\infty} t^{8n+1} \quad \Rightarrow \quad \displaystyle\int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}$. The series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1 \quad \Leftrightarrow \quad |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1-t^8}\,dt$ also has $R = 1$.

25. By Example 7, $\tan^{-1} x = \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$x - \tan^{-1} x = x - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots\right) = \dfrac{x^3}{3} - \dfrac{x^5}{5} + \dfrac{x^7}{7} - \cdots = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^{2n+1}}{2n+1}$ and

$\dfrac{x - \tan^{-1} x}{x^3} = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1}$, so

$\displaystyle\int \frac{x - \tan^{-1} x}{x^3}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1}$. By Theorem 2, $R = 1$.

27. $\dfrac{1}{1+x^5} = \dfrac{1}{1-(-x^5)} = \sum\limits_{n=0}^{\infty}(-x^5)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^{5n} \quad \Rightarrow$

$\displaystyle\int \frac{1}{1+x^5}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{5n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+1}}{5n+1}$. Thus,

$I = \displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx = \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \cdots\right]_0^{0.2} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$. So $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$ to six decimal places.

29. We substitute $3x$ for x in Example 7, and find that

$\displaystyle\int x\arctan(3x)\,dx = \int x\sum_{n=0}^{\infty}(-1)^n\frac{(3x)^{2n+1}}{2n+1}\,dx = \int \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}x^{2n+2}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}x^{2n+3}}{(2n+1)(2n+3)}$

So

$$\begin{aligned}\int_0^{0.1} x\arctan(3x)\,dx &= \left[\frac{3x^3}{1\cdot 3} - \frac{3^3x^5}{3\cdot 5} + \frac{3^5x^7}{5\cdot 7} - \frac{3^7x^9}{7\cdot 9} + \cdots\right]_0^{0.1}\\ &= \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} - \frac{2187}{63\times 10^9} + \cdots.\end{aligned}$$

The series is alternating, so if we use three terms, the error is at most $\dfrac{2187}{63\times 10^9} \approx 3.5\times 10^{-8}$. So

$\displaystyle\int_0^{0.1} x\arctan(3x)\,dx \approx \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} \approx 0.000\,983$ to six decimal places.

31. Using the result of Example 6, $\ln(1-x) = -\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$, with $x = -0.1$, we have

$\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \dfrac{0.01}{2} + \dfrac{0.001}{3} - \dfrac{0.0001}{4} + \dfrac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\dfrac{0.00001}{5} = 0.000002$. So $\ln 1.1 \approx 0.1 - \dfrac{0.01}{2} + \dfrac{0.001}{3} - \dfrac{0.0001}{4} \approx 0.09531$.

33. (a) $J_0(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}
x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}\left[(n-1)!\right]^2} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{-1}2^2n^2x^{2n}}{2^{2n}(n!)^2} \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1)+2n-2^2n^2}{2^{2n}(n!)^2}\right] x^{2n} \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2-2n+2n-4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0
\end{aligned}$$

(b) $$\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

35. (a) $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} \quad\Rightarrow\quad f'(x) = \sum\limits_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum\limits_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 9.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

37. If $a_n = \frac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum\limits_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n^2} \quad\Rightarrow\quad f'(x) = \sum\limits_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty} \frac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum\limits_{n=1}^{\infty} \frac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1, 1)$.

39. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have $\dfrac{\pi}{6} = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \dfrac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{3}\right)^n \dfrac{1}{\sqrt{3}} \dfrac{1}{2n+1}$, so

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

11.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

3. Since $f^{(n)}(0) = (n+1)!$, Equation 7 gives the Maclaurin series

$\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \dfrac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (n+1)x^n$. Applying the Ratio Test with $a_n = (n+1)x^n$ gives us

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \lim_{n\to\infty} \dfrac{n+2}{n+1} = |x| \cdot 1 = |x|$. For convergence, we must have $|x| < 1$, so the radius of convergence $R = 1$.

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1-x)^{-2} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + 2x + \tfrac{6}{2}x^2 + \tfrac{24}{6}x^3 + \tfrac{120}{24}x^4 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n\end{aligned}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \lim_{n\to\infty} \frac{n+2}{n+1} = |x|\,(1) = |x| < 1$$

for convergence, so $R = 1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin \pi x$	0
1	$\pi \cos \pi x$	π
2	$-\pi^2 \sin \pi x$	0
3	$-\pi^3 \cos \pi x$	$-\pi^3$
4	$\pi^4 \sin \pi x$	0
5	$\pi^5 \cos \pi x$	π^5
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sin \pi x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= 0 + \pi x + 0 - \frac{\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + \cdots \\ &= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} x^{2n+1}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\pi^{2n+3}\, x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\pi^{2n+1}\, x^{2n+1}}\right| = \lim_{n\to\infty} \frac{\pi^2\, x^2}{(2n+3)(2n+2)} \\ &= 0 < 1 \quad \text{for all } x\text{, so } R = \infty.\end{aligned}$$

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n|x|^n} \right] = \lim_{n\to\infty} \frac{5|x|}{n+1} = 0 < 1$$

for all x, so $R = \infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \qquad \text{so } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Use the Ratio Test to find R. If $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)}$$

$$= 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

13.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
6	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite series expansion about $a = 1$.

$$f(x) = x^4 - 3x^2 + 1 = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

A finite series converges for all x, so $R = \infty$.

15. $f(x) = e^x \;\Rightarrow\; f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum_{n=0}^{\infty} \dfrac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n} \right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

17.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\pi|^{2n}} \right] = \lim_{n\to\infty} \frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\frac{1}{\sqrt{x}} &= \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!} \\ &\quad - \frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!} + \cdots \\ &= \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\cdot n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,|x-9|^n}\right] \\ &= \lim_{n\to\infty}\left[\frac{(2n+1)\,|x-9|}{2\cdot 3^2(n+1)}\right] = \frac{1}{9}|x-9| < 1\end{aligned}$$

for convergence, so $|x-9| < 9$ and $R = 9$.

21. If $f(x) = \sin\pi x$, then $f^{(n+1)}(x) = \pm\pi^{n+1}\sin\pi x$ or $\pm\pi^{n+1}\cos\pi x$. In each case, $\left|f^{(n+1)}(x)\right| \le \pi^{n+1}$, so by Formula 9 with $a = 0$ and $M = \pi^{n+1}$, $|R_n(x)| \le \dfrac{\pi^{n+1}}{(n+1)!}\,|x|^{n+1} = \dfrac{|\pi x|^{n+1}}{(n+1)!}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 7 represents $\sin\pi x$ for all x.

23. If $f(x) = \sinh x$, then for all n, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x, we have $\left|f^{(n+1)}(x)\right| \le \cosh x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| \le \cosh x \le \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \le \dfrac{\cosh d}{(n+1)!}\,|x|^{n+1}$. It follows that $|R_n(x)| \to 0$ as $n \to \infty$ for $|x| \le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x.

25. The general binomial series in (17) is

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2\cdot 2!} + \frac{1\cdot 3\cdot x^3}{2^3\cdot 3!} - \frac{1\cdot 3\cdot 5\cdot x^4}{2^4\cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!} \quad \text{for } |x| < 1, \quad \text{so } R = 1.\end{aligned}$$

27. $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+x/2)]^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3} = \dfrac{1}{8}\displaystyle\sum_{n=0}^{\infty}\binom{-3}{n}\left(\dfrac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot[-(n+2)]}{n!}$$
$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)(n+2)}{2\cdot n!} = \frac{(-1)^n(n+1)(n+2)}{2}$$

Thus, $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\displaystyle\sum_{n=0}^{\infty}\dfrac{(-1)^n(n+1)(n+2)}{2}\dfrac{x^n}{2^n} = \sum_{n=0}^{\infty}\dfrac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\dfrac{x}{2}\right| < 1 \;\Leftrightarrow\; |x| < 2$, so $R = 2$.

29. $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \;\Rightarrow\; f(x) = \sin(\pi x) = \sum_{n=0}^{\infty}(-1)^n\frac{(\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}x^{2n+1}$, $R = \infty$.

31. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \;\Rightarrow\; e^{2x} = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$, so $f(x) = e^x + e^{2x} = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n + \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{2^n+1}{n!}x^n$,
$R = \infty$.

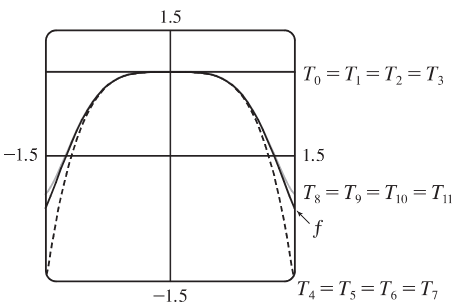
33. $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \;\Rightarrow\; \cos\left(\tfrac{1}{2}x^2\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{2^{2n}(2n)!}$, so
$f(x) = x\cos\left(\tfrac{1}{2}x^2\right) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n}(2n)!}x^{4n+1}$, $R = \infty$.

35. We must write the binomial in the form (1+ expression), so we'll factor out a 4.

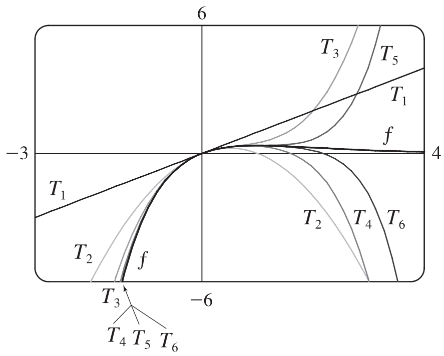
$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$
$$= \frac{x}{2}\left[1 + \left(-\tfrac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$
$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$$
$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \;\Leftrightarrow\; \frac{|x|}{2} < 1 \;\Leftrightarrow\; |x| < 2, \quad \text{so } R = 2.$$

37. $\sin^2 x = \dfrac{1}{2}(1-\cos 2x) = \dfrac{1}{2}\left[1 - \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \dfrac{1}{2}\left[1 - 1 - \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$,
$R = \infty$

39. $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \;\Rightarrow\; f(x) = \cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$



Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.



41. $e^x \overset{(11)}{=} \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-x} = \sum\limits_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^n}{n!}$, so

$$f(x) = xe^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{n+1}$$
$$= x - x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{6}x^4 + \tfrac{1}{24}x^5 - \tfrac{1}{120}x^6 + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{(n-1)!}$$

The series for e^x converges for all x, so the same is true of the series for $f(x)$; that is, $R = \infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.

43. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-0.2} = \sum\limits_{n=0}^{\infty} \dfrac{(-0.2)^n}{n!} = 1 - 0.2 + \dfrac{1}{2!}(0.2)^2 - \dfrac{1}{3!}(0.2)^3 + \dfrac{1}{4!}(0.2)^4 - \dfrac{1}{5!}(0.2)^5 + \dfrac{1}{6!}(0.2)^6 - \cdots$.

But $\dfrac{1}{6!}(0.2)^6 = 8.\overline{8} \times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2} \approx \sum\limits_{n=0}^{5} \dfrac{(-0.2)^n}{n!} \approx 0.81873$, correct to five decimal places.

45. (a) $1/\sqrt{1-x^2} = \left[1 + (-x^2)\right]^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}$$

(b) $\sin^{-1} x = \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$

$$= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1} \quad \text{since } 0 = \sin^{-1} 0 = C.$$

47. $\cos x \overset{(16)}{=} \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \cos(x^3) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^3)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{6n}}{(2n)!} \quad \Rightarrow$

$x\cos(x^3) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{6n+1}}{(2n)!} \quad \Rightarrow \quad \displaystyle\int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R = \infty$.

49. $\cos x \overset{(16)}{=} \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \cos x - 1 = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \dfrac{\cos x - 1}{x} = \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^{2n-1}}{(2n)!} \quad \Rightarrow$

$\displaystyle\int \frac{\cos x - 1}{x}\,dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}$, with $R = \infty$.

51. By Exercise 47, $\displaystyle\int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so

$\displaystyle\int_0^1 x\cos(x^3)\,dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} - \frac{1}{20 \cdot 6!} + \cdots$, but

$\dfrac{1}{20 \cdot 6!} = \dfrac{1}{14{,}400} \approx 0.000\,069$, so $\displaystyle\int_0^1 x\cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

53. $\sqrt{1+x^4} = (1+x^4)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (x^4)^n$, so $\int \sqrt{1+x^4}\,dx = C + \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{x^{4n+1}}{4n+1}$ and hence, since $0.4 < 1$,

we have

$$\begin{aligned} I &= \int_0^{0.4} \sqrt{1+x^4}\,dx = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{(0.4)^{4n+1}}{4n+1} \\ &= (1)\frac{(0.4)^1}{0!} + \frac{\frac{1}{2}}{1!}\frac{(0.4)^5}{5} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{(0.4)^9}{9} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\frac{(0.4)^{13}}{13} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}\frac{(0.4)^{17}}{17} + \cdots \\ &= 0.4 + \frac{(0.4)^5}{10} - \frac{(0.4)^9}{72} + \frac{(0.4)^{13}}{208} - \frac{5(0.4)^{17}}{2176} + \cdots \end{aligned}$$

Now $\frac{(0.4)^9}{72} \approx 3.6 \times 10^{-6} < 5 \times 10^{-6}$, so by the Alternating Series Estimation Theorem, $I \approx 0.4 + \frac{(0.4)^5}{10} \approx 0.40102$ (correct to five decimal places).

55.
$$\begin{aligned} \lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} &= \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3} \\ &= \lim_{x\to 0}\left(\tfrac{1}{3} - \tfrac{1}{5}x^2 + \tfrac{1}{7}x^4 - \cdots\right) = \tfrac{1}{3} \end{aligned}$$

since power series are continuous functions.

57.
$$\begin{aligned} \lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5} \\ &= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots\right) = \frac{1}{5!} = \frac{1}{120} \end{aligned}$$

since power series are continuous functions.

59. From Equation 11, we have $e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$ and we know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x = \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right)$. Writing only the terms with degree ≤ 4, we get $e^{-x^2}\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

61. $\dfrac{x}{\sin x} \overset{(15)}{=} \dfrac{x}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}$.

$$\begin{array}{r|l} & 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots \\ \hline x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots & x \\ & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\ \cline{2-2} & \quad \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots \\ & \quad \frac{1}{6}x^3 - \frac{1}{36}x^5 + \cdots \\ \cline{2-2} & \qquad \frac{7}{360}x^5 + \cdots \\ & \qquad \frac{7}{360}x^5 + \cdots \\ \cline{2-2} & \qquad\qquad \cdots \end{array}$$

From the long division above, $\frac{x}{\sin x} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$.

63. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

65. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, by (15).

67. $\displaystyle 3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots=\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots=\sum_{n=1}^{\infty}\frac{3^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}-1=e^3-1$, by (11).

69. Assume that $|f'''(x)|\le M$, so $f'''(x)\le M$ for $a\le x\le a+d$. Now $\int_a^x f'''(t)\,dt\le\int_a^x M\,dt \quad\Rightarrow$

$f''(x)-f''(a)\le M(x-a) \quad\Rightarrow\quad f''(x)\le f''(a)+M(x-a)$. Thus, $\int_a^x f''(t)\,dt\le\int_a^x[f''(a)+M(t-a)]\,dt \quad\Rightarrow$

$f'(x)-f'(a)\le f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \quad\Rightarrow\quad f'(x)\le f'(a)+f''(a)(x-a)+\frac{1}{2}M(x-a)^2 \quad\Rightarrow$

$\int_a^x f'(t)\,dt\le\int_a^x\left[f'(a)+f''(a)(t-a)+\frac{1}{2}M(t-a)^2\right]dt \quad\Rightarrow$

$f(x)-f(a)\le f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}M(x-a)^3$. So

$f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2\le\frac{1}{6}M(x-a)^3$. But

$R_2(x)=f(x)-T_2(x)=f(x)-f(a)-f'(a)(x-a)-\frac{1}{2}f''(a)(x-a)^2$, so $R_2(x)\le\frac{1}{6}M(x-a)^3$.

A similar argument using $f'''(x)\ge -M$ shows that $R_2(x)\ge-\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)|\le\frac{1}{6}M\,|x-a|^3$.

Although we have assumed that $x>a$, a similar calculation shows that this inequality is also true if $x<a$.

71. (a) $\displaystyle g(x)=\sum_{n=0}^{\infty}\binom{k}{n}x^n \quad\Rightarrow\quad g'(x)=\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}$, so

$$\begin{aligned}(1+x)g'(x)&=(1+x)\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}=\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}+\sum_{n=1}^{\infty}\binom{k}{n}nx^n\\&=\sum_{n=0}^{\infty}\binom{k}{n+1}(n+1)x^n+\sum_{n=0}^{\infty}\binom{k}{n}nx^n \qquad \left[\begin{matrix}\text{Replace } n \text{ with } n+1\\ \text{in the first series}\end{matrix}\right]\\&=\sum_{n=0}^{\infty}(n+1)\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}x^n+\sum_{n=0}^{\infty}\left[(n)\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]x^n\\&=\sum_{n=0}^{\infty}\frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}\left[(k-n)+n\right]x^n\\&=k\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n=k\sum_{n=0}^{\infty}\binom{k}{n}x^n=kg(x)\end{aligned}$$

Thus, $g'(x)=\dfrac{kg(x)}{1+x}$.

(b) $h(x)=(1+x)^{-k}\,g(x) \quad\Rightarrow$

$$\begin{aligned}h'(x)&=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}\,g'(x) &&\text{[Product Rule]}\\&=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}\,\frac{kg(x)}{1+x} &&\text{[from part (a)]}\\&=-k(1+x)^{-k-1}g(x)+k(1+x)^{-k-1}g(x)=0\end{aligned}$$

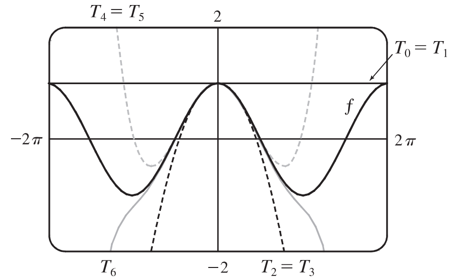
(c) From part (b) we see that $h(x)$ must be constant for $x\in(-1,1)$, so $h(x)=h(0)=1$ for $x\in(-1,1)$.

Thus, $h(x)=1=(1+x)^{-k}\,g(x) \quad\Leftrightarrow\quad g(x)=(1+x)^k$ for $x\in(-1,1)$.

11.11 Applications of Taylor Polynomials

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1 - \frac{1}{2}x^2$
3	$\sin x$	0	$1 - \frac{1}{2}x^2$
4	$\cos x$	1	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
5	$-\sin x$	0	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$
6	$-\cos x$	-1	$1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$



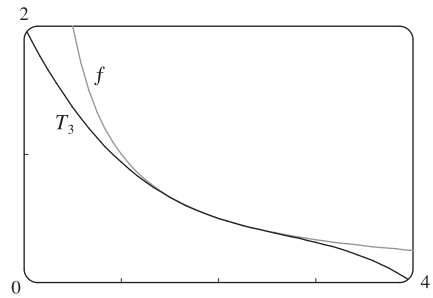
(b)

x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

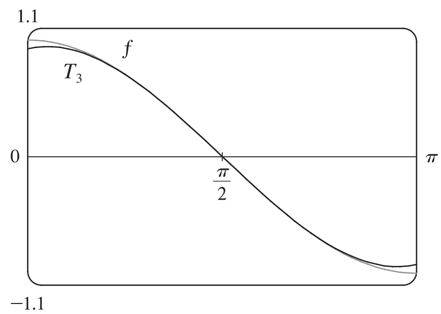
n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1/x$	$\frac{1}{2}$
1	$-1/x^2$	$-\frac{1}{4}$
2	$2/x^3$	$\frac{1}{4}$
3	$-6/x^4$	$-\frac{3}{8}$



$$
\begin{aligned}
T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \frac{\frac{1}{2}}{0!} - \frac{\frac{1}{4}}{1!}(x-2) + \frac{\frac{1}{4}}{2!}(x-2)^2 - \frac{\frac{3}{8}}{3!}(x-2)^3 \\
&= \tfrac{1}{2} - \tfrac{1}{4}(x-2) + \tfrac{1}{8}(x-2)^2 - \tfrac{1}{16}(x-2)^3
\end{aligned}
$$

5.

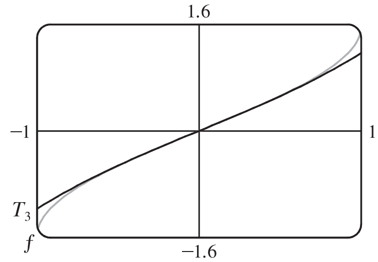
n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1



$$
\begin{aligned}
T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(\pi/2)}{n!}\left(x - \tfrac{\pi}{2}\right)^n \\
&= -\left(x - \tfrac{\pi}{2}\right) + \tfrac{1}{6}\left(x - \tfrac{\pi}{2}\right)^3
\end{aligned}
$$

7.

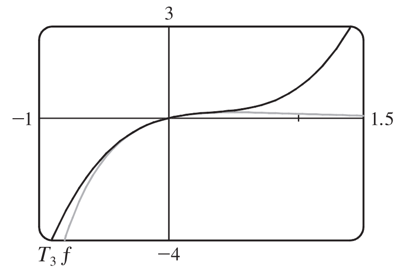
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$\dfrac{1}{\sqrt{1-x^2}}$	1
2	$\dfrac{x}{(1-x^2)^{3/2}}$	0
3	$\dfrac{2x^2+1}{(1-x^2)^{5/2}}$	1



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = x + \frac{x^3}{6}$$

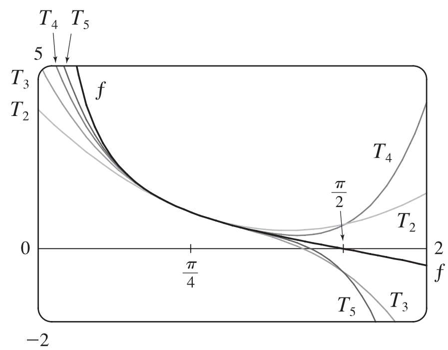
9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = \tfrac{0}{1} \cdot 1 + \tfrac{1}{1}x^1 + \tfrac{-4}{2}x^2 + \tfrac{12}{6}x^3 = x - 2x^2 + 2x^3$$

11. You may be able to simply find the Taylor polynomials for $f(x) = \cot x$ using your CAS. We will list the values of $f^{(n)}(\pi/4)$ for $n = 0$ to $n = 5$.



n	0	1	2	3	4	5
$f^{(n)}(\pi/4)$	1	-2	4	-16	80	-512

$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(\pi/4)}{n!} \left(x - \tfrac{\pi}{4}\right)^n$$

$$= 1 - 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 - \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4 - \tfrac{64}{15}\left(x - \tfrac{\pi}{4}\right)^5$$

For $n = 2$ to $n = 5$, $T_n(x)$ is the polynomial consisting of all the terms up to and including the $\left(x - \frac{\pi}{4}\right)^n$ term.

13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

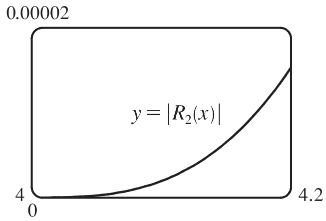
(a) $f(x) = \sqrt{x} \approx T_2(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1/32}{2!}(x-4)^2$

$$= 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2$$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $4 \le x \le 4.2 \Rightarrow |x-4| \le 0.2 \Rightarrow |x-4|^3 \le 0.008$. Since $f'''(x)$ is decreasing on $[4, 4.2]$, we can take $M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}$, so

$$|R_2(x)| \le \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000\,015\,625.$$

(c)



From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

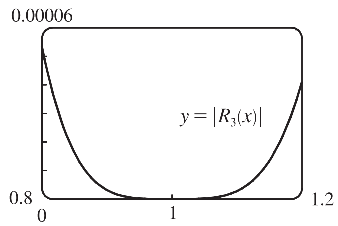
(a) $f(x) = x^{2/3} \approx T_3(x) = 1 + \frac{2}{3}(x-1) - \dfrac{2/9}{2!}(x-1)^2 + \dfrac{8/27}{3!}(x-1)^3$

$$= 1 + \tfrac{2}{3}(x-1) - \tfrac{1}{9}(x-1)^2 + \tfrac{4}{81}(x-1)^3$$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.8 \le x \le 1.2 \;\Rightarrow\; |x-1| \le 0.2 \;\Rightarrow\; |x-1|^4 \le 0.0016$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8, 1.2]$, we can take $M = \left|f^{(4)}(0.8)\right| = \frac{56}{81}(0.8)^{-10/3}$, so

$$|R_3(x)| \le \frac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.000\,096\,97.$$

(c)



From the graph of $|R_3(x)| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than $0.000\,053\,3$ on $[0.8, 1.2]$.

17.

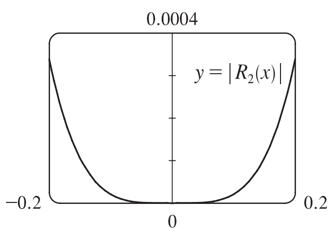
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x\,(2\sec^2 x - 1)$	1
3	$\sec x \tan x\,(6\sec^2 x - 1)$	

(a) $f(x) = \sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$. Now $-0.2 \le x \le 0.2 \;\Rightarrow\; |x| \le 0.2 \;\Rightarrow\; |x|^3 \le (0.2)^3$.

$f^{(3)}(x)$ is an odd function and it is increasing on $[0, 0.2]$ since $\sec x$ and $\tan x$ are increasing on $[0, 0.2]$, so $\left|f^{(3)}(x)\right| \le f^{(3)}(0.2) \approx 1.085\,158\,892$. Thus, $|R_2(x)| \le \dfrac{f^{(3)}(0.2)}{3!}(0.2)^3 \approx 0.001\,447$.

(c)



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it seems that the error is less than $0.000\,339$ on $[-0.2, 0.2]$.

19.

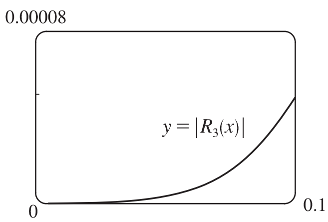
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

(a) $f(x)=e^{x^2}\approx T_3(x)=1+\dfrac{2}{2!}x^2=1+x^2$

(b) $|R_3(x)|\le\dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right|\le M$. Now $0\le x\le 0.1\quad\Rightarrow$ $x^4\le(0.1)^4$, and letting $x=0.1$ gives

$$|R_3(x)|\le\frac{e^{0.01}\,(12+0.48+0.0016)}{24}(0.1)^4\approx 0.00006.$$

(c)



From the graph of $|R_3(x)|=\left|e^{x^2}-T_3(x)\right|$, it appears that the error is less than 0.000 051 on $[0,0.1]$.

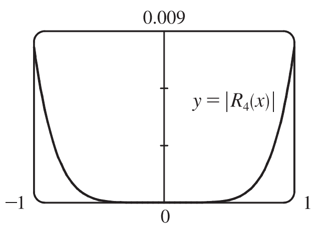
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\sin x$	0
1	$\sin x+x\cos x$	0
2	$2\cos x-x\sin x$	2
3	$-3\sin x-x\cos x$	0
4	$-4\cos x+x\sin x$	-4
5	$5\sin x+x\cos x$	

(a) $f(x)=x\sin x\approx T_4(x)=\dfrac{2}{2!}(x-0)^2+\dfrac{-4}{4!}(x-0)^4=x^2-\dfrac{1}{6}x^4$

(b) $|R_4(x)|\le\dfrac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right|\le M$. Now $-1\le x\le 1\quad\Rightarrow$ $|x|\le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right|\le 5$ for $-1\le x\le 1$. Thus, we can take $M=5$ and get $|R_4(x)|\le\dfrac{5}{5!}\cdot 1^5=\dfrac{1}{24}=0.041\overline{6}$.

(c)



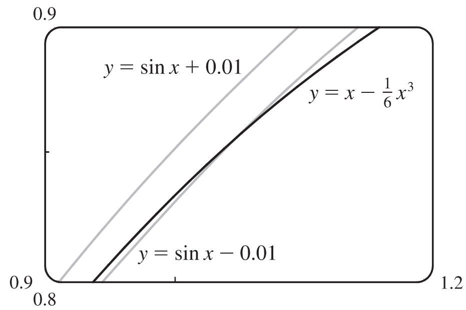
From the graph of $|R_4(x)|=|x\sin x-T_4(x)|$, it seems that the error is less than 0.0082 on $[-1,1]$.

23. From Exercise 5, $\cos x=-\left(x-\frac{\pi}{2}\right)+\frac{1}{6}\left(x-\frac{\pi}{2}\right)^3+R_3(x)$, where $|R_3(x)|\le\dfrac{M}{4!}\left|x-\frac{\pi}{2}\right|^4$ with $\left|f^{(4)}(x)\right|=|\cos x|\le M=1$. Now $x=80^\circ=(90^\circ-10^\circ)=\left(\frac{\pi}{2}-\frac{\pi}{18}\right)=\frac{4\pi}{9}$ radians, so the error is $\left|R_3\left(\frac{4\pi}{9}\right)\right|\le\frac{1}{24}\left(\frac{\pi}{18}\right)^4\approx 0.000\,039$, which means our estimate would *not* be accurate to five decimal places. However, $T_3=T_4$, so we can use $\left|R_4\left(\frac{4\pi}{9}\right)\right|\le\frac{1}{120}\left(\frac{\pi}{18}\right)^5\approx 0.000\,001$. Therefore, to five decimal places, $\cos 80^\circ\approx-\left(-\frac{\pi}{18}\right)+\frac{1}{6}\left(-\frac{\pi}{18}\right)^3\approx 0.17365$.

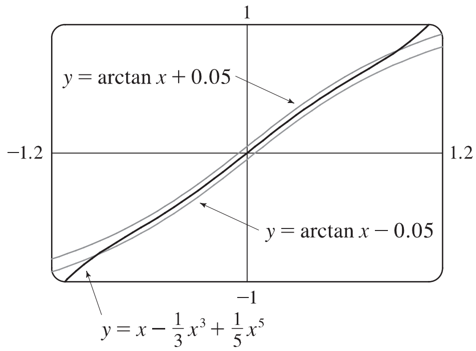
25. All derivatives of e^x are e^x, so $|R_n(x)|\le\dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0<x<0.1$. Letting $x=0.1$, $R_n(0.1)\le\dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1}<0.00001$, and by trial and error we find that $n=3$ satisfies this inequality since

$R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$, and 3, we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1} \approx 1.10517$.)

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow$ $|x^5| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.



29. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{7}x^7\right| < 0.05 \Leftrightarrow$ $|x^7| < 0.35 \Leftrightarrow |x| < (0.35)^{1/7} \approx 0.8607$. The curves $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ and $y = \arctan x + 0.05$ intersect at $x \approx 0.9245$, so the graph confirms our estimate. Since both the arctangent function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-0.86 < x < 0.86$.



31. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance traveled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be $140\text{ m/s} \approx 313\text{ mi/h}$!).

33. $E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - \frac{q}{D^2(1+d/D)^2} = \frac{q}{D^2}\left[1 - \left(1 + \frac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right] = \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right]$$

$$\approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

35. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad \Leftrightarrow \quad v \approx \sqrt{gL/(2\pi)}$.

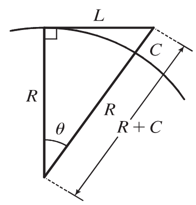
(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi} \cdot \dfrac{2\pi d}{L} \quad \Leftrightarrow \quad v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$.

If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi \cdot \dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\dfrac{gL}{2\pi} \cdot \dfrac{\pi^3}{375} \approx 0.0132gL$.

37. (a) L is the length of the arc subtended by the angle θ, so $L = R\theta \quad \Rightarrow$ $\theta = L/R$. Now $\sec\theta = (R + C)/R \quad \Rightarrow \quad R\sec\theta = R + C \quad \Rightarrow$ $C = R\sec\theta - R = R\sec(L/R) - R$.



(b) Extending the result in Exercise 17, we have $f^{(4)}(x) = \sec x\,(18\sec^2 x\,\tan^2 x + 6\sec^4 x - \sec^2 x - \tan^2 x)$, so $f^{(4)}(0) = 5$, and $\sec x \approx T_4(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. By part (a),

$$C \approx R\left[1 + \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4\right] - R = R + \frac{1}{2}R \cdot \frac{L^2}{R^2} + \frac{5}{24}R \cdot \frac{L^4}{R^4} - R = \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

(c) Taking $L = 100$ km and $R = 6370$ km, the formula in part (a) says that $C = R\sec(L/R) - R = 6370\sec(100/6370) - 6370 \approx 0.785\,009\,965\,44$ km.

The formula in part (b) says that $C \approx \dfrac{L^2}{2R} + \dfrac{5L^4}{24R^3} = \dfrac{100^2}{2 \cdot 6370} + \dfrac{5 \cdot 100^4}{24 \cdot 6370^3} \approx 0.785\,009\,957\,36$ km.

The difference between these two results is only $0.000\,000\,008\,08$ km, or $0.000\,008\,08$ m!

39. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f, so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side gives us $f'(x_n)(x_n - r) - f(x_n) = R_1(r)$. Dividing by $f'(x_n)$, we get $x_n - r - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{R_1(r)}{f'(x_n)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left|\dfrac{R_1(r)}{f'(x_n)}\right|$. Taylor's Inequality gives us $|R_1(r)| \le \dfrac{|f''(r)|}{2!}\,|r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \le M$ and $|f'(x)| \ge K$ gives us $|x_{n+1} - r| \le \dfrac{M}{2K}\,|x_n - r|^2$.

11 Review

CONCEPT CHECK

1. (a) See Definition 11.1.1.

(b) See Definition 11.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 11.1.11.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 11.1.12, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 11.2.

(b) The p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent if $p > 1$.

4. If $\sum a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. (a) *Test for Divergence:* If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) *Integral Test:* Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:
(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) *Comparison Test:* Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

(d) *Limit Comparison Test:* Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} (a_n/b_n) = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge.

(e) *Alternating Series Test:* If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad [b_n > 0]$ satisfies (i) $b_{n+1} \leq b_n$ for all n and (ii) $\lim\limits_{n\to\infty} b_n = 0$, then the series is convergent.

(f) *Ratio Test:*
(i) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(ii) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

(g) *Root Test:*

(i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive..

6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

(b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

(c) A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

7. (a) Use (3) in Section 11.3.

(b) See Example 5 in Section 11.4.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 712.

8. (a) $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$

(b) Given the power series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 11.9.2.

10. (a) $T_n(x) = \sum\limits_{i=0}^{n} \dfrac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 11.10.8.

(e) See Taylor's Inequality (11.10.9).

11. (a)–(e) See Table 1 on page 743.

12. See the binomial series (11.10.17) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 11.2.6.

3. True. If $\lim_{n\to\infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. If $n > N$, then $2n+1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim_{n\to\infty} a_{2n+1} = L$.

5. False. For example, take $c_n = (-1)^n/(n6^n)$.

7. False, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^3}\cdot\frac{n^3}{1}\right| = \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\cdot\frac{1/n^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1$.

9. False. See the note after Example 2 in Section 11.4.

11. True. See (9) in Section 11.1.

13. True. By Theorem 11.10.5 the coefficient of x^3 is $\frac{f'''(0)}{3!} = \frac{1}{3} \quad\Rightarrow\quad f'''(0) = 2$.

Or: Use Theorem 11.9.2 to differentiate f three times.

15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_nb_n = 1$, so $\{a_nb_n\}$ is convergent.

17. True by Theorem 11.6.3. $\left[\sum(-1)^n a_n\right.$ is absolutely convergent and hence convergent.$\left.\right]$

19. True. $0.99999\ldots = 0.9 + 0.9(0.1)^1 + 0.9(0.1)^2 + 0.9(0.1)^3 + \cdots = \sum_{n=1}^{\infty}(0.9)(0.1)^{n-1} = \frac{0.9}{1-0.1} = 1$ by the formula for the sum of a geometric series $[S = a_1/(1-r)]$ with ratio r satisfying $|r| < 1$.

EXERCISES

1. $\left\{\frac{2+n^3}{1+2n^3}\right\}$ converges since $\lim_{n\to\infty}\frac{2+n^3}{1+2n^3} = \lim_{n\to\infty}\frac{2/n^3+1}{1/n^3+2} = \frac{1}{2}$.

3. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^3}{1+n^2} = \lim_{n\to\infty}\frac{n}{1/n^2+1} = \infty$, so the sequence diverges.

5. $|a_n| = \left|\frac{n\sin n}{n^2+1}\right| \le \frac{n}{n^2+1} < \frac{1}{n}$, so $|a_n| \to 0$ as $n\to\infty$. Thus, $\lim_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

7. $\left\{\left(1+\frac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y = \left(1+\frac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}4x\ln(1+3/x) = \lim_{x\to\infty}\frac{\ln(1+3/x)}{1/(4x)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+3/x}\left(-\frac{3}{x^2}\right)}{-1/(4x^2)} = \lim_{x\to\infty}\frac{12}{1+3/x} = 12, \text{ so}$$

$$\lim_{x\to\infty} y = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n} = e^{12}.$$

9. We use induction, hypothesizing that $a_{n-1} < a_n < 2$. Note first that $1 < a_2 = \frac{1}{3}(1+4) = \frac{5}{3} < 2$, so the hypothesis holds for $n = 2$. Now assume that $a_{k-1} < a_k < 2$. Then $a_k = \frac{1}{3}(a_{k-1}+4) < \frac{1}{3}(a_k+4) < \frac{1}{3}(2+4) = 2$. So $a_k < a_{k+1} < 2$, and the induction is complete. To find the limit of the sequence, we note that $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \quad\Rightarrow$ $L = \frac{1}{3}(L+4) \quad\Rightarrow\quad L = 2$.

11. $\dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $[p = 2 > 1]$.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^3}{5^{n+1}} \cdot \dfrac{5^n}{n^3}\right] = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^3 \cdot \dfrac{1}{5} = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{5^n}$ converges by the Ratio Test.

15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \left[u = \ln x,\, du = \tfrac{1}{x}\,dx\right] \quad = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-1/2}\,du = \lim_{t\to\infty} \left[2\sqrt{u}\right]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty,$$

so the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

17. $|a_n| = \left|\dfrac{\cos 3n}{1 + (1.2)^n}\right| \le \dfrac{1}{1 + (1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6}\right)^n$, so $\sum\limits_{n=1}^{\infty} |a_n|$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ $\left[r = \frac{5}{6} < 1\right]$. It follows that $\sum\limits_{n=1}^{\infty} a_n$ converges (by Theorem 3 in Section 11.6).

19. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!} \cdot \dfrac{5^n\, n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \lim\limits_{n\to\infty} \dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series converges by the Ratio Test.

21. $b_n = \dfrac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

23. Consider the series of absolute values: $\sum\limits_{n=1}^{\infty} n^{-1/3}$ is a p-series with $p = \frac{1}{3} \le 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \dfrac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\, n^{-1/3}$ converges. Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\, n^{-1/3}$ is conditionally convergent.

25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)3^{n+1}}{2^{2n+3}} \cdot \dfrac{2^{2n+1}}{(-1)^n(n+1)3^n}\right| = \dfrac{n+2}{n+1} \cdot \dfrac{3}{4} = \dfrac{1 + (2/n)}{1 + (1/n)} \cdot \dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n \to \infty$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n(n+1)3^n}{2^{2n+1}}$ is absolutely convergent.

27. $$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(2^3)^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^n} = \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^{n-1}} = \frac{1}{8} \sum_{n=1}^{\infty} \left(-\frac{3}{8}\right)^{n-1} = \frac{1}{8}\left(\frac{1}{1 - (-3/8)}\right)$$

$$= \frac{1}{8} \cdot \frac{8}{11} = \frac{1}{11}$$

29. $\displaystyle\sum_{n=1}^{\infty}[\tan^{-1}(n+1)-\tan^{-1}n] = \lim_{n\to\infty} s_n$

$$= \lim_{n\to\infty}[(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+\cdots+(\tan^{-1}(n+1)-\tan^{-1}n)]$$

$$= \lim_{n\to\infty}[\tan^{-1}(n+1)-\tan^{-1}1] = \tfrac{\pi}{2}-\tfrac{\pi}{4}=\tfrac{\pi}{4}$$

31. $\displaystyle 1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{e^n}{n!}=\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}=e^{-e}$ since $\displaystyle e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

33. $\displaystyle \cosh x=\frac{1}{2}(e^x+e^{-x})=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$

$$=\frac{1}{2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots\right)\right]$$

$$=\frac{1}{2}\left(2+2\cdot\frac{x^2}{2!}+2\cdot\frac{x^4}{4!}+\cdots\right)=1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!}\geq 1+\frac{1}{2}x^2 \quad \text{for all } x$$

35. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}=1-\frac{1}{32}+\frac{1}{243}-\frac{1}{1024}+\frac{1}{3125}-\frac{1}{7776}+\frac{1}{16{,}807}-\frac{1}{32{,}768}+\cdots.$

Since $\displaystyle b_8=\frac{1}{8^5}=\frac{1}{32{,}768}<0.000031$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\approx\sum_{n=1}^{7}\frac{(-1)^{n+1}}{n^5}\approx 0.9721.$

37. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+5^n}\approx\sum_{n=1}^{8}\frac{1}{2+5^n}\approx 0.18976224$. To estimate the error, note that $\displaystyle\frac{1}{2+5^n}<\frac{1}{5^n}$, so the remainder term is

$\displaystyle R_8=\sum_{n=9}^{\infty}\frac{1}{2+5^n}<\sum_{n=9}^{\infty}\frac{1}{5^n}=\frac{1/5^9}{1-1/5}=6.4\times10^{-7}$ [geometric series with $a=\frac{1}{5^9}$ and $r=\frac{1}{5}$].

39. Use the Limit Comparison Test. $\displaystyle\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{n}\right)a_n}{a_n}\right|=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1>0.$

Since $\sum|a_n|$ is convergent, so is $\displaystyle\sum\left|\left(\frac{n+1}{n}\right)a_n\right|$, by the Limit Comparison Test.

41. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\frac{n\,4^n}{|x+2|^n}\right]=\lim_{n\to\infty}\left[\frac{n}{n+1}\,\frac{|x+2|}{4}\right]=\frac{|x+2|}{4}<1 \;\Leftrightarrow\; |x+2|<4$, so $R=4$.

$|x+2|<4 \;\Leftrightarrow\; -4<x+2<4 \;\Leftrightarrow\; -6<x<2$. If $x=-6$, then the series $\displaystyle\sum_{n=1}^{\infty}\frac{(x+2)^n}{n\,4^n}$ becomes

$\displaystyle\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x=2$, the series becomes the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges. Thus, $I=[-6,2)$.

43. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{2^n(x-3)^n}\right|=2\,|x-3|\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}}=2\,|x-3|<1 \;\Leftrightarrow\; |x-3|<\tfrac{1}{2}$,

so $R=\frac{1}{2}$. $|x-3|<\frac{1}{2} \;\Leftrightarrow\; -\frac{1}{2}<x-3<\frac{1}{2} \;\Leftrightarrow\; \frac{5}{2}<x<\frac{7}{2}$. For $x=\frac{7}{2}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes

$\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}}=\sum_{n=3}^{\infty}\frac{1}{n^{1/2}}$, which diverges $\left[p=\frac{1}{2}\leq 1\right]$, but for $x=\frac{5}{2}$, we get $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I=\left[\frac{5}{2},\frac{7}{2}\right)$.

45.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sin x &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots \\ &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots\right] \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\end{aligned}$$

47. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1 \quad\Rightarrow\quad \dfrac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R = 1$.

49. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1 \quad\Rightarrow\quad \ln(1-x) = -\displaystyle\int \frac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n\, dx = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$.

$\ln(1-0) = C - 0 \quad\Rightarrow\quad C = 0 \quad\Rightarrow\quad \ln(1-x) = -\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{-x^n}{n}$ with $R = 1$.

51. $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad\Rightarrow\quad \sin(x^4) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

53. $f(x) = \dfrac{1}{\sqrt[4]{16-x}} = \dfrac{1}{\sqrt[4]{16(1-x/16)}} = \dfrac{1}{\sqrt[4]{16}\left(1-\frac{1}{16}x\right)^{1/4}} = \frac{1}{2}\left(1-\frac{1}{16}x\right)^{-1/4}$

$$\begin{aligned}&= \frac{1}{2}\left[1 + \left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\frac{x}{16}\right)^3 + \cdots\right] \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1\cdot 5\cdot 9\cdot \cdots \cdot (4n-3)}{2\cdot 4^n\cdot n!\cdot 16^n}x^n = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1\cdot 5\cdot 9\cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!}x^n\end{aligned}$$

for $\left|-\dfrac{x}{16}\right| < 1 \quad\Leftrightarrow\quad |x| < 16$, so $R = 16$.

55. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $\dfrac{e^x}{x} = \dfrac{1}{x}\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n-1}}{n!} = x^{-1} + \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$ and

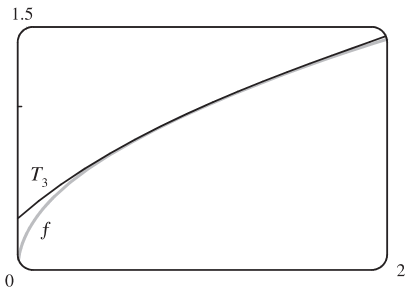
$\displaystyle\int \frac{e^x}{x}\,dx = C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$.

57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sqrt{x} \approx T_3(x) &= 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3\end{aligned}$$

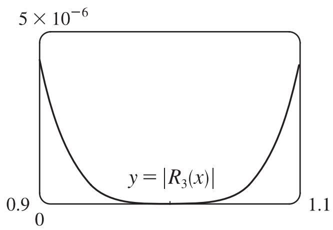
(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \quad \Rightarrow$ $-0.1 \le x - 1 \le 0.1 \quad \Rightarrow \quad (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$\begin{aligned}|R_3(x)| \le \frac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 &\approx 0.000\,005\,648 \\ &\approx 0.000\,006 = 6 \times 10^{-6}\end{aligned}$$

(d)



From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

59. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, so $\sin x - x = -\dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and $\dfrac{\sin x - x}{x^3} = -\dfrac{1}{3!} + \dfrac{x^2}{5!} - \dfrac{x^4}{7!} + \cdots$. Thus, $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^3} = \lim\limits_{x \to 0}\left(-\dfrac{1}{6} + \dfrac{x^2}{120} - \dfrac{x^4}{5040} + \cdots\right) = -\dfrac{1}{6}$.

61. $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad f(-x) = \sum\limits_{n=0}^{\infty} c_n(-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum\limits_{n=0}^{\infty} (-1)^n c_n x^n = \sum\limits_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 11.10.5), so $(-1)^n c_n = -c_n$.

If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0, that is, $c_0 = c_2 = c_4 = \cdots = 0$.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum\limits_{n=0}^{\infty} (-1)^n c_n x^n = \sum\limits_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad (-1)^n c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \cdots = 0$.

PROBLEMS PLUS

1. It would be far too much work to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. We start with the Maclaurin series for sin: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$. Then $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \cdots$, and so the coefficient of x^{15} is $\dfrac{f^{(15)}(0)}{15!} = \dfrac{1}{5!}$. Therefore, $f^{(15)}(0) = \dfrac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10{,}897{,}286{,}400$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, so $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta}$ $\Rightarrow$

$2\cot 2\theta = \dfrac{1 - \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or $\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of x, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}\tan\dfrac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\qquad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad \text{[telescoping sum]} \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n\sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$ for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\cot x + \frac{1}{2^n}\cot\frac{x}{2^n}\right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

5. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3 \cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
$\vdots$	$\vdots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is $A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is $A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\dfrac{4}{9}$, so $A = a + \dfrac{a/3}{1 - \frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \dfrac{1}{2} \cdot 1 \cdot \sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\dfrac{8}{5} \cdot \dfrac{\sqrt{3}}{4} = \dfrac{2\sqrt{3}}{5}$.

7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\, \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy}$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a-b)) = \arctan \dfrac{x-y}{1+xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

(b) From part (a) we have

$$\arctan \tfrac{120}{119} - \arctan \tfrac{1}{239} = \arctan \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \arctan \frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$. So

$$4 \arctan \tfrac{1}{5} = 2\left(\arctan \tfrac{1}{5} + \arctan \tfrac{1}{5}\right) = 2 \arctan \frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \cdot \frac{1}{5}} = 2 \arctan \tfrac{5}{12} = \arctan \tfrac{5}{12} + \arctan \tfrac{5}{12}$$

$$= \arctan \frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \cdot \frac{5}{12}} = \arctan \tfrac{120}{119}$$

Thus, from part (b), we have $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 11.9 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} - \frac{1}{11 \cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan \frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan \dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3 \cdot 239^3} + \dfrac{1}{5 \cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan \frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan \frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$, so from parts (d) and (e) we have

$$16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \quad \Rightarrow$$

$3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

9. We start with the geometric series $\sum\limits_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$

for $|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{d}{dx} \frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - (x^2+x)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad \Rightarrow$$

$\sum\limits_{n=1}^{\infty} n^3 x^n = \dfrac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3 (\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

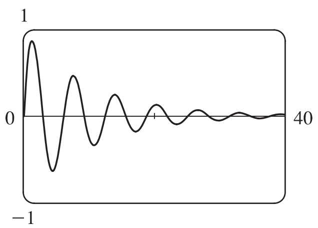
11.
$$\begin{aligned} \ln\left(1 - \frac{1}{n^2}\right) &= \ln\left(\frac{n^2-1}{n^2}\right) = \ln \frac{(n+1)(n-1)}{n^2} = \ln[(n+1)(n-1)] - \ln n^2 \\ &= \ln(n+1) + \ln(n-1) - 2\ln n = \ln(n-1) - \ln n - \ln n + \ln(n+1) \\ &= \ln \frac{n-1}{n} - [\ln n - \ln(n+1)] = \ln \frac{n-1}{n} - \ln \frac{n}{n+1}. \end{aligned}$$

Let $s_k = \sum\limits_{n=2}^{k} \ln\left(1 - \dfrac{1}{n^2}\right) = \sum\limits_{n=2}^{k} \left(\ln \dfrac{n-1}{n} - \ln \dfrac{n}{n+1}\right)$ for $k \geq 2$. Then

$$s_k = \left(\ln \frac{1}{2} - \ln \frac{2}{3}\right) + \left(\ln \frac{2}{3} - \ln \frac{3}{4}\right) + \cdots + \left(\ln \frac{k-1}{k} - \ln \frac{k}{k+1}\right) = \ln \frac{1}{2} - \ln \frac{k}{k+1}, \text{ so}$$

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(\ln \frac{1}{2} - \ln \frac{k}{k+1}\right) = \ln \frac{1}{2} - \ln 1 = \ln 1 - \ln 2 - \ln 1 = -\ln 2.$$

13. (a)



The x-intercepts of the curve occur where $\sin x = 0 \quad \Leftrightarrow \quad x = n\pi$, n an integer. So using the formula for disks (and either a CAS or $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and Formula 99 to evaluate the integral), the volume of the nth bead is

$$\begin{aligned} V_n &= \pi \int_{(n-1)\pi}^{n\pi} (e^{-x/10} \sin x)^2 \, dx = \pi \int_{(n-1)\pi}^{n\pi} e^{-x/5} \sin^2 x \, dx \\ &= \tfrac{250\pi}{101} \left(e^{-(n-1)\pi/5} - e^{-n\pi/5}\right) \end{aligned}$$

(b) The total volume is

$$\pi \int_0^\infty e^{-x/5}\sin^2 x\,dx = \sum_{n=1}^{\infty} V_n = \tfrac{250\pi}{101}\sum_{n=1}^{\infty}\left[e^{-(n-1)\pi/5} - e^{-n\pi/5}\right] = \tfrac{250\pi}{101} \quad \text{[telescoping sum]}.$$

Another method: If the volume in part (a) has been written as $V_n = \frac{250\pi}{101}e^{-n\pi/5}(e^{\pi/5}-1)$, then we recognize $\sum\limits_{n=1}^{\infty} V_n$ as a geometric series with $a = \frac{250\pi}{101}(1-e^{-\pi/5})$ and $r = e^{-\pi/5}$.

15. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L\cdot\frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$.

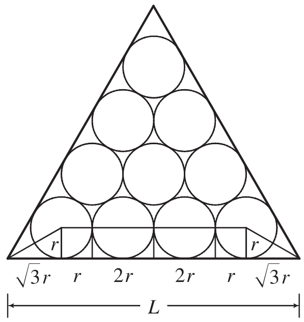
Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that

$$L = \sqrt{3}\,r + r + (n-2)(2r) + r + \sqrt{3}\,r = r\big(2n - 2 + 2\sqrt{3}\big), \text{ so } r = \frac{L}{2\big(n+\sqrt{3}-1\big)}.$$

The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, and so the total area of the circles is

$$\begin{aligned} A_n &= \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\,\pi\,\frac{L^2}{4\big(n+\sqrt{3}-1\big)^2} \\ &= \frac{n(n+1)}{2}\,\pi\,\frac{4A/\sqrt{3}}{4\big(n+\sqrt{3}-1\big)^2} = \frac{n(n+1)}{\big(n+\sqrt{3}-1\big)^2}\,\frac{\pi A}{2\sqrt{3}} \quad\Rightarrow \end{aligned}$$

$$\begin{aligned} \frac{A_n}{A} &= \frac{n(n+1)}{\big(n+\sqrt{3}-1\big)^2}\,\frac{\pi}{2\sqrt{3}} \\ &= \frac{1+1/n}{\big[1+\big(\sqrt{3}-1\big)/n\big]^2}\,\frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \quad \text{as } n\to\infty \end{aligned}$$



17. As in Section 11.9 we have to integrate the function x^x by integrating series. Writing $x^x = (e^{\ln x})^x = e^{x\ln x}$ and using the Maclaurin series for e^x, we have $x^x = (e^{\ln x})^x = e^{x\ln x} = \sum\limits_{n=0}^{\infty}\dfrac{(x\ln x)^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{x^n(\ln x)^n}{n!}$. As with power series, we can integrate this series term-by-term: $\displaystyle\int_0^1 x^x\,dx = \sum_{n=0}^{\infty}\int_0^1 \frac{x^n(\ln x)^n}{n!}\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^n(\ln x)^n\,dx$. We integrate by parts with $u = (\ln x)^n$, $dv = x^n\,dx$, so $du = \dfrac{n(\ln x)^{n-1}}{x}\,dx$ and $v = \dfrac{x^{n+1}}{n+1}$:

$$\begin{aligned} \int_0^1 x^n(\ln x)^n\,dx &= \lim_{t\to0^+}\int_t^1 x^n(\ln x)^n\,dx = \lim_{t\to0^+}\left[\frac{x^{n+1}}{n+1}(\ln x)^n\right]_t^1 - \lim_{t\to0^+}\int_t^1 \frac{n}{n+1}x^n(\ln x)^{n-1}\,dx \\ &= 0 - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}\,dx \end{aligned}$$

(where l'Hospital's Rule was used to help evaluate the first limit). Further integration by parts gives

$\displaystyle\int_0^1 x^n(\ln x)^k\,dx = -\frac{k}{n+1}\int_0^1 x^n(\ln x)^{k-1}\,dx$ and, combining these steps, we get

$$\int_0^1 x^n(\ln x)^n\,dx = \frac{(-1)^n\,n!}{(n+1)^n}\int_0^1 x^n\,dx = \frac{(-1)^n\,n!}{(n+1)^{n+1}} \quad\Rightarrow$$

$$\int_0^1 x^x\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^n(\ln x)^n\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{(-1)^n\,n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}.$$

19. Let $f(x) = \sum\limits_{m=0}^{\infty} c_m x^m$ and $g(x) = e^{f(x)} = \sum\limits_{n=0}^{\infty} d_n x^n$. Then $g'(x) = \sum\limits_{n=0}^{\infty} n d_n x^{n-1}$, so nd_n occurs as the coefficient of x^{n-1}. But also

$$\begin{aligned} g'(x) = e^{f(x)} f'(x) &= \left(\sum_{n=0}^{\infty} d_n x^n\right)\left(\sum_{m=1}^{\infty} m c_m x^{m-1}\right) \\ &= \left(d_0 + d_1 x + d_2 x^2 + \cdots + d_{n-1}x^{n-1} + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots\right) \end{aligned}$$

so the coefficient of x^{n-1} is $c_1 d_{n-1} + 2c_2 d_{n-2} + 3c_3 d_{n-3} + \cdots + nc_n d_0 = \sum\limits_{i=1}^{n} i c_i d_{n-i}$. Therefore, $nd_n = \sum\limits_{i=1}^{n} i c_i d_{n-i}$.

21. Call the series S. We group the terms according to the number of digits in their denominators:

$$S = \underbrace{\left(\tfrac{1}{1} + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + \tfrac{1}{9}\right)}_{g_1} + \underbrace{\left(\tfrac{1}{11} + \cdots + \tfrac{1}{99}\right)}_{g_2} + \underbrace{\left(\tfrac{1}{111} + \cdots + \tfrac{1}{999}\right)}_{g_3} + \cdots$$

Now in the group g_n, since we have 9 choices for each of the n digits in the denominator, there are 9^n terms. Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$ [except for the first term in g_1]. So $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$.

Now $\sum\limits_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$. Therefore, by the Comparison Test,

$S = \sum\limits_{n=1}^{\infty} g_n < \sum\limits_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} = \frac{9}{1 - 9/10} = 90.$

23. $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$.

Use the Ratio Test to show that the series for u, v, and w have positive radii of convergence (∞ in each case), so Theorem 11.9.2 applies, and hence, we may differentiate each of these series:

$$\frac{du}{dx} = \frac{3x^2}{3!} + \frac{6x^5}{6!} + \frac{9x^8}{9!} + \cdots = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots = w$$

Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$.

So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\begin{aligned} \frac{d}{dx}(u^3 + v^3 + w^3 - 3uvw) &= 3u^2u' + 3v^2v' + 3w^2w' - 3(u'vw + uv'w + uvw') \\ &= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \quad \Rightarrow \end{aligned}$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \quad \Rightarrow \quad C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

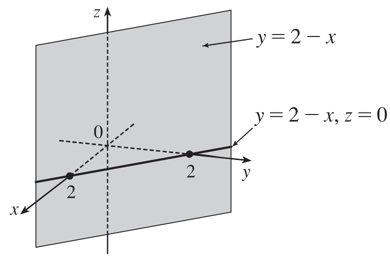
13 □ VECTORS AND THE GEOMETRY OF SPACE □ ET 12

13.1 Three-Dimensional Coordinate Systems ET 12.1

1. We start at the origin, which has coordinates $(0, 0, 0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4, 0, -3)$.

3. The distance from a point to the xz-plane is the absolute value of the y-coordinate of the point. $Q(-5, -1, 4)$ has the y-coordinate with the smallest absolute value, so Q is the point closest to the xz-plane. $R(0, 3, 8)$ must lie in the yz-plane since the distance from R to the yz-plane, given by the x-coordinate of R, is 0.

5. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$.



7. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{(7-3)^2 + [0-(-2)]^2 + [1-(-3)]^2} = \sqrt{16+4+16} = 6$$
$$|QR| = \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{36+4+0} = \sqrt{40} = 2\sqrt{10}$$
$$|RP| = \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{4+16+16} = 6$$

The longest side is QR, but the Pythagorean Theorem is not satisfied: $|PQ|^2 + |RP|^2 \neq |QR|^2$. Thus PQR is not a right triangle. PQR is isosceles, as two sides have the same length.

9. (a) First we find the distances between points:

$$|AB| = \sqrt{(3-2)^2 + (7-4)^2 + (-2-2)^2} = \sqrt{26}$$
$$|BC| = \sqrt{(1-3)^2 + (3-7)^2 + [3-(-2)]^2} = \sqrt{45} = 3\sqrt{5}$$
$$|AC| = \sqrt{(1-2)^2 + (3-4)^2 + (3-2)^2} = \sqrt{3}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $\sqrt{26} + \sqrt{3} \neq 3\sqrt{5}$, the three points do not lie on a straight line.

(b) First we find the distances between points:

$$|DE| = \sqrt{(1-0)^2 + [-2-(-5)]^2 + (4-5)^2} = \sqrt{11}$$

$$|EF| = \sqrt{(3-1)^2 + [4-(-2)]^2 + (2-4)^2} = \sqrt{44} = 2\sqrt{11}$$

$$|DF| = \sqrt{(3-0)^2 + [4-(-5)]^2 + (2-5)^2} = \sqrt{99} = 3\sqrt{11}$$

Since $|DE| + |EF| = |DF|$, the three points lie on a straight line.

11. An equation of the sphere with center $(1,-4,3)$ and radius 5 is $(x-1)^2 + [y-(-4)]^2 + (z-3)^2 = 5^2$ or $(x-1)^2 + (y+4)^2 + (z-3)^2 = 25$. The intersection of this sphere with the xz-plane is the set of points on the sphere whose y-coordinate is 0. Putting $y = 0$ into the equation, we have $(x-1)^2 + 4^2 + (z-3)^2 = 25$, $y = 0$ or $(x-1)^2 + (z-3)^2 = 9$, $y = 0$, which represents a circle in the xz-plane with center $(1,0,3)$ and radius 3.

13. The radius of the sphere is the distance between $(4,3,-1)$ and $(3,8,1)$: $r = \sqrt{(3-4)^2 + (8-3)^2 + [1-(-1)]^2} = \sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

15. Completing squares in the equation $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$ gives $(x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2 - 2z + 1) = 11 + 9 + 4 + 1 \quad\Rightarrow\quad (x-3)^2 + (y+2)^2 + (z-1)^2 = 25$, which we recognize as an equation of a sphere with center $(3,-2,1)$ and radius 5.

17. Completing squares in the equation $2x^2 - 8x + 2y^2 + 2z^2 + 24z = 1$ gives $2(x^2 - 4x + 4) + 2y^2 + 2(z^2 + 12z + 36) = 1 + 8 + 72 \quad\Rightarrow\quad 2(x-2)^2 + 2y^2 + 2(z+6)^2 = 81 \quad\Rightarrow$ $(x-2)^2 + y^2 + (z+6)^2 = \frac{81}{2}$, which we recognize as an equation of a sphere with center $(2,0,-6)$ and radius $\sqrt{\frac{81}{2}} = 9/\sqrt{2}$.

19. (a) If the midpoint of the line segment from $P_1(x_1,y_1,z_1)$ to $P_2(x_2,y_2,z_2)$ is $Q = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$,

then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$\begin{aligned}
|P_1Q| &= \sqrt{\left[\tfrac{1}{2}(x_1+x_2) - x_1\right]^2 + \left[\tfrac{1}{2}(y_1+y_2) - y_1\right]^2 + \left[\tfrac{1}{2}(z_1+z_2) - z_1\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]} = \tfrac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \\
&= \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

$$\begin{aligned}
|QP_2| &= \sqrt{\left[x_2 - \tfrac{1}{2}(x_1+x_2)\right]^2 + \left[y_2 - \tfrac{1}{2}(y_1+y_2)\right]^2 + \left[z_2 - \tfrac{1}{2}(z_1+z_2)\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} = \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2},1,4\right)$, $P_2\left(1,\frac{1}{2},5\right)$ and $P_3\left(\frac{5}{2},\frac{3}{2},4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$|AP_2| = \sqrt{0^2 + \left(\tfrac{1}{2}-2\right)^2 + (5-3)^2} = \sqrt{\tfrac{9}{4}+4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}$$

$$|BP_3| = \sqrt{\left(\tfrac{5}{2}+2\right)^2 + \left(\tfrac{3}{2}\right)^2 + (4-5)^2} = \sqrt{\tfrac{81}{4}+\tfrac{9}{4}+1} = \sqrt{\tfrac{94}{4}} = \tfrac{1}{2}\sqrt{94}$$

$$|CP_1| = \sqrt{\left(-\tfrac{1}{2}-4\right)^2 + (1-1)^2 + (4-5)^2} = \sqrt{\tfrac{81}{4}+1} = \tfrac{1}{2}\sqrt{85}$$

21. (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(2,-3,6)$, to the xy-plane, namely 6. Therefore $r=6$ and an equation of the sphere is $(x-2)^2+(y+3)^2+(z-6)^2=6^2=36$.

(b) The radius of this sphere is the distance from its center $(2,-3,6)$ to the yz-plane, which is 2. Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=4$.

(c) Here the radius is the distance from the center $(2,-3,6)$ to the xz-plane, which is 3. Therefore, an equation is $(x-2)^2+(y+3)^2+(z-6)^2=9$.

23. The equation $y=-4$ represents a plane parallel to the xz-plane and 4 units to the left of it.

25. The inequality $x>3$ represents a half-space consisting of all points in front of the plane $x=3$.

27. The inequality $0\leq z\leq 6$ represents all points on or between the horizontal planes $z=0$ (the xy-plane) and $z=6$.

29. The inequality $x^2+y^2+z^2\leq 3$ is equivalent to $\sqrt{x^2+y^2+z^2}\leq\sqrt{3}$, so the region consists of those points whose distance from the origin is at most $\sqrt{3}$. This is the set of all points on or inside the sphere with radius $\sqrt{3}$ and center $(0,0,0)$.

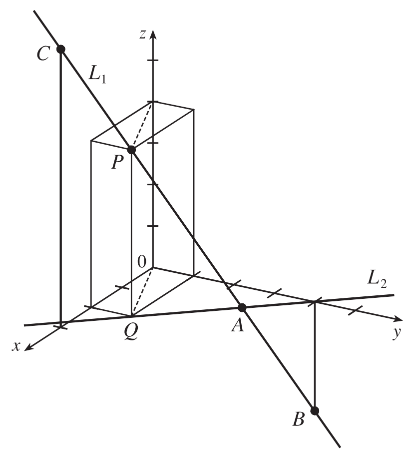
31. Here $x^2+z^2\leq 9$ or equivalently $\sqrt{x^2+z^2}\leq 3$ which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 3. Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y-axis.

33. This describes all points whose x-coordinate is between 0 and 5, that is, $0<x<5$.

35. This describes a region all of whose points have a distance to the origin which is greater than r, but smaller than R. So inequalities describing the region are $r<\sqrt{x^2+y^2+z^2}<R$, or $r^2<x^2+y^2+z^2<R^2$.

37. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2,1,4)$.

(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.



39. We need to find a set of points $\{P(x,y,z) \mid |AP| = |BP|\}$.

$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} = \sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \quad \Rightarrow$

$(x+1)^2+(y-5)+(z-3)^2 = (x-6)^2+(y-2)^2+(z+2)^2 \quad \Rightarrow$

$x^2+2x+1+y^2-10y+25+z^2-6z+9 = x^2-12x+36+y^2-4y+4+z^2+4z+4 \quad \Rightarrow \quad 14x-6y-10z=9.$

Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

13.2 Vectors ET 12.2

1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

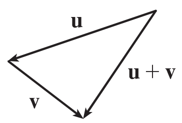
(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

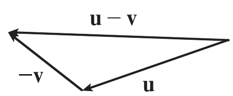
(d) The population of the world is a scalar, because it has only magnitude.

3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

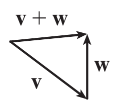
5. (a)



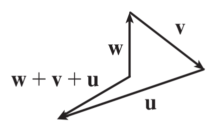
(b)



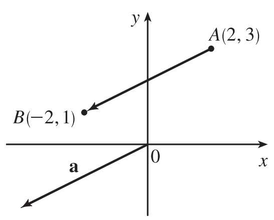
(c)



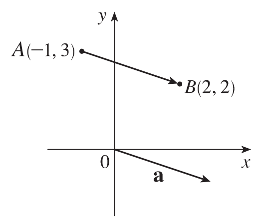
(d)



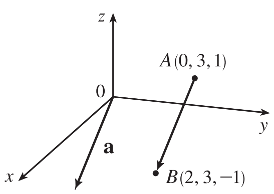
7. $\mathbf{a} = \langle -2-2, 1-3\rangle = \langle -4, -2\rangle$



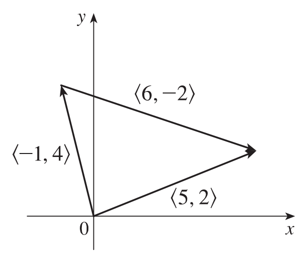
9. $\mathbf{a} = \langle 2-(-1), 2-3\rangle = \langle 3, -1\rangle$



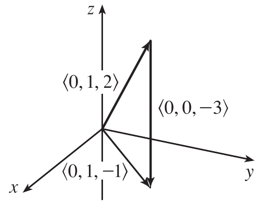
11. $\mathbf{a} = \langle 2-0, 3-3, -1-1\rangle = \langle 2, 0, -2\rangle$



13. $\langle -1, 4\rangle + \langle 6, -2\rangle = \langle -1+6, 4+(-2)\rangle = \langle 5, 2\rangle$



15. $\langle 0,1,2\rangle + \langle 0,0,-3\rangle = \langle 0+0, 1+0, 2+(-3)\rangle$
$= \langle 0,1,-1\rangle$



17. $\mathbf{a}+\mathbf{b} = \langle 5+(-3), -12+(-6)\rangle = \langle 2,-18\rangle$

$2\mathbf{a}+3\mathbf{b} = \langle 10,-24\rangle + \langle -9,-18\rangle = \langle 1,-42\rangle$

$|\mathbf{a}| = \sqrt{5^2+(-12)^2} = \sqrt{169} = 13$

$|\mathbf{a}-\mathbf{b}| = |\langle 5-(-3), -12-(-6)\rangle| = |\langle 8,-6\rangle| = \sqrt{8^2+(-6)^2} = \sqrt{100} = 10$

19. $\mathbf{a}+\mathbf{b} = (\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k}) + (-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k}) = -\,\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$2\mathbf{a}+3\mathbf{b} = 2\,(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k}) + 3\,(-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k}) = 2\,\mathbf{i}+4\,\mathbf{j}-6\,\mathbf{k}-6\,\mathbf{i}-3\,\mathbf{j}+15\,\mathbf{k} = -\,4\,\mathbf{i}+\mathbf{j}+9\mathbf{k}$

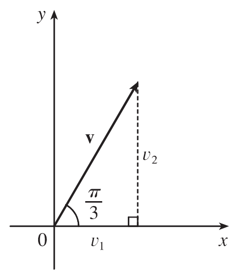
$|\mathbf{a}| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$

$|\mathbf{a}-\mathbf{b}| = |(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k}) - (-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k})| = |3\,\mathbf{i}+3\,\mathbf{j}-8\,\mathbf{k}| = \sqrt{3^2+3^2+(-8)^2} = \sqrt{82}$

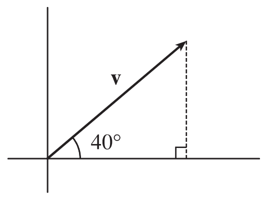
21. $|-3\,\mathbf{i}+7\,\mathbf{j}| = \sqrt{(-3)^2+7^2} = \sqrt{58}$, so $\mathbf{u} = \dfrac{1}{\sqrt{58}}\,(-3\,\mathbf{i}+7\,\mathbf{j}) = -\dfrac{3}{\sqrt{58}}\,\mathbf{i} + \dfrac{7}{\sqrt{58}}\,\mathbf{j}$.

23. The vector $8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ has length $|8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}| = \sqrt{8^2+(-1)^2+4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}) = \frac{8}{9}\,\mathbf{i} - \frac{1}{9}\,\mathbf{j} + \frac{4}{9}\,\mathbf{k}$.

25. From the figure, we see that the x-component of $\mathbf{v}$ is
$v_1 = |\mathbf{v}|\cos(\pi/3) = 4\cdot\frac{1}{2} = 2$ and the y-component is
$v_2 = |\mathbf{v}|\sin(\pi/3) = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$. Thus
$\mathbf{v} = \langle v_1, v_2\rangle = \langle 2, 2\sqrt{3}\,\rangle$.



27. The velocity vector $\mathbf{v}$ makes an angle of 40° with the horizontal and has magnitude equal to the speed at which the football was thrown. From the figure, we see that the horizontal component of $\mathbf{v}$ is $|\mathbf{v}|\cos 40^\circ = 60\cos 40^\circ \approx 45.96$ ft/s and the vertical component is $|\mathbf{v}|\sin 40^\circ = 60\sin 40^\circ \approx 38.57$ ft/s.



29. The given force vectors can be expressed in terms of their horizontal and vertical components as $-300\,\mathbf{i}$ and $200\cos 60^\circ\,\mathbf{i} + 200\sin 60^\circ\,\mathbf{j} = 200\left(\frac{1}{2}\right)\,\mathbf{i} + 200\left(\frac{\sqrt{3}}{2}\right)\,\mathbf{j} = 100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = (-300+100)\,\mathbf{i} + \left(0+100\sqrt{3}\,\right)\mathbf{j} = -200\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. Then we have

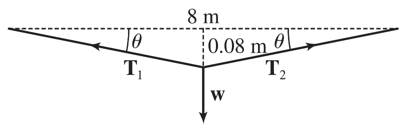
$|\mathbf{F}| \approx \sqrt{(-200)^2 + \left(100\sqrt{3}\right)^2} = \sqrt{70{,}000} = 100\sqrt{7} \approx 264.6$ N. Let θ be the angle $\mathbf{F}$ makes with the positive x-axis.

Then $\tan\theta = \dfrac{100\sqrt{3}}{-200} = -\dfrac{\sqrt{3}}{2}$ and the terminal point of $\mathbf{F}$ lies in the second quadrant, so

$\theta = \tan^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) + 180^\circ \approx -40.9^\circ + 180^\circ = 139.1^\circ$.

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y-direction, then $\mathbf{v} = \langle 0, 22\rangle + \langle -3, 0\rangle = \langle -3, 22\rangle$. The woman's speed is $|\mathbf{v}| = \sqrt{9 + 484} \approx 22.2$ mi/h. The vector $\mathbf{v}$ makes an angle θ with the east, where $\theta = \tan^{-1}\left(\frac{22}{-3}\right) \approx 98^\circ$. Therefore, the woman's direction is about N$(98 - 90)^\circ$W = N8°W.

33. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write



$\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j}$ $[a, b > 0]$. By similar triangles, $\dfrac{b}{a} = \dfrac{0.08}{4} \Rightarrow a = 50b$. The force due to gravity acting on the shirt has magnitude $0.8g \approx (0.8)(9.8) = 7.84$ N, hence we have $\mathbf{w} = -7.84\,\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensile forces counterbalances $\mathbf{w}$, so $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \Rightarrow (-a\,\mathbf{i} + b\,\mathbf{j}) + (a\,\mathbf{i} + b\,\mathbf{j}) = 7.84\,\mathbf{j} \Rightarrow$ $(-50b\,\mathbf{i} + b\,\mathbf{j}) + (50b\,\mathbf{i} + b\,\mathbf{j}) = 2b\,\mathbf{j} = 7.84\,\mathbf{j} \Rightarrow b = \frac{7.84}{2} = 3.92$ and $a = 50b = 196$. Thus the tensions are $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j} = -196\,\mathbf{i} + 3.92\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j} = 196\,\mathbf{i} + 3.92\,\mathbf{j}$.

Alternatively, we can find the value of θ and proceed as in Example 7.

35. The slope of the tangent line to the graph of $y = x^2$ at the point $(2, 4)$ is

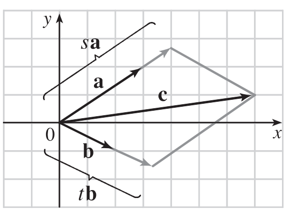
$$\left.\frac{dy}{dx}\right|_{x=2} = 2x\Big|_{x=2} = 4$$

and a parallel vector is $\mathbf{i} + 4\,\mathbf{j}$ which has length $|\mathbf{i} + 4\,\mathbf{j}| = \sqrt{1^2 + 4^2} = \sqrt{17}$, so unit vectors parallel to the tangent line are $\pm\frac{1}{\sqrt{17}}(\mathbf{i} + 4\,\mathbf{j})$.

37. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$.

So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

39. (a), (b)



(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \Leftrightarrow 7 = 3s + 2t$ and $1 = 2s - t$.

Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

41. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate method: $|\mathbf{r} - \mathbf{r}_0| = 1 \Leftrightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \Leftrightarrow$ $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

43. $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = \langle a_1,a_2\rangle + (\langle b_1,b_2\rangle + \langle c_1,c_2\rangle) = \langle a_1,a_2\rangle + \langle b_1+c_1, b_2+c_2\rangle$

$$= \langle a_1+b_1+c_1, a_2+b_2+c_2\rangle = \langle (a_1+b_1)+c_1, (a_2+b_2)+c_2\rangle$$

$$= \langle a_1+b_1, a_2+b_2\rangle + \langle c_1,c_2\rangle = (\langle a_1,a_2\rangle + \langle b_1,b_2\rangle) + \langle c_1,c_2\rangle$$

$$= (\mathbf{a}+\mathbf{b})+\mathbf{c}$$

45. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ **(1)** and $\overrightarrow{DB}+\overrightarrow{BE}=\overrightarrow{DE}$ **(2)**. However, $\overrightarrow{DB}=\frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE}=\frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into **(2)** gives $\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}=\overrightarrow{DE}$. Comparing this with **(1)** gives $\overrightarrow{DE}=\frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

13.3 The Dot Product ET 12.3

1. (a) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ has no meaning.

(b) $(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b}\cdot\mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b}\cdot\mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ are vectors, so the dot product $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ has meaning.

(e) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$ has no meaning.

3. $\mathbf{a}\cdot\mathbf{b} = \left\langle -2,\frac{1}{3}\right\rangle\cdot\langle -5,12\rangle = (-2)(-5)+\left(\frac{1}{3}\right)(12) = 10+4 = 14$

5. $\mathbf{a}\cdot\mathbf{b} = \left\langle 4,1,\frac{1}{4}\right\rangle\cdot\langle 6,-3,-8\rangle = (4)(6)+(1)(-3)+\left(\frac{1}{4}\right)(-8) = 19$

7. $\mathbf{a}\cdot\mathbf{b} = (\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k})\cdot(5\,\mathbf{i}+9\,\mathbf{k}) = (1)(5)+(-2)(0)+(3)(9) = 32$

9. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = (6)(5)\cos\frac{2\pi}{3} = 30\left(-\frac{1}{2}\right) = -15$

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u}\cdot\mathbf{w} = |\mathbf{u}|\,|\mathbf{w}|\cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

13. (a) $\mathbf{i}\cdot\mathbf{j} = \langle 1,0,0\rangle\cdot\langle 0,1,0\rangle = (1)(0)+(0)(1)+(0)(0) = 0$. Similarly, $\mathbf{j}\cdot\mathbf{k} = (0)(0)+(1)(0)+(0)(1) = 0$ and $\mathbf{k}\cdot\mathbf{i} = (0)(1)+(0)(0)+(1)(0) = 0$.

Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2} = 0$.

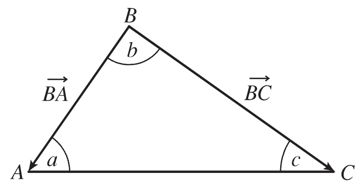
(b) By Property 1 of the dot product, $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}\cdot\mathbf{k} = |\mathbf{k}|^2 = 1$.

15. $|\mathbf{a}| = \sqrt{(-8)^2+6^2} = 10$, $|\mathbf{b}| = \sqrt{\left(\sqrt{7}\right)^2+3^2} = 4$, and $\mathbf{a}\cdot\mathbf{b} = (-8)\left(\sqrt{7}\right)+(6)(3) = 18-8\sqrt{7}$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{18-8\sqrt{7}}{10\cdot 4} = \dfrac{9-4\sqrt{7}}{20}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{9-4\sqrt{7}}{20}\right) \approx 95^\circ$.

17. $|\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 5^2} = \sqrt{35}$, $|\mathbf{b}| = \sqrt{(-2)^2 + 4^2 + 3^2} = \sqrt{29}$, and $\mathbf{a} \cdot \mathbf{b} = (3)(-2) + (-1)(4) + (5)(3) = 5$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{\sqrt{35} \cdot \sqrt{29}} = \dfrac{5}{\sqrt{1015}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{5}{\sqrt{1015}}\right) \approx 81^\circ$.

19. $|\mathbf{a}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, and $\mathbf{a} \cdot \mathbf{b} = (0)(1) + (1)(2) + (1)(-3) = -1$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{2} \cdot \sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{2\sqrt{7}}\right) \approx 101^\circ$.

21. Let a, b, and c be the angles at vertices A, B, and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|} = \dfrac{\langle 2, 6\rangle \cdot \langle -2, 4\rangle}{\sqrt{2^2 + 6^2}\,\sqrt{(-2)^2 + 4^2}} = \dfrac{1}{\sqrt{40}\,\sqrt{20}}(-4 + 24) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$ and $a = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45^\circ$. Similarly, $\cos b = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\,\left|\overrightarrow{BC}\right|} = \dfrac{\langle -2, -6\rangle \cdot \langle -4, -2\rangle}{\sqrt{4 + 36}\,\sqrt{16 + 4}} = \dfrac{1}{\sqrt{40}\,\sqrt{20}}(8 + 12) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$ so $b = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45^\circ$ and $c = 180^\circ - (45^\circ + 45^\circ) = 90^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a = \dfrac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|}$, $\cos b = \dfrac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{BC}\right|}$, and $\cos c = \dfrac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\,\left|\overrightarrow{BC}\right|}$.

23. (a) $\mathbf{a} \cdot \mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(b) $\mathbf{a} \cdot \mathbf{b} = (4)(-3) + (6)(2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(c) $\mathbf{a} \cdot \mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(d) Because $\mathbf{a} = -\frac{2}{3}\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

25. $\overrightarrow{QP} = \langle -1, -3, 2\rangle$, $\overrightarrow{QR} = \langle 4, -2, -1\rangle$, and $\overrightarrow{QP} \cdot \overrightarrow{QR} = -4 + 6 - 2 = 0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

27. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$. Then $\mathbf{a} \cdot (\mathbf{i} + \mathbf{j}) = 0 \quad\Leftrightarrow\quad a_1 + a_2 = 0$ and $\mathbf{a} \cdot (\mathbf{i} + \mathbf{k}) = 0 \quad\Leftrightarrow\quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

29. Since $|\langle 3,4,5\rangle| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$, using Equations 8 and 9 we have $\cos\alpha = \frac{3}{5\sqrt{2}}$, $\cos\beta = \frac{4}{5\sqrt{2}}$, and $\cos\gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{3}{5\sqrt{2}}\right) \approx 65^\circ$, $\beta = \cos^{-1}\left(\frac{4}{5\sqrt{2}}\right) \approx 56^\circ$, and $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$.

31. Since $|2\,\mathbf{i} + 3\,\mathbf{j} - 6\,\mathbf{k}| = \sqrt{4+9+36} = \sqrt{49} = 7$, Equations 8 and 9 give $\cos\alpha = \frac{2}{7}$, $\cos\beta = \frac{3}{7}$, and $\cos\gamma = \frac{-6}{7}$, while $\alpha = \cos^{-1}\left(\frac{2}{7}\right) \approx 73^\circ$, $\beta = \cos^{-1}\left(\frac{3}{7}\right) \approx 65^\circ$, and $\gamma = \cos^{-1}\left(-\frac{6}{7}\right) \approx 149^\circ$.

33. $|\langle c,c,c\rangle| = \sqrt{c^2+c^2+c^2} = \sqrt{3}c$ [since $c > 0$], so $\cos\alpha = \cos\beta = \cos\gamma = \dfrac{c}{\sqrt{3}c} = \dfrac{1}{\sqrt{3}}$ and $\alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

35. $|\mathbf{a}| = \sqrt{3^2 + (-4)^2} = 5$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{3\cdot 5 + (-4)\cdot 0}{5} = 3$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = 3\cdot\frac{1}{5}\langle 3,-4\rangle = \left\langle \frac{9}{5}, -\frac{12}{5}\right\rangle$.

37. $|\mathbf{a}| = \sqrt{9+36+4} = 7$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{1}{7}\,(3+12-6) = \frac{9}{7}$. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{9}{7}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{9}{7}\cdot\frac{1}{7}\,\langle 3,6,-2\rangle = \frac{9}{49}\,\langle 3,6,-2\rangle = \left\langle \frac{27}{49}, \frac{54}{49}, -\frac{18}{49}\right\rangle$.

39. $|\mathbf{a}| = \sqrt{4+1+16} = \sqrt{21}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{0-1+2}{\sqrt{21}} = \dfrac{1}{\sqrt{21}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{1}{\sqrt{21}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{21}}\cdot\dfrac{2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}}{\sqrt{21}} = \frac{1}{21}(2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) = \frac{2}{21}\,\mathbf{i} - \frac{1}{21}\,\mathbf{j} + \frac{4}{21}\,\mathbf{k}$.

41. $(\operatorname{orth}_{\mathbf{a}}\mathbf{b})\cdot\mathbf{a} = (\mathbf{b} - \operatorname{proj}_{\mathbf{a}}\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - (\operatorname{proj}_{\mathbf{a}}\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,|\mathbf{a}|^2 = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0$.

So they are orthogonal by (7).

43. $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 2 \quad\Leftrightarrow\quad \mathbf{a}\cdot\mathbf{b} = 2\,|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$. In general, $\mathbf{b} = \left\langle s, t, 3s - 2\sqrt{10}\right\rangle$, $s, t \in \mathbb{R}$.

45. The displacement vector is $\mathbf{D} = (6-0)\,\mathbf{i} + (12-10)\,\mathbf{j} + (20-8)\,\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$ so by Equation 12 the work done is $W = \mathbf{F}\cdot\mathbf{D} = (8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k})\cdot(6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}) = 48 - 12 + 108 = 144$ joules.

47. Here $|\mathbf{D}| = 80$ ft, $|\mathbf{F}| = 30$ lb, and $\theta = 40^\circ$. Thus
$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (30)(80)\cos 40^\circ = 2400\cos 40^\circ \approx 1839$ ft-lb.

49. First note that $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then $\mathbf{n}\cdot\overrightarrow{Q_1Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line.

Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection

of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\text{comp}_{\mathbf{n}}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n}\cdot\langle x_2-x_1, y_2-y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2-ax_1+by_2-by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

since $ax_2+by_2=-c$. The required distance is $\dfrac{|3\cdot -2 + -4\cdot 3+5|}{\sqrt{3^2+4^2}} = \dfrac{13}{5}$.

51. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1,1,1)$ has vector representation $\langle 1,1,1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is, $\langle 1,0,0\rangle$] is given by $\cos\theta = \dfrac{\langle 1,1,1\rangle\cdot\langle 1,0,0\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle|} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\!\left(\frac{1}{\sqrt{3}}\right)\approx 55^\circ$.

53. Consider the H—C—H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H—C—H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1-\frac{1}{2}, 0-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left\langle 0-\frac{1}{2}, 1-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The bond angle, θ, is therefore given by

$$\cos\,\theta = \frac{\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\cdot\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle\right|} = \frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\!\left(-\tfrac{1}{3}\right)\approx 109.5^\circ.$$

55. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $= \langle b_1, b_2, b_3\rangle$.

Property 2: $\mathbf{a}\cdot\mathbf{b} = \langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle = a_1b_1+a_2b_2+a_3b_3$

$\qquad = b_1a_1+b_2a_2+b_3a_3 = \langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle = \mathbf{b}\cdot\mathbf{a}$

Property 4: $(c\,\mathbf{a})\cdot\mathbf{b} = \langle ca_1,ca_2,ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle = (ca_1)b_1+(ca_2)b_2+(ca_3)b_3$

$\qquad = c\,(a_1b_1+a_2b_2+a_3b_3) = c\,(\mathbf{a}\cdot\mathbf{b}) = a_1(cb_1)+a_2(cb_2)+a_3(cb_3)$

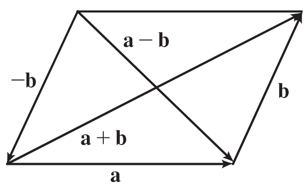
$\qquad = \langle a_1,a_2,a_3\rangle\cdot\langle cb_1,cb_2,cb_3\rangle = \mathbf{a}\cdot(c\,\mathbf{b})$

Property 5: $\mathbf{0}\cdot\mathbf{a} = \langle 0,0,0\rangle\cdot\langle a_1,a_2,a_3\rangle = (0)(a_1)+(0)(a_2)+(0)(a_3) = 0$

57. $|\mathbf{a}\cdot\mathbf{b}| = \big|\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta\big| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta|$. Since $|\cos\theta|\le 1$, $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\,\theta| \le |\mathbf{a}|\,|\mathbf{b}|$.

Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

59. (a)



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$ and $|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$.

Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2$.

13.4 The Cross Product ET 12.4

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix}\mathbf{k}$

$= [0-(-16)]\,\mathbf{i} - (0-0)\,\mathbf{j} + (48-0)\,\mathbf{k} = 16\,\mathbf{i} + 48\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 16,0,48\rangle\cdot\langle 6,0,-2\rangle = 96+0-96 = 0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 16,0,48\rangle\cdot\langle 0,8,0\rangle = 0+0+0 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}\mathbf{k}$

$= (15-0)\,\mathbf{i} - (5-2)\,\mathbf{j} + [0-(-3)]\,\mathbf{k} = 15\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (15\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k})\cdot(\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}) = 15 - 9 - 6 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (15\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k})\cdot(-\mathbf{i} + 5\,\mathbf{k}) = -15 + 0 + 15 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & \frac{1}{2} \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}\mathbf{k}$

$= \left[-\frac{1}{2} - (-1)\right]\mathbf{i} - \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right]\mathbf{j} + \left[1 - \left(-\frac{1}{2}\right)\right]\,\mathbf{k} = \frac{1}{2}\,\mathbf{i} - \mathbf{j} + \frac{3}{2}\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \left(\frac{1}{2}\,\mathbf{i} - \mathbf{j} + \frac{3}{2}\,\mathbf{k}\right)\cdot(\mathbf{i} - \mathbf{j} - \mathbf{k}) = \frac{1}{2} + 1 - \frac{3}{2} = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \left(\frac{1}{2}\,\mathbf{i} - \mathbf{j} + \frac{3}{2}\,\mathbf{k}\right)\cdot\left(\frac{1}{2}\,\mathbf{i} + \mathbf{j} + \frac{1}{2}\,\mathbf{k}\right) = \frac{1}{4} - 1 + \frac{3}{4} = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}\mathbf{k}$

$= (3t^4 - 2t^4)\,\mathbf{i} - (3t^3 - t^3)\,\mathbf{j} + (2t^2 - t^2)\,\mathbf{k} = t^4\,\mathbf{i} - 2t^3\,\mathbf{j} + t^2\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \left\langle t^4, -2t^3, t^2\right\rangle\cdot\left\langle t, t^2, t^3\right\rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \left\langle t^4, -2t^3, t^2\right\rangle\cdot\left\langle 1, 2t, 3t^2\right\rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

9. According to the discussion preceding Theorem 8, $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, so $(\mathbf{i}\times\mathbf{j})\times\mathbf{k} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ [by Example 2].

11. $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i}) = (\mathbf{j}-\mathbf{k})\times\mathbf{k} + (\mathbf{j}-\mathbf{k})\times(-\mathbf{i})$ by Property 3 of Theorem 8

$= \mathbf{j}\times\mathbf{k} + (-\mathbf{k})\times\mathbf{k} + \mathbf{j}\times(-\mathbf{i}) + (-\mathbf{k})\times(-\mathbf{i})$ by Property 4 of Theorem 8

$= (\mathbf{j}\times\mathbf{k}) + (-1)(\mathbf{k}\times\mathbf{k}) + (-1)(\mathbf{j}\times\mathbf{i}) + (-1)^2(\mathbf{k}\times\mathbf{i})$ by Property 2 of Theorem 8

$= \mathbf{i} + (-1)\,\mathbf{0} + (-1)(-\mathbf{k}) + \mathbf{j} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ by Example 2 and the discussion preceding Theorem 8

13. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is meaningful and results in another vector.

(d) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, so the cross product $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ is meaningless.

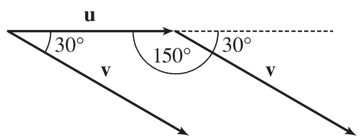
(e) Since $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{c}\cdot\mathbf{d})$ are both scalars, the cross product $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ is meaningless.

(f) $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both vectors, so the dot product $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is meaningful and is a scalar.

15. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30°. Using Theorem 6, we have

$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin 30^\circ = (6)(8)\left(\frac{1}{2}\right) = 24.$

By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.



17. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\mathbf{k} = (6-1)\,\mathbf{i} - (3-0)\,\mathbf{j} + (1-0)\,\mathbf{k} = 5\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k} = (1-6)\,\mathbf{i} - (0-3)\,\mathbf{j} + (0-1)\,\mathbf{k} = -5\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$$

Notice $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Theorem 8.

19. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\langle 1,-1,1\rangle \times \langle 0,4,4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = -8\,\mathbf{i} - 4\mathbf{j} + 4\,\mathbf{k}.$$

So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8,-4,4\rangle}{\sqrt{64+16+16}} = \pm\dfrac{\langle -8,-4,4\rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle$.

21. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$. Then

$$\mathbf{0}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix}\mathbf{k} = \mathbf{0},$$

$$\mathbf{a}\times\mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}\mathbf{k} = \mathbf{0}.$$

23. $$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \\ &= \langle (-1)(b_2a_3 - b_3a_2), (-1)(b_3a_1 - b_1a_3), (-1)(b_1a_2 - b_2a_1)\rangle \\ &= -\langle b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1\rangle = -\mathbf{b}\times\mathbf{a}\end{aligned}$$

25. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$$= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2),\, a_3(b_1 + c_1) - a_1(b_3 + c_3),\, a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle$$

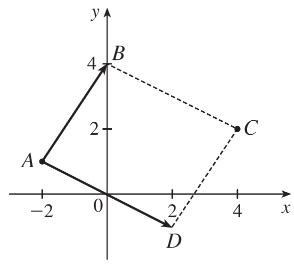
$$= \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1 \rangle$$

$$= \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2),\ (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3),\ (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1) \rangle$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1 \rangle$$

$$= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

27. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2, 3 \rangle$ and $\overrightarrow{AD} = \langle 4, -2 \rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2, 3, 0 \rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{matrix} \right\| = |(0)\,\mathbf{i} - (0)\,\mathbf{j} + (-4 - 12)\,\mathbf{k}| = |-16\,\mathbf{k}| = 16$$

29. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(3) - (0)(0), (0)(-1) - (-1)(3), (-1)(0) - (2)(-1) \rangle = \langle 6, 3, 2 \rangle$$

Therefore, $\langle 6, 3, 2 \rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P, Q, and R.

(b) Note that the area of the triangle determined by P, Q, and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle 6, 3, 2 \rangle| = \sqrt{36 + 9 + 4} = 7$, so the area of the triangle is $\frac{1}{2}(7) = \frac{7}{2}$.

31. (a) $\overrightarrow{PQ} = \langle 4, 3, -2 \rangle$ and $\overrightarrow{PR} = \langle 5, 5, 1 \rangle$, so a vector orthogonal to the plane through P, Q, and R is

$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(1) - (-2)(5), (-2)(5) - (4)(1), (4)(5) - (3)(5) \rangle = \langle 13, -14, 5 \rangle$ [or any scalar mutiple thereof].

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle 13, -14, 5 \rangle| = \sqrt{13^2 + (-14)^2 + 5^2} = \sqrt{390}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{390}$.

33. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product, which is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} = 6(5 + 4) - 3(0 - 8) - (0 - 4) = 82.$

Thus the volume of the parallelepiped is 82 cubic units.

35. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 1, 1 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 1, -1, 2 \rangle$, and $\mathbf{c} = \overrightarrow{PS} = \langle 0, -2, 3 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = 2 - 3 - 2 = -3,$$

so the volume of the parallelepiped is 3 cubic units.

37. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4 + 60 - 64 = 0$, which says that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, and thus these three vectors are coplanar.

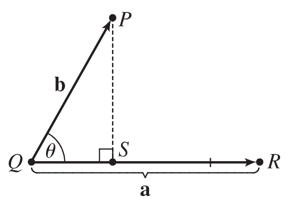
39. The magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.18 \text{ m})(60 \text{ N}) \sin(70 + 10)^\circ = 10.8 \sin 80^\circ \approx 10.6 \text{ N·m}$.

41. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0 \rangle$ and $\mathbf{F}$ has direction $\langle 0, 3, -4 \rangle$. The angle θ between them can be determined by

$\cos\theta = \dfrac{\langle 0, 0.3, 0 \rangle \cdot \langle 0, 3, -4 \rangle}{|\langle 0, 0.3, 0 \rangle|\,|\langle 0, 3, -4 \rangle|} \Rightarrow \cos\theta = \dfrac{0.9}{(0.3)(5)} \Rightarrow \cos\theta = 0.6 \Rightarrow \theta \approx 53.1^\circ$. Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta \Rightarrow$

$100 = 0.3\,|\mathbf{F}| \sin 53.1^\circ \Rightarrow |\mathbf{F}| \approx 417 \text{ N}$.

43. (a)



The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right| = d$. But referring to triangle PQS, $d = \left|\overrightarrow{PS}\right| = \left|\overrightarrow{QP}\right| \sin\theta = |\mathbf{b}| \sin\theta$. But θ is the angle between $\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by Theorem 6, $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$ and so $d = |\mathbf{b}| \sin\theta = \dfrac{|\mathbf{b}|\,|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, -2, -1 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1, -5, -7 \rangle$. Then

$\mathbf{a} \times \mathbf{b} = \langle (-2)(-7) - (-1)(-5), (-1)(1) - (-1)(-7), (-1)(-5) - (-2)(1) \rangle = \langle 9, -8, 7 \rangle$.

Thus the distance is $d = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}} \sqrt{81 + 64 + 49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$.

45.
$$\begin{aligned}
(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} && \text{by Property 3 of Theorem 8} \\
&= \mathbf{a} \times \mathbf{a} + (-\mathbf{b}) \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + (-\mathbf{b}) \times \mathbf{b} && \text{by Property 4 of Theorem 8} \\
&= (\mathbf{a} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b}) && \text{by Property 2 of Theorem 8 (with } c = -1) \\
&= \mathbf{0} - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - \mathbf{0} && \text{by Example 2} \\
&= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) && \text{by Property 1 of Theorem 8} \\
&= 2(\mathbf{a} \times \mathbf{b})
\end{aligned}$$

47. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$
$$\begin{aligned}
&= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] && \text{by Exercise 46} \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{0}
\end{aligned}$$

49. (a) No. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$. For example, let $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 1, 0, 0 \rangle$ and $\mathbf{c} = \langle 0, 1, 0 \rangle$.

(b) No. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$, which of course can happen if $\mathbf{b} \neq \mathbf{c}$.

(c) Yes. Since $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b} - \mathbf{c}$. Thus since $\mathbf{a} \neq \mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b} - \mathbf{c}$, we have $\mathbf{b} - \mathbf{c} = \mathbf{0}$, so $\mathbf{b} = \mathbf{c}$.

13.5 Equations of Lines and Planes ET 12.5

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \leq \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

3. For this line, we have $\mathbf{r}_0 = 2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}) + t(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}) = (2 + 3t)\,\mathbf{i} + (2.4 + 2t)\,\mathbf{j} + (3.5 - t)\,\mathbf{k}$ and parametric equations are $x = 2 + 3t$, $y = 2.4 + 2t$, $z = 3.5 - t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\,\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\,\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) = (1 + t)\,\mathbf{i} + 3t\,\mathbf{j} + (6 + t)\,\mathbf{k}$, and parametric equations are $x = 1 + t$, $y = 3t$, $z = 6 + t$.

7. The vector $\mathbf{v} = \langle -4 - 1, 3 - 3, 0 - 2 \rangle = \langle -5, 0, -2 \rangle$ is parallel to the line. Letting $P_0 = (1, 3, 2)$, parametric equations are $x = 1 - 5t$, $y = 3 + 0t = 3$, $z = 2 - 2t$, while symmetric equations are $\dfrac{x - 1}{-5} = \dfrac{z - 2}{-2}$, $y = 3$. Notice here that the direction number $b = 0$, so rather than writing $\dfrac{y - 3}{0}$ in the symmetric equation we must write the equation $y = 3$ separately.

9. $\mathbf{v} = \left\langle 2-0, 1-\frac{1}{2}, -3-1 \right\rangle = \left\langle 2, \frac{1}{2}, -4 \right\rangle$, and letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$, while symmetric equations are $\dfrac{x-2}{2} = \dfrac{y-1}{1/2} = \dfrac{z+3}{-4}$ or $\dfrac{x-2}{2} = 2y - 2 = \dfrac{z+3}{-4}$.

11. The line has direction $\mathbf{v} = \langle 1, 2, 1 \rangle$. Letting $P_0 = (1, -1, 1)$, parametric equations are $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$ and symmetric equations are $x - 1 = \dfrac{y+1}{2} = z - 1$.

13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2-(-4), 0-(-6), -3-1 \rangle = \langle 2, 6, -4 \rangle$ and $\mathbf{v}_2 = \langle 5-10, 3-18, 14-4 \rangle = \langle -5, -15, 10 \rangle$, and since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors and thus the lines are parallel.

15. (a) The line passes through the point $(1, -5, 6)$ and a direction vector for the line is $\langle -1, 2, -3 \rangle$, so symmetric equations for the line are $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{z-6}{-3}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{0-6}{-3}$ or $\dfrac{x-1}{-1} = 2 \;\Rightarrow\; x = -1$, $\dfrac{y+5}{2} = 2 \;\Rightarrow\; y = -1$. Thus the point of intersection with the xy-plane is $(-1, -1, 0)$. Similarly for the yz-plane, we need $x = 0 \;\Rightarrow\; 1 = \dfrac{y+5}{2} = \dfrac{z-6}{-3} \;\Rightarrow\; y = -3$, $z = 3$. Thus the line intersects the yz-plane at $(0, -3, 3)$. For the xz-plane, we need $y = 0 \;\Rightarrow\; \dfrac{x-1}{-1} = \dfrac{5}{2} = \dfrac{z-6}{-3} \;\Rightarrow\; x = -\frac{3}{2}$, $z = -\frac{3}{2}$. So the line intersects the xz-plane at $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$.

17. From Equation 4, the line segment from $\mathbf{r}_0 = 2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ to $\mathbf{r}_1 = 4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}$ is
$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)(2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}) = (2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(2\,\mathbf{i} + 7\,\mathbf{j} - 3\,\mathbf{k})$, $0 \le t \le 1$.

19. Since the direction vectors are $\mathbf{v}_1 = \langle -6, 9, -3 \rangle$ and $\mathbf{v}_2 = \langle 2, -3, 1 \rangle$, we have $\mathbf{v}_1 = -3\mathbf{v}_2$ so the lines are parallel.

21. Since the direction vectors $\langle 1, 2, 3 \rangle$ and $\langle -4, -3, 2 \rangle$ are not scalar multiples of each other, the lines are not parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1: $x = t$, $y = 1 + 2t$, $z = 2 + 3t$ and L_2: $x = 3 - 4s$, $y = 2 - 3s$, $z = 1 + 2s$. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $t = 3 - 4s$, $1 + 2t = 2 - 3s$, $2 + 3t = 1 + 2s$. Solving the first two equations we get $t = -1$, $s = 1$ and checking, we see that these values don't satisfy the third equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

23. Since the plane is perpendicular to the vector $\langle -2, 1, 5 \rangle$, we can take $\langle -2, 1, 5 \rangle$ as a normal vector to the plane. $(6, 3, 2)$ is a point on the plane, so setting $a = -2$, $b = 1$, $c = 5$ and $x_0 = 6$, $y_0 = 3$, $z_0 = 2$ in Equation 7 gives $-2(x-6) + 1(y-3) + 5(z-2) = 0$ or $-2x + y + 5z = 1$ to be an equation of the plane.

25. $\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 1, 1, -1 \rangle$ is a normal vector to the plane and $(1, -1, 1)$ is a point on the plane, so setting $a = 1$, $b = 1$, $c = -1$, $x_0 = 1$, $y_0 = -1$, $z_0 = 1$ in Equation 7 gives $1\,(x-1) + 1[y-(-1)] - 1(z-1) = 0$ or $x + y - z = -1$ to be an equation of the plane.

27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 2, -1, 3 \rangle$, and an equation of the plane is $2(x-0) - 1(y-0) + 3(z-0) = 0$ or $2x - y + 3z = 0$.

29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 3, 0, -7 \rangle$, and an equation of the plane is $3(x-4) + 0[y-(-2)] - 7(z-3) = 0$ or $3x - 7z = -9$.

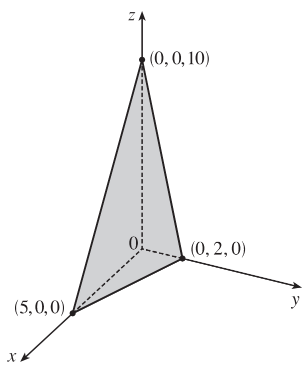
31. Here the vectors $\mathbf{a} = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$ and $\mathbf{b} = \langle 1-0, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1-0, 0+1, 0+1 \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x-0) + 1(y-1) + 1(z-1) = 0$ or $x + y + z = 2$.

33. Here the vectors $\mathbf{a} = \langle 8-3, 2-(-1), 4-2 \rangle = \langle 5, 3, 2 \rangle$ and $\mathbf{b} = \langle -1-3, -2-(-1), -3-2 \rangle = \langle -4, -1, -5 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -15+2, -8+25, -5+12 \rangle = \langle -13, 17, 7 \rangle$ and an equation of the plane is $-13(x-3) + 17[y-(-1)] + 7(z-2) = 0$ or $-13x + 17y + 7z = -42$.

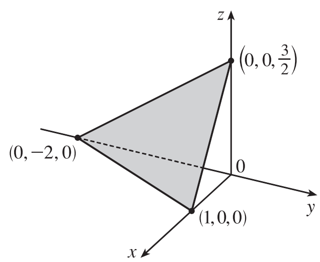
35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4 \rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6-4, 0-3, -2-7 \rangle = \langle 2, -3, -9 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45+12, 8-18, 6-10 \rangle = \langle -33, -10, -4 \rangle$. Thus, an equation of the plane is $-33(x-6) - 10(y-0) - 4[z-(-2)] = 0$ or $33x + 10y + 4z = 190$.

37. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1 \rangle \times \langle 2, -1, 3 \rangle = \langle 2, -5, -3 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $\left(0, \frac{7}{2}, \frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle$. Then a normal vector to the plane is $\mathbf{n} = \langle 2, -5, -3 \rangle \times \left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle = \langle -2, 4, -8 \rangle$ and an equation of the plane is $-2(x+1) + 4(y-2) - 8(z-1) = 0$ or $x - 2y + 4z = -1$.

39. To find the x-intercept we set $y = z = 0$ in the equation $2x + 5y + z = 10$ and obtain $2x = 10 \quad \Rightarrow \quad x = 5$ so the x-intercept is $(5, 0, 0)$. When $x = z = 0$ we get $5y = 10 \quad \Rightarrow \quad y = 2$, so the y-intercept is $(0, 2, 0)$. Setting $x = y = 0$ gives $z = 10$, so the z-intercept is $(0, 0, 10)$ and we graph the portion of the plane that lies in the first octant.



41. Setting $y = z = 0$ in the equation $6x - 3y + 4z = 6$ gives $6x = 6 \quad \Rightarrow$ $x = 1$, when $x = z = 0$ we have $-3y = 6 \quad \Rightarrow \quad y = -2$, and $x = y = 0$ implies $4z = 6 \quad \Rightarrow \quad z = \frac{3}{2}$, so the intercepts are $(1, 0, 0)$, $(0, -2, 0)$, and $\left(0, 0, \frac{3}{2}\right)$. The figure shows the portion of the plane cut off by the coordinate planes.



43. Substitute the parametric equations of the line into the equation of the plane: $(3-t)-(2+t)+2(5t)=9 \quad\Rightarrow$ $8t=8 \quad\Rightarrow\quad t=1$. Therefore, the point of intersection of the line and the plane is given by $x=3-1=2$, $y=2+1=3$, and $z=5(1)=5$, that is, the point $(2,3,5)$.

45. Parametric equations for the line are $x=t$, $y=1+t$, $z=\frac{1}{2}t$ and substituting into the equation of the plane gives $4(t)-(1+t)+3\left(\frac{1}{2}t\right)=8 \quad\Rightarrow\quad \frac{9}{2}t=9 \quad\Rightarrow\quad t=2$. Thus $x=2$, $y=1+2=3$, $z=\frac{1}{2}(2)=1$ and the point of intersection is $(2,3,1)$.

47. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are $1, 0, -1$.

49. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,4,-3\rangle$ and $\mathbf{n}_2=\langle -3,6,7\rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1\cdot\mathbf{n}_2=-3+24-21=0$, so the normals (and thus the planes) are perpendicular.

51. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,1,1\rangle$ and $\mathbf{n}_2=\langle 1,-1,1\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1\cdot\mathbf{n}_2=1-1+1=1\neq 0$, so the planes aren't perpendicular. The angle between them is given by

$$\cos\theta=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}=\frac{1}{\sqrt{3}\,\sqrt{3}}=\frac{1}{3} \quad\Rightarrow\quad \theta=\cos^{-1}\left(\tfrac{1}{3}\right)\approx 70.5^\circ.$$

53. The normals are $\mathbf{n}_1=\langle 1,-4,2\rangle$ and $\mathbf{n}_2=\langle 2,-8,4\rangle$. Since $\mathbf{n}_2=2\mathbf{n}_1$, the normals (and thus the planes) are parallel.

55. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z=0$. (This will fail if the line of intersection does not cross the xy-plane; in that case, try setting x or y equal to 0.) The equations of the two planes reduce to $x+y=1$ and $x+2y=1$. Solving these two equations gives $x=1$, $y=0$. Thus a point on the line is $(1,0,0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so we can take $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 1,1,1\rangle\times\langle 1,2,2\rangle=\langle 2-2,1-2,2-1\rangle=\langle 0,-1,1\rangle$. By Equations 2, parametric equations for the line are $x=1$, $y=-t$, $z=t$.

(b) The angle between the planes satisfies $\cos\theta=\dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}=\dfrac{1+2+2}{\sqrt{3}\,\sqrt{9}}=\dfrac{5}{3\sqrt{3}}$. Therefore $\theta=\cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right)\approx 15.8^\circ$.

57. Setting $z=0$, the equations of the two planes become $5x-2y=1$ and $4x+y=6$. Solving these two equations gives $x=1$, $y=2$ so a point on the line of intersection is $(1,2,0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes. So we can use $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 5,-2,-2\rangle\times\langle 4,1,1\rangle=\langle 0,-13,13\rangle$ or equivalently we can take $\mathbf{v}=\langle 0,-1,1\rangle$, and symmetric equations for the line are $x=1$, $\dfrac{y-2}{-1}=\dfrac{z}{1}$ or $x=1$, $y-2=-z$.

59. The distance from a point (x,y,z) to $(1,0,-2)$ is $d_1=\sqrt{(x-1)^2+y^2+(z+2)^2}$ and the distance from (x,y,z) to $(3,4,0)$ is $\sqrt{(x-3)^2+(y-4)^2+z^2}$. The plane consists of all points (x,y,z) where $d_1=d_2 \quad\Rightarrow\quad d_1^2=d_2^2 \quad\Leftrightarrow$

$(x-1)^2+y^2+(z+2)^2=(x-3)^2+(y-4)^2+z^2 \quad\Leftrightarrow$

$x^2-2x+y^2+z^2+4z+5=x^2-6x+y^2-8y+z^2+25 \quad\Leftrightarrow\quad 4x+8y+4z=20$ so an equation for the plane is $4x+8y+4z=20$ or equivalently $x+2y+z=5$.

Alternatively, you can argue that the segment joining points $(1,0,-2)$ and $(3,4,0)$ is perpendicular to the plane and the plane includes the midpoint of the segment.

61. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

63. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

65. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 4, -2, 6 \rangle$, $\mathbf{n}_2 = \langle 4, -2, -2 \rangle$, $\mathbf{n}_3 = \langle -6, 3, -9 \rangle$, $\mathbf{n}_4 = \langle 2, -1, -1 \rangle$. Now $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_3$, so $\mathbf{n}_1$ and $\mathbf{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\mathbf{n}_2 = 2\mathbf{n}_4$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $\left(0, 0, \frac{1}{2}\right)$ lies on P_1, but not on P_3, so they are not the same plane, but both P_2 and P_4 contain the point $(0, 0, -3)$, so these two planes are identical.

67. Let $Q = (1, 3, 4)$ and $R = (2, 1, 1)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (4, 1, -2)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -2, -3 \rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle 3, -2, -6 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -2, -3 \rangle \times \langle 3, -2, -6 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{|\langle 6, -3, 4 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{\sqrt{6^2 + (-3)^2 + 4^2}}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\sqrt{61}}{\sqrt{14}} = \sqrt{\frac{64}{14}}.$$

69. By Equation 9, the distance is $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \dfrac{|18|}{\sqrt{49}} = \dfrac{18}{7}.$

71. Put $y = z = 0$ in the equation of the first plane to get the point $(2, 0, 0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(2, 0, 0)$ to the second plane. By Equation 9,

$$D = \frac{|4(2) - 6(0) + 2(0) - 3|}{\sqrt{4^2 + (-6)^2 + (2)^2}} = \frac{5}{\sqrt{56}} = \frac{5}{2\sqrt{14}} \text{ or } \frac{5\sqrt{14}}{28}.$$

73. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9,

$$D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

75. L_1: $x = y = z \;\Rightarrow\; x = y$ **(1)**. L_2: $x + 1 = y/2 = z/3 \;\Rightarrow\; x + 1 = y/2$ **(2)**. The solution of **(1)** and **(2)** is $x = y = -2$. However, when $x = -2$, $x = z \;\Rightarrow\; z = -2$, but $x + 1 = z/3 \;\Rightarrow\; z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$, the direction vectors of the two lines. So set

$\mathbf{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 3-2, -3+1, 2-1\rangle = \langle 1,-2,1\rangle$. From above, we know that $(-2,-2,-2)$ and $(-2,-2,-3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2)-2(-2)+1(-2)+d_1=0 \quad\Rightarrow\quad d_1=0$ and $1(-2)-2(-2)+1(-3)+d_2=0 \quad\Rightarrow\quad d_2=1$.

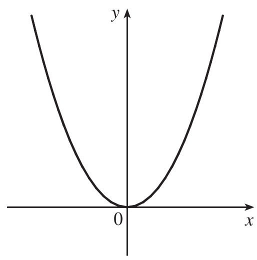
By Exercise 73, the distance between these two skew lines is $D = \dfrac{|0-1|}{\sqrt{1+4+1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1,1,1\rangle \times \langle 1,2,3\rangle = \langle 1,-2,1\rangle$. Pick any point on each of the lines, say $(-2,-2,-2)$ and $(-2,-2,-3)$, and form the vector $\mathbf{b} = \langle 0,0,1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1\cdot 0 - 2\cdot 0 + 1\cdot 1|}{\sqrt{1+4+1}} = \dfrac{1}{\sqrt{6}}$.

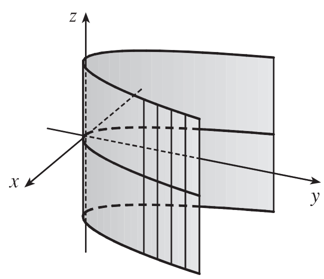
77. If $a \neq 0$, then $ax+by+cz+d=0 \quad\Rightarrow\quad a(x+d/a)+b(y-0)+c(z-0)=0$ which by (7) is the scalar equation of the plane through the point $(-d/a,0,0)$ with normal vector $\langle a,b,c\rangle$. Similarly, if $b \neq 0$ (or if $c \neq 0$) the equation of the plane can be rewritten as $a(x-0)+b(y+d/b)+c(z-0)=0$ [or as $a(x-0)+b(y-0)+c(z+d/c)=0$] which by (7) is the scalar equation of a plane through the point $(0,-d/b,0)$ [or the point $(0,0,-d/c)$] with normal vector $\langle a,b,c\rangle$.

13.6 Cylinders and Quadric Surfaces ET 12.6

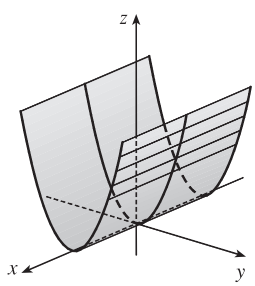
1. (a) In $\mathbb{R}^2$, the equation $y=x^2$ represents a parabola.



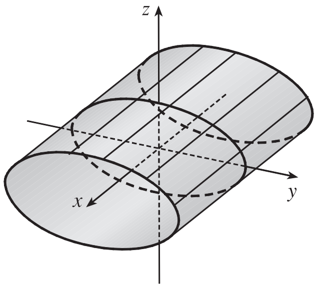
(b) In $\mathbb{R}^3$, the equation $y=x^2$ doesn't involve z, so any horizontal plane with equation $z=k$ intersects the graph in a curve with equation $y=x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



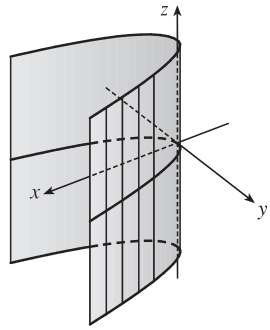
(c) In $\mathbb{R}^3$, the equation $z=y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z=y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



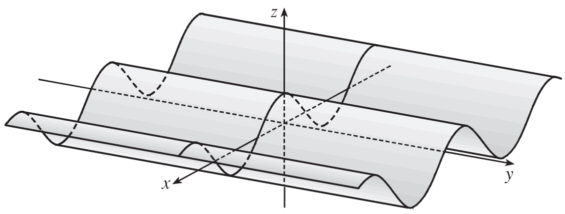
3. Since x is missing from the equation, the vertical traces $y^2 + 4z^2 = 4$, $x = k$, are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 4z^2 = 4$ is an elliptic cylinder with rulings parallel to the x-axis.



5. Since z is missing, each horizontal trace $x = y^2$, $z = k$, is a copy of the same parabola in the plane $z = k$. Thus, the surface $x - y^2 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.

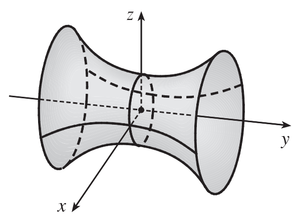


7. Since y is missing, each vertical trace $z = \cos x$, $y = k$ is a copy of a cosine curve in the plane $y = k$. Thus, the surface $z = \cos x$ is a cylindrical surface with rulings parallel to the y-axis.

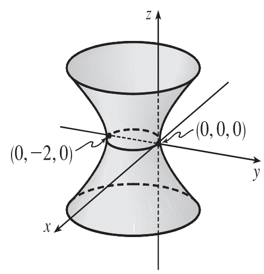


9. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

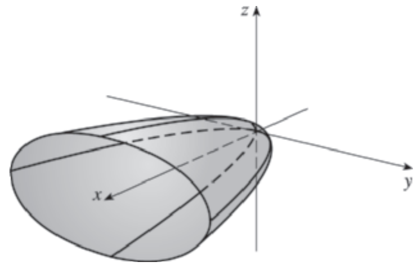
(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.



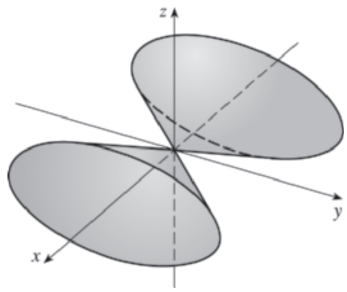
(c) Completing the square in y gives $x^2 + (y + 1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.



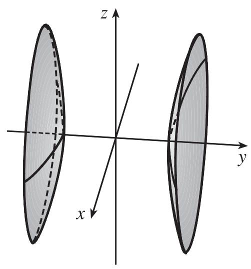
11. For $x = y^2 + 4z^2$, the traces in $x = k$ are $y^2 + 4z^2 = k$. When $k > 0$ we have a family of ellipses. When $k = 0$ we have just a point at the origin, and the trace is empty for $k < 0$. The traces in $y = k$ are $x = 4z^2 + k^2$, a family of parabolas opening in the positive x-direction. Similarly, the traces in $z = k$ are $x = y^2 + 4k^2$, a family of parabolas opening in the positive x-direction. We recognize the graph as an elliptic paraboloid with axis the x-axis and vertex the origin.



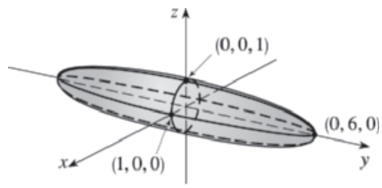
13. $x^2 = y^2 + 4z^2$. The traces in $x = k$ are the ellipses $y^2 + 4z^2 = k^2$. The traces in $y = k$ are $x^2 - 4z^2 = k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. Similarly, the traces in $z = k$ are $x^2 - y^2 = 4k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. We recognize the graph as an elliptic cone with axis the x-axis and vertex the origin.



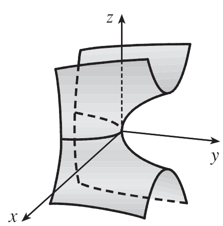
15. $-x^2 + 4y^2 - z^2 = 4$. The traces in $x = k$ are the hyperbolas $4y^2 - z^2 = 4 + k^2$. The traces in $y = k$ are $x^2 + z^2 = 4k^2 - 4$, a family of circles for $|k| > 1$, and the traces in $z = k$ are $4y^2 - x^2 = 4 + k^2$, a family of hyperbolas. Thus the surface is a hyperboloid of two sheets with axis the y-axis.



17. $36x^2 + y^2 + 36z^2 = 36$. The traces in $x = k$ are $y^2 + 36z^2 = 36(1 - k^2)$, a family of ellipses for $|k| < 1$. (The traces are a single point for $|k| = 1$ and are empty for $|k| > 1$.) The traces in $y = k$ are the circles $36x^2 + 36z^2 = 36 - k^2 \quad \Leftrightarrow \quad x^2 + z^2 = 1 - \frac{1}{36}k^2$, $|k| < 6$, and the traces in $z = k$ are the ellipses $36x^2 + y^2 = 36(1 - k^2)$, $|k| < 1$. The graph is an ellipsoid centered at the origin with intercepts $x = \pm 1$, $y = \pm 6$, $z = \pm 1$.



19. $y = z^2 - x^2$. The traces in $x = k$ are the parabolas $y = z^2 - k^2$; the traces in $y = k$ are $k = z^2 - x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k > 0$ than for $k < 0$); and the traces in $z = k$ are the parabolas $y = k^2 - x^2$. Thus, $\dfrac{y}{1} = \dfrac{z^2}{1^2} - \dfrac{x^2}{1^2}$ is a hyperbolic paraboloid.



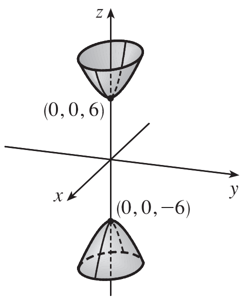
21. This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \dfrac{y^2}{(1/2)^2} + \dfrac{z^2}{(1/3)^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

23. This is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis, hence the correct graph is II.

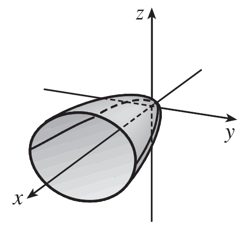
25. There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. So the graph is VIII.

29. $z^2 = 4x^2 + 9y^2 + 36$ or $-4x^2 - 9y^2 + z^2 = 36$ or $-\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{36} = 1$ represents a hyperboloid of two sheets with axis the z-axis.



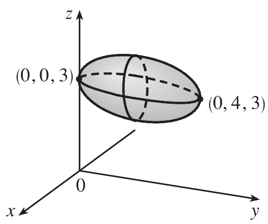
31. $x = 2y^2 + 3z^2$ or $x = \dfrac{y^2}{1/2} + \dfrac{z^2}{1/3}$ or $\dfrac{x}{6} = \dfrac{y^2}{3} + \dfrac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0, 0, 0)$ and axis the x-axis.



33. Completing squares in y and z gives

$4x^2 + (y - 2)^2 + 4(z - 3)^2 = 4$ or

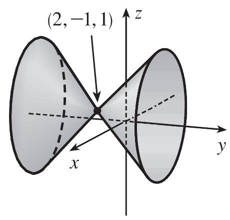
$x^2 + \dfrac{(y - 2)^2}{4} + (z - 3)^2 = 1$, an ellipsoid with center $(0, 2, 3)$.



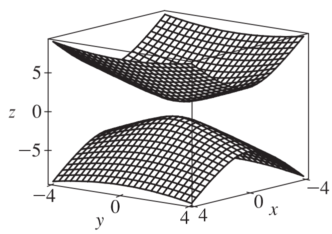
35. Completing squares in all three variables gives

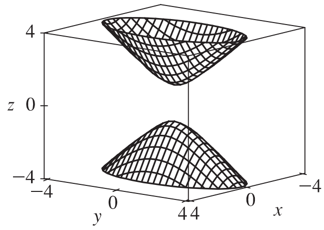
$(x - 2)^2 - (y + 1)^2 + (z - 1)^2 = 0$ or

$(y + 1)^2 = (x - 2)^2 + (z - 1)^2$, a circular cone with center $(2, -1, 1)$ and axis the horizontal line $x = 2$, $z = 1$.



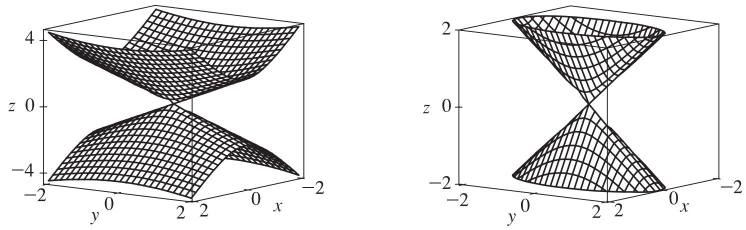
37. Solving the equation for z we get $z = \pm\sqrt{1 + 4x^2 + y^2}$, so we plot separately $z = \sqrt{1 + 4x^2 + y^2}$ and $z = -\sqrt{1 + 4x^2 + y^2}$.



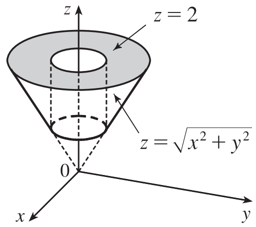


To restrict the z-range as in the second graph, we can use the option `view = -4..4` in Maple's `plot3d` command, or `PlotRange -> {-4,4}` in Mathematica's `Plot3D` command.

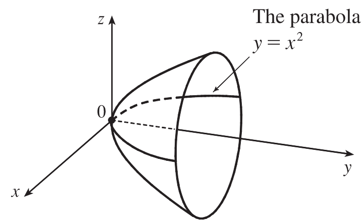
39. Solving the equation for z we get $z = \pm\sqrt{4x^2 + y^2}$, so we plot separately $z = \sqrt{4x^2 + y^2}$ and $z = -\sqrt{4x^2 + y^2}$.



41.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y-axis and opens to the right. Thus the trace in the yz-plane is also a parabola: $y = z^2$, $x = 0$. The equation is $y = x^2 + z^2$.



45. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x - 1| / \sqrt{1^2} = |x - 1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x - 1| = \sqrt{(x+1)^2 + y^2 + z^2} \quad\Leftrightarrow\quad (x-1)^2 = (x+1)^2 + y^2 + z^2 \quad\Leftrightarrow$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \quad\Leftrightarrow\quad -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative direction.

47. (a) An equation for an ellipsoid centered at the origin with intercepts $x = \pm a$, $y = \pm b$, and $z = \pm c$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Here the poles of the model intersect the z-axis at $z = \pm 6356.523$ and the equator intersects the x- and y-axes at $x = \pm 6378.137$, $y = \pm 6378.137$, so an equation is

$$\frac{x^2}{(6378.137)^2} + \frac{y^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} = 1$$

(b) Traces in $z = k$ are the circles $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} = 1 - \dfrac{k^2}{(6356.523)^2} \quad\Leftrightarrow$

$x^2 + y^2 = (6378.137)^2 - \left(\dfrac{6378.137}{6356.523}\right)^2 k^2$.

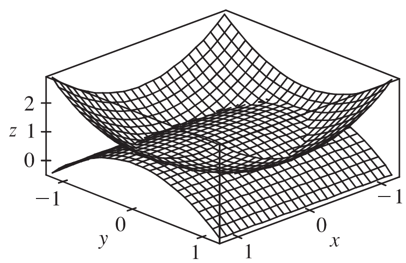
(c) To identify the traces in $y = mx$ we substitute $y = mx$ into the equation of the ellipsoid:

$$\frac{x^2}{(6378.137)^2} + \frac{(mx)^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} = 1$$
$$\frac{(1+m^2)x^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} = 1$$
$$\frac{x^2}{(6378.137)^2/(1+m^2)} + \frac{z^2}{(6356.523)^2} = 1$$

As expected, this is a family of ellipses.

49. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \quad \Rightarrow$ $c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \quad \Rightarrow \quad c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \quad \Rightarrow$ $c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \quad \Rightarrow \quad c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

51.



The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 = 1 \quad \Leftrightarrow$ $x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$. This is an equation of an ellipse.

13 Review — ET 12

CONCEPT CHECK

1. A scalar is a real number, while a vector is a quantity that has both a real-valued magnitude and a direction.

2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 3 and 4 in Section 13.2 [ET 12.2]. Algebraically, we add the corresponding components of the vectors.

3. For $c > 0$, $c\,\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the opposite direction as $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. (See Figures 7 and 15 in Section 13.2 [ET 12.2].) Algebraically, to find $c\,\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c.

4. See (1) in Section 13.2 [ET 12.2].

5. See Theorem 13.3.3 [ET 12.3.3] and Definition 13.3.1 [ET 12.3.1].

6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.

7. See the boxed equations on page 819 [ET 783] as well as Figures 4 and 5 and the accompanying discussion on pages 818–19 [ET 782–83] .

8. See Theorem 13.4.6 [ET 12.4.6] and the preceding discussion; use either (1) or (4) in Section 13.4 [ET 12.4].

9. The cross product can be used to create a vector orthogonal to two given vectors as well as to determine if two vectors are parallel. The cross product can also be used to find the area of a parallelogram determined by two vectors. In addition, the cross product can be used to determine torque if the force and position vectors are known.

10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a} \times \mathbf{b}|$.

(b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

11. If an equation of the plane is known, it can be written as $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c \rangle$ (or any scalar multiple of $\langle a, b, c \rangle$). If an equation is not known, we can use points on the plane to find two non-parallel vectors which lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.

12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 13.3.6 [ET 12.3.6].

13. See (1), (2), and (3) in Section 13.5 [ET 12.5].

14. See (5), (6), and (7) in Section 13.5 [ET 12.5].

15. (a) Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$.

(b) Two vectors are perpendicular if and only if their dot product is 0.

(c) Two planes are parallel if and only if their normal vectors are parallel.

16. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3 \rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3 \rangle = t \langle b_1, b_2, b_3 \rangle$, then the vectors are parallel and the points must all lie on the same line.

Alternatively, if $\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.

Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

(b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 5 in Section 13.4 [ET 12.4].)

Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

17. (a) See Exercise 13.4.43 [ET 12.4.43].

(b) See Example 8 in Section 13.5 [ET 12.5].

(c) See Example 10 in Section 13.5 [ET 12.5].

18. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

19. See Table 1 in Section 13.6 [ET 12.6].

TRUE-FALSE QUIZ

1. True, by Theorem 13.3.2 [ET 12.3.2], property 2.

3. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 13.4.6 [ET 12.4.6],

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = |\mathbf{v}|\,|\mathbf{u}| \sin\theta = |\mathbf{v} \times \mathbf{u}|$.

(Or, by Theorem 13.4.8 [ET 12.4.8], $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1|\,|\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)

5. Theorem 13.4.8 [ET 12.4.8], property 2 tells us that this is true.

7. This is true by Theorem 13.4.8 [ET 12.4.8], property 5.

9. This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 13.4.5 [ET 12.4.5]), and the dot product of two orthogonal vectors is 0.

11. If $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$ and θ is the angle between these two vectors (so $0 \leq \theta \leq \pi$), then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = \sin\theta$, which is equal to 1 if and only if $\theta = \frac{\pi}{2}$ (that is, if and only if the two vectors are orthogonal). Therefore, the assertion that the cross product of two unit vectors is a unit vector is false.

13. This is false. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle, but $\{(x, y, z) \mid x^2 + y^2 = 1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

15. False. For example, $\mathbf{i} \cdot \mathbf{j} = \mathbf{0}$ but $\mathbf{i} \neq \mathbf{0}$ and $\mathbf{j} \neq \mathbf{0}$.

17. This is true. If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then by (7) in Section 13.3 [ET 12.3], $\mathbf{u} \cdot \mathbf{v} = 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. But $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel (see Corollary 13.4.7 [ET 12.4.7]). Two nonzero vectors can't be both parallel and orthogonal, so at least one of $\mathbf{u}$, $\mathbf{v}$ must be $\mathbf{0}$.

EXERCISES

1. (a) The radius of the sphere is the distance between the points $(-1, 2, 1)$ and $(6, -2, 3)$, namely, $\sqrt{[6-(-1)]^2 + (-2-2)^2 + (3-1)^2} = \sqrt{69}$. By the formula for an equation of a sphere (see page 804 [ET 768]), an equation of the sphere with center $(-1, 2, 1)$ and radius $\sqrt{69}$ is $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$.

(b) The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $(y-2)^2 + (z-1)^2 = 68, x = 0$ which represents a circle in the yz-plane with center $(0, 2, 1)$ and radius $\sqrt{68}$.

(c) Completing squares gives $(x-4)^2 + (y+1)^2 + (z+3)^2 = -1 + 16 + 1 + 9 = 25$. Thus the sphere is centered at $(4, -1, -3)$ and has radius 5.

3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$.
By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

5. For the two vectors to be orthogonal, we need $\langle 3, 2, x\rangle \cdot \langle 2x, 4, x\rangle = 0 \quad \Leftrightarrow \quad (3)(2x) + (2)(4) + (x)(x) = 0 \quad \Leftrightarrow$ $x^2 + 6x + 8 = 0 \quad \Leftrightarrow \quad (x+2)(x+4) = 0 \quad \Leftrightarrow \quad x = -2$ or $x = -4$.

7. (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$

(b) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{u} \cdot [-(\mathbf{v} \times \mathbf{w})] = -\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -2$

(c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} = -(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = -2$

(d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{0} = 0$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0, 0, 0)$ to $(1, 1, 1)$ and $(1, 0, 0)$ to $(0, 1, 1)$ are $\langle 1, 1, 1\rangle$ and $\langle -1, 1, 1\rangle$. Let θ be the angle between these two vectors. $\langle 1, 1, 1\rangle \cdot \langle -1, 1, 1\rangle = -1 + 1 + 1 = 1 = |\langle 1, 1, 1\rangle|\,|\langle -1, 1, 1\rangle| \cos\theta = 3\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{1}{3} \quad \Rightarrow$ $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

11. $\overrightarrow{AB} = \langle 1, 0, -1\rangle$, $\overrightarrow{AC} = \langle 0, 4, 3\rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0\rangle = \langle 4, -3, 4\rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ **(1)**, and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \quad \Rightarrow \quad F_1 = F_2 \dfrac{\sin 30^\circ}{\sin 20^\circ}$ **(2)**. Substituting **(2)** into **(1)** gives $F_2(\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \quad \Rightarrow \quad F_2 \approx 114$ N. Substituting this into **(2)** gives $F_1 \approx 166$ N.

15. The line has direction $\mathbf{v} = \langle -3, 2, 3\rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are
$x = 4 - 3t,\ y = -1 + 2t,\ z = 2 + 3t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5\rangle$, and parametric equations for the line are
$x = -2 + 2t,\ y = 2 - t,\ z = 4 + 5t$.

19. Here the vectors $\mathbf{a} = \langle 4-3, 0-(-1), 2-1\rangle = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 6-3, 3-(-1), 1-1\rangle = \langle 3, 4, 0\rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a}\times\mathbf{b} = \langle -4, 3, 1\rangle$ is a normal vector to the plane and an equation of the plane is $-4(x-3) + 3(y-(-1)) + 1(z-1) = 0$ or $-4x + 3y + z = -14$.

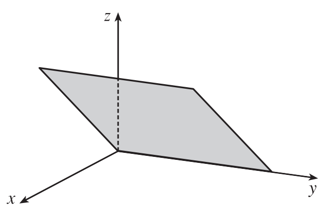
21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2-t) - (1+3t) + 4t = 2 \quad\Rightarrow\quad -t + 3 = 2 \quad\Rightarrow\quad t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

23. Since the direction vectors $\langle 2, 3, 4\rangle$ and $\langle 6, -1, 2\rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

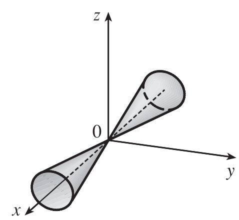
25. $\mathbf{n}_1 = \langle 1, 0, -1\rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2\rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1\rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2\rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, the normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4-1, 1+2, 1+2\rangle = 3\langle 1, 1, 1\rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x-1) + (y-3) + z = 0 \quad\Leftrightarrow\quad x + y + z = 4$.

27. By Exercise 13.5.73 [ET 12.5.73], $D = \dfrac{|2-24|}{\sqrt{26}} = \dfrac{22}{\sqrt{26}}$.

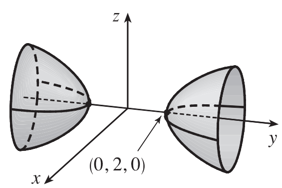
29. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



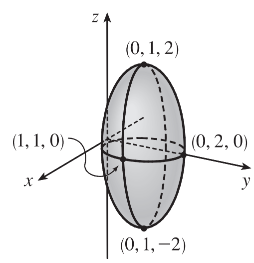
31. The equation $x^2 = y^2 + 4z^2$ represents a (right elliptical) cone with vertex at the origin and axis the x-axis.



33. An equivalent equation is $-x^2 + \dfrac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



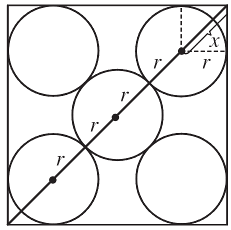
35. Completing the square in y gives $4x^2 + 4(y-1)^2 + z^2 = 4$ or $x^2 + (y-1)^2 + \dfrac{z^2}{4} = 1$, an ellipsoid centered at $(0, 1, 0)$.



37. $4x^2 + y^2 = 16 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. The equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1 \quad \Leftrightarrow$ $4x^2 + y^2 + z^2 = 16$.

PROBLEMS PLUS

1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.



The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore

$x = \sqrt{2}\,r \quad\Rightarrow\quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = \left(4 + 2\sqrt{2}\right)r \quad\Rightarrow\quad r = \frac{\sqrt{2}}{4 + 2\sqrt{2}}$.

Let's use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore

$x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r \quad\Rightarrow\quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = \left(4 + 2\sqrt{3}\right)r$. Thus $r = \dfrac{\sqrt{3}}{4 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{2}$.

The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

3. (a) We find the line of intersection L as in Example 13.5.7(b) [ET 12.5.7(b)]. Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \left\langle 2c, -c^2 + 1, -c^2 - 1 \right\rangle$. So symmetric equations of L can be written as

$\dfrac{x + 1}{-2c} = \dfrac{y - c}{c^2 - 1} = \dfrac{z - c}{c^2 + 1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

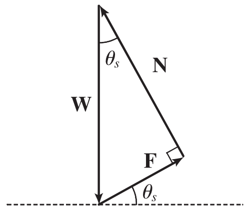
(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t - c)(-2c)}{c^2 + 1}$,

$y - c = \dfrac{(t - c)(c^2 - 1)}{c^2 + 1} \quad\Rightarrow\quad x = \dfrac{-2ct + (c^2 - 1)}{c^2 + 1},\ y = \dfrac{(c^2 - 1)t + 2c}{c^2 + 1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2 + 1}$.

(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2 + 1)$, so $V = \int_0^1 A(z)dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

5. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have

$$\tan(\theta_s) = \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{\mu_s n}{n} = \mu_s.$$



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis.

$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad\qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j} \qquad\qquad \mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad\Rightarrow\quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \qquad \textbf{(1)}$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad\Rightarrow\quad h_{\min}\sin\theta + mg\cos\theta = n \qquad \textbf{(2)}$$

(c) Since **(2)** is solved for n, we substitute into **(1)**:

$$h_{\min}\cos\theta + \mu_s(h_{\min}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg\tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the

steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90^\circ \quad \Rightarrow \quad h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg\sin\theta + h_{\max}\cos\theta = 0 \quad \Rightarrow \quad h_{\max}\cos\theta - \mu_s n = mg\sin\theta$$

$$n - mg\cos\theta - h_{\max}\sin\theta = 0 \quad \Rightarrow \quad h_{\max}\sin\theta + mg\cos\theta = n$$

Then substituting,

$$h_{\max}\cos\theta - \mu_s(h_{\max}\sin\theta + mg\cos\theta) = mg\sin\theta \quad \Rightarrow$$

$$h_{\max}\cos\theta - h_{\max}\mu_s\sin\theta = mg\sin\theta + mg\mu_s\cos\theta \quad \Rightarrow$$

$$\begin{aligned} h_{\max} &= mg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right) \\ &= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s\tan\theta}\right) = mg\tan(\theta + \theta_s) \end{aligned}$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \le \theta \le 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

14.1 Vector Functions and Space Curves ET 13.1

1. The component functions $\sqrt{4-t^2}$, e^{-3t}, and $\ln(t+1)$ are all defined when $4-t^2 \geq 0 \quad \Rightarrow \quad -2 \leq t \leq 2$ and $t+1>0 \quad \Rightarrow \quad t>-1$, so the domain of $\mathbf{r}$ is $(-1,2]$.

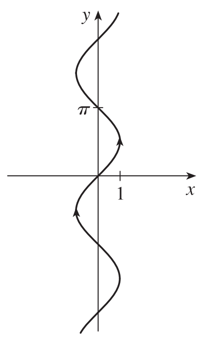
3. $\lim\limits_{t\to 0^+} \cos t = \cos 0 = 1$, $\lim\limits_{t\to 0^+} \sin t = \sin 0 = 0$, $\lim\limits_{t\to 0^+} t \ln t = \lim\limits_{t\to 0^+} \dfrac{\ln t}{1/t} = \lim\limits_{t\to 0^+} \dfrac{1/t}{-1/t^2} = \lim\limits_{t\to 0^+} -t = 0$

[by l'Hospital's Rule]. Thus $\lim\limits_{t\to 0^+} \langle \cos t, \sin t, t\ln t\rangle = \left\langle \lim\limits_{t\to 0^+} \cos t, \lim\limits_{t\to 0^+} \sin t, \lim\limits_{t\to 0^+} t\ln t \right\rangle = \langle 1,0,0\rangle$.

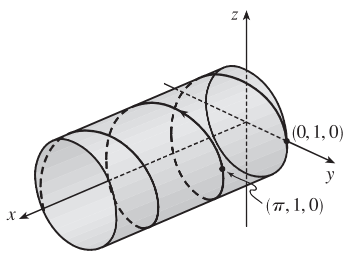
5. $\lim\limits_{t\to 0} e^{-3t} = e^0 = 1$, $\lim\limits_{t\to 0} \dfrac{t^2}{\sin^2 t} = \lim\limits_{t\to 0} \dfrac{1}{\dfrac{\sin^2 t}{t^2}} = \dfrac{1}{\lim\limits_{t\to 0} \dfrac{\sin^2 t}{t^2}} = \dfrac{1}{\left(\lim\limits_{t\to 0} \dfrac{\sin t}{t}\right)^2} = \dfrac{1}{1^2} = 1$

and $\lim\limits_{t\to 0} \cos 2t = \cos 0 = 1$. Thus the given limit equals $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

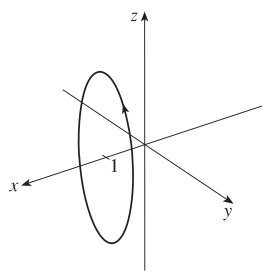
7. The corresponding parametric equations for this curve are $x = \sin t$, $y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \quad \Rightarrow \quad x = \sin y$, with $y \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



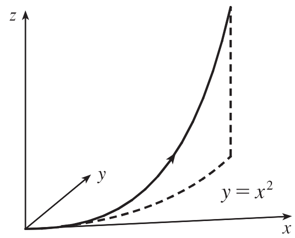
9. The corresponding parametric equations are $x = t$, $y = \cos 2t$, $z = \sin 2t$. Note that $y^2+z^2 = \cos^2 2t + \sin^2 2t = 1$, so the curve lies on the circular cylinder $y^2+z^2=1$. Since $x = t$, the curve is a helix.



11. The corresponding parametric equations are $x = 1$, $y = \cos t$, $z = 2\sin t$. Eliminating the parameter in y and z gives $y^2 + (z/2)^2 = \cos^2 t + \sin^2 t = 1$ or $y^2 + z^2/4 = 1$. Since $x = 1$, the curve is an ellipse centered at $(1,0,0)$ in the plane $x = 1$.



13. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. On the xz-plane $z = x^3$, $z > 0$, a half cubic, and the yz-plane, $y^3 = z^2$.



15. Taking $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$ and $\mathbf{r}_1 = \langle 1, 2, 3 \rangle$, we have from Equation 13.5.4 [ET 12.5.4]

$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle 0, 0, 0 \rangle + t\,\langle 1, 2, 3 \rangle$, $0 \leq t \leq 1$ or $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$, $0 \leq t \leq 1$.

Parametric equations are $x = t,\ y = 2t,\ z = 3t,\ 0 \leq t \leq 1$.

17. Taking $\mathbf{r}_0 = \langle 1, -1, 2 \rangle$ and $\mathbf{r}_1 = \langle 4, 1, 7 \rangle$, we have

$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)\,\langle 1, -1, 2 \rangle + t\,\langle 4, 1, 7 \rangle$, $0 \leq t \leq 1$ or $\mathbf{r}(t) = \langle 1 + 3t, -1 + 2t, 2 + 5t \rangle$, $0 \leq t \leq 1$.

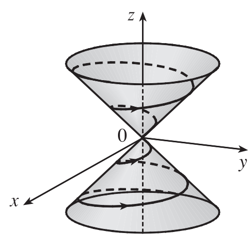
Parametric equations are $x = 1 + 3t,\ y = -1 + 2t,\ z = 2 + 5t,\ 0 \leq t \leq 1$.

19. $x = \cos 4t,\ y = t,\ z = \sin 4t$. At any point (x, y, z) on the curve, $x^2 + z^2 = \cos^2 4t + \sin^2 4t = 1$. So the curve lies on a circular cylinder with axis the y-axis. Since $y = t$, this is a helix. So the graph is VI.

21. $x = t,\ y = 1/(1 + t^2),\ z = t^2$. Note that y and z are positive for all t. The curve passes through $(0, 1, 0)$ when $t = 0$. As $t \to \infty$, $(x, y, z) \to (\infty, 0, \infty)$, and as $t \to -\infty$, $(x, y, z) \to (-\infty, 0, \infty)$. So the graph is IV.

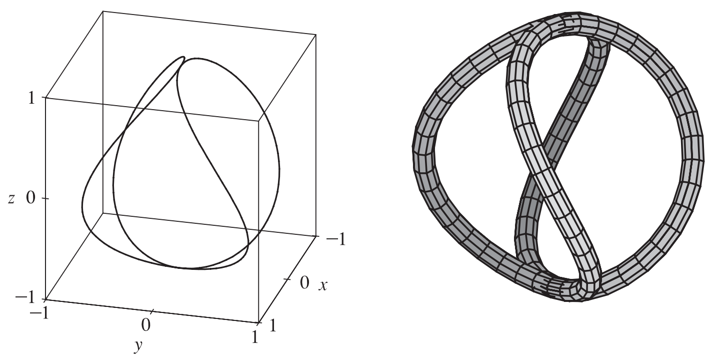
23. $x = \cos t,\ y = \sin t,\ z = \sin 5t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the z-axis. Each of x, y and z is periodic, and at $t = 0$ and $t = 2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

25. If $x = t \cos t,\ y = t \sin t,\ z = t$, then $x^2 + y^2 = t^2 \cos^2 t + t^2 \sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.

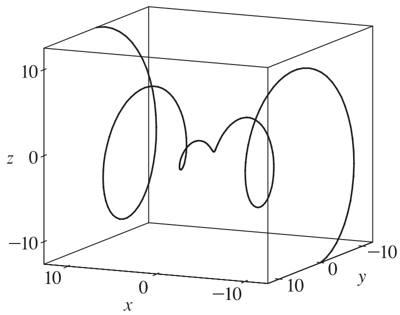


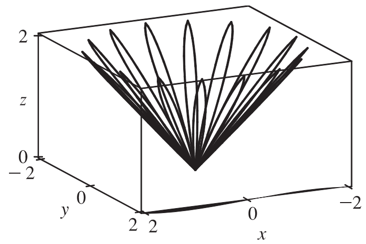
27. Parametric equations for the curve are $x = t,\ y = 0,\ z = 2t - t^2$. Substituting into the equation of the paraboloid gives $2t - t^2 = t^2 \quad \Rightarrow \quad 2t = 2t^2 \quad \Rightarrow \quad t = 0, 1$. Since $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$, the points of intersection are $(0, 0, 0)$ and $(1, 0, 1)$.

29. $\mathbf{r}(t) = \langle \cos t\ \sin 2t, \sin t\ \sin 2t, \cos 2t \rangle$. We include both a regular plot and a plot showing a tube of radius 0.08 around the curve.



31. $\mathbf{r}(t) = \langle t, t\sin t, t\cos t\rangle$



33. 

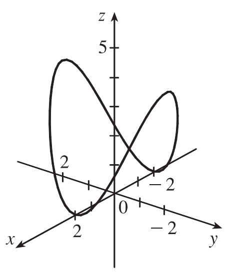
$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

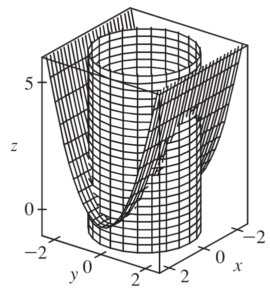
$$\begin{aligned} x^2 + y^2 &= (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t \\ &= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone } x^2 + y^2 = z^2. \end{aligned}$$

From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

35. If $t = -1$, then $x = 1$, $y = 4$, $z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9$, $y = -8$, $z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \quad\Rightarrow$ $t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the curve.

37. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2 + y^2} = 1 + y \quad\Rightarrow\quad x^2 + y^2 = 1 + 2y + y^2 \quad\Rightarrow$ $x^2 = 1 + 2y \quad\Rightarrow\quad y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$. Thus a vector function representing C is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$.

39. 



The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for C are $x = 2\cos t$, $y = 2\sin t$, $z = 4\cos^2 t$, $0 \le t \le 2\pi$.

41. For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \quad\Leftrightarrow\quad \langle t^2, 7t - 12, t^2\rangle = \langle 4t - 3, t^2, 5t - 6\rangle$. Equating components gives $t^2 = 4t - 3$, $7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \quad\Leftrightarrow\quad (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

43. (a) $\lim\limits_{t\to a} \mathbf{u}(t) + \lim\limits_{t\to a} \mathbf{v}(t) = \left\langle \lim\limits_{t\to a} u_1(t), \lim\limits_{t\to a} u_2(t), \lim\limits_{t\to a} u_3(t)\right\rangle + \left\langle \lim\limits_{t\to a} v_1(t), \lim\limits_{t\to a} v_2(t), \lim\limits_{t\to a} v_3(t)\right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors

and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}\lim_{t\to a}\mathbf{u}(t)+\lim_{t\to a}\mathbf{v}(t)&=\left\langle\lim_{t\to a}u_1(t)+\lim_{t\to a}v_1(t),\lim_{t\to a}u_2(t)+\lim_{t\to a}v_2(t),\lim_{t\to a}u_3(t)+\lim_{t\to a}v_3(t)\right\rangle\\&=\left\langle\lim_{t\to a}\left[u_1(t)+v_1(t)\right],\lim_{t\to a}\left[u_2(t)+v_2(t)\right],\lim_{t\to a}\left[u_3(t)+v_3(t)\right]\right\rangle\\&=\lim_{t\to a}\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\rangle\qquad\text{[using (1) backward]}\\&=\lim_{t\to a}\left[\mathbf{u}(t)+\mathbf{v}(t)\right]\end{aligned}$$

(b) $$\begin{aligned}\lim_{t\to a}c\mathbf{u}(t)&=\lim_{t\to a}\langle cu_1(t),cu_2(t),cu_3(t)\rangle=\left\langle\lim_{t\to a}cu_1(t),\lim_{t\to a}cu_2(t),\lim_{t\to a}cu_3(t)\right\rangle\\&=\left\langle c\lim_{t\to a}u_1(t),c\lim_{t\to a}u_2(t),c\lim_{t\to a}u_3(t)\right\rangle=c\left\langle\lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\\&=c\lim_{t\to a}\langle u_1(t),u_2(t),u_3(t)\rangle=c\lim_{t\to a}\mathbf{u}(t)\end{aligned}$$

(c) $$\begin{aligned}\lim_{t\to a}\mathbf{u}(t)\cdot\lim_{t\to a}\mathbf{v}(t)&=\left\langle\lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\cdot\left\langle\lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle\\&=\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_1(t)\right]+\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_2(t)\right]+\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_3(t)\right]\\&=\lim_{t\to a}u_1(t)v_1(t)+\lim_{t\to a}u_2(t)v_2(t)+\lim_{t\to a}u_3(t)v_3(t)\\&=\lim_{t\to a}\left[u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right]=\lim_{t\to a}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]\end{aligned}$$

(d) $$\begin{aligned}\lim_{t\to a}\mathbf{u}(t)\times\lim_{t\to a}\mathbf{v}(t)&=\left\langle\lim_{t\to a}u_1(t),\lim_{t\to a}u_2(t),\lim_{t\to a}u_3(t)\right\rangle\times\left\langle\lim_{t\to a}v_1(t),\lim_{t\to a}v_2(t),\lim_{t\to a}v_3(t)\right\rangle\\&=\left\langle\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_3(t)\right]-\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_2(t)\right],\right.\\&\qquad\left[\lim_{t\to a}u_3(t)\right]\left[\lim_{t\to a}v_1(t)\right]-\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_3(t)\right],\\&\qquad\quad\left.\left[\lim_{t\to a}u_1(t)\right]\left[\lim_{t\to a}v_2(t)\right]-\left[\lim_{t\to a}u_2(t)\right]\left[\lim_{t\to a}v_1(t)\right]\right\rangle\\&=\left\langle\lim_{t\to a}\left[u_2(t)v_3(t)-u_3(t)v_2(t)\right],\lim_{t\to a}\left[u_3(t)v_1(t)-u_1(t)v_3(t)\right],\right.\\&\qquad\qquad\left.\lim_{t\to a}\left[u_1(t)v_2(t)-u_2(t)v_1(t)\right]\right\rangle\\&=\lim_{t\to a}\langle u_2(t)v_3(t)-u_3(t)v_2(t),u_3(t)v_1(t)-u_1(t)v_3(t),u_1(t)v_2(t)-u_2(t)v_1(t)\rangle\\&=\lim_{t\to a}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]\end{aligned}$$

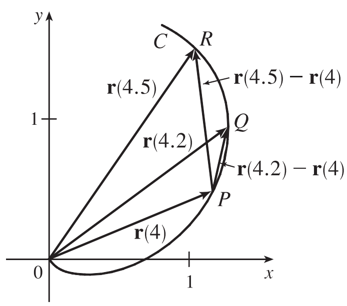
45. Let $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$. If $\lim_{t\to a}\mathbf{r}(t)=\mathbf{b}$, then $\lim_{t\to a}\mathbf{r}(t)$ exists, so by (1), $\mathbf{b}=\lim_{t\to a}\mathbf{r}(t)=\left\langle\lim_{t\to a}f(t),\lim_{t\to a}g(t),\lim_{t\to a}h(t)\right\rangle$. By the definition of equal vectors we have $\lim_{t\to a}f(t)=b_1$, $\lim_{t\to a}g(t)=b_2$ and $\lim_{t\to a}h(t)=b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\varepsilon>0$ there exists $\delta_1>0$, $\delta_2>0$, $\delta_3>0$ so that if $0<|t-a|<\delta_1$ then $|f(t)-b_1|<\varepsilon/3$, if $0<|t-a|<\delta_2$ then $|g(t)-b_2|<\varepsilon/3$, and if $0<|t-a|<\delta_3$ then $|h(t)-b_3|<\varepsilon/3$. Letting $\delta=$ minimum of $\{\delta_1,\delta_2,\delta_3\}$, then if $0<|t-a|<\delta$ we have $|f(t)-b_1|+|g(t)-b_2|+|h(t)-b_3|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$. But

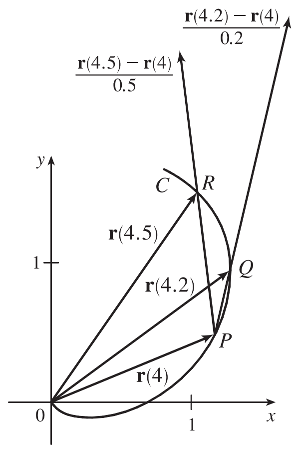
$$\begin{aligned}|\mathbf{r}(t)-\mathbf{b}|&=|\langle f(t)-b_1,g(t)-b_2,h(t)-b_3\rangle|=\sqrt{(f(t)-b_1)^2+(g(t)-b_2)^2+(h(t)-b_3)^2}\\&\leq\sqrt{[f(t)-b_1]^2}+\sqrt{[g(t)-b_2]^2}+\sqrt{[h(t)-b_3]^2}=|f(t)-b_1|+|g(t)-b_2|+|h(t)-b_3|\end{aligned}$$

Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |t-a| < \delta$ then $|\mathbf{r}(t) - \mathbf{b}| \le |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \varepsilon$. Conversely, suppose for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |t-a| < \delta$ then $|\mathbf{r}(t) - \mathbf{b}| < \varepsilon \;\Leftrightarrow\; |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3\rangle| < \varepsilon \;\Leftrightarrow$ $\sqrt{[f(t)-b_1]^2 + [g(t)-b_2]^2 + [h(t)-b_3]^2} < \varepsilon \;\Leftrightarrow\; [f(t)-b_1]^2 + [g(t)-b_2]^2 + [h(t)-b_3]^2 < \varepsilon^2$. But each term on the left side of the last inequality is positive, so if $0 < |t-a| < \delta$, then $[f(t)-b_1]^2 < \varepsilon^2$, $[g(t)-b_2]^2 < \varepsilon^2$ and $[h(t)-b_3]^2 < \varepsilon^2$ or, taking the square root of both sides in each of the above, $|f(t)-b_1| < \varepsilon$, $|g(t)-b_2| < \varepsilon$ and $|h(t)-b_3| < \varepsilon$. And by definition of limits of real-valued functions we have $\lim\limits_{t\to a} f(t) = b_1$, $\lim\limits_{t\to a} g(t) = b_2$ and $\lim\limits_{t\to a} h(t) = b_3$. But by (1), $\lim\limits_{t\to a} \mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t)\right\rangle$, so $\lim\limits_{t\to a} \mathbf{r}(t) = \langle b_1, b_2, b_3\rangle = \mathbf{b}$.

14.2 Derivatives and Integrals of Vector Functions ET 13.2

1. (a)



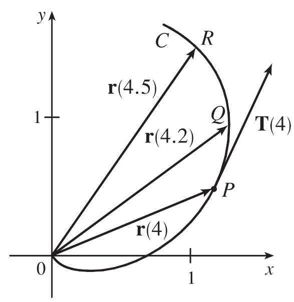


(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$.

$\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.

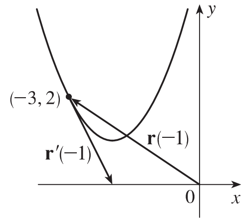
(c) By Definition 1, $\mathbf{r}'(4) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$. $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.



3. Since $(x+2)^2 = t^2 = y - 1 \quad \Rightarrow$ $y = (x+2)^2 - 1$, the curve is a parabola.

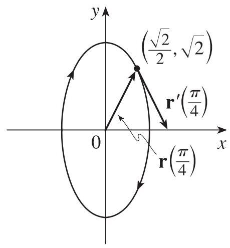
(a), (c)



(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$, $\mathbf{r}'(-1) = \langle 1, -2 \rangle$

5. $x = \sin t, \quad y = 2\cos t$ so $x^2 + (y/2)^2 = 1$ and the curve is an ellipse.

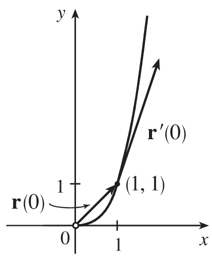
(a), (c)



(b) $\mathbf{r}'(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$, $\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$

7. Since $y = e^{3t} = (e^t)^3 = x^3$, the curve is part of a cubic cuve. Note that here, $x > 0$.

(a), (c)



(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 3e^{3t}\,\mathbf{j}$, $\mathbf{r}'(0) = \mathbf{i} + 3\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\left[t\sin t\right], \dfrac{d}{dt}\left[t^2\right], \dfrac{d}{dt}\left[t\cos 2t\right] \right\rangle = \langle t\cos t + \sin t, 2t, t(-\sin 2t)\cdot 2 + \cos 2t \rangle$

$= \langle t\cos t + \sin t, 2t, \cos 2t - 2t\sin 2t \rangle$

11. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = 0\,\mathbf{i} + 0\,\mathbf{j} + 4e^{4t}\,\mathbf{k} = 4e^{4t}\,\mathbf{k}$

13. $\mathbf{r}(t) = e^{t^2}\,\mathbf{i} - \mathbf{j} + \ln(1+3t)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + \dfrac{3}{1+3t}\,\mathbf{k}$

15. $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

17. $\mathbf{r}'(t) = \left\langle -te^{-t} + e^{-t}, 2/(1+t^2), 2e^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 2, 2 \rangle$. So $|\mathbf{r}'(0)| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$ and

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 1, 2, 2 \rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$.

19. $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. Thus

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \dfrac{1}{\sqrt{0^2 + 3^2 + 4^2}}(3\,\mathbf{j} + 4\,\mathbf{k}) = \frac{1}{5}(3\,\mathbf{j} + 4\,\mathbf{k}) = \frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$.

21. $\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle$. Then $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{14}}\langle 1, 2, 3 \rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$. $\quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$, so

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{k}$$

$$= (12t^2 - 6t^2)\,\mathbf{i} - (6t - 0)\,\mathbf{j} + (2 - 0)\,\mathbf{k} = \left\langle 6t^2, -6t, 2 \right\rangle$$

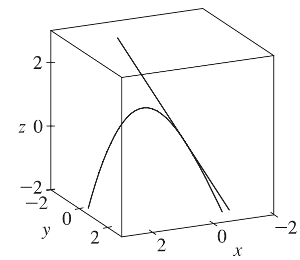
23. The vector equation for the curve is $\mathbf{r}(t) = \langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t \rangle$, so $\mathbf{r}'(t) = \langle 1/\sqrt{t}, 3t^2 - 1, 3t^2 + 1 \rangle$. The point $(3, 0, 2)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 1, 2, 4 \rangle$. Thus, the tangent line goes through the point $(3, 0, 2)$ and is parallel to the vector $\langle 1, 2, 4 \rangle$. Parametric equations are $x = 3 + t$, $y = 2t$, $z = 2 + 4t$.

25. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t} \rangle$, so

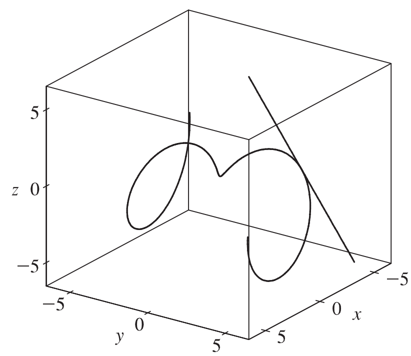
$$\begin{aligned} \mathbf{r}'(t) &= \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t}) \rangle \\ &= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle \end{aligned}$$

The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0 \rangle = \langle -1, 1, -1 \rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1 \rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1 \cdot t = t$, $z = 1 + (-1)t = 1 - t$.

27. $\mathbf{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle$. At $(0, 1, 0)$, $t = 0$ and $\mathbf{r}'(0) = \langle 1, -1, 2 \rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 1 - t$, $z = 2t$.



29. $\mathbf{r}(t) = \langle t\cos t, t, t\sin t \rangle \Rightarrow \mathbf{r}'(t) = \langle \cos t - t\sin t, 1, t\cos t + \sin t \rangle$. At $(-\pi, \pi, 0)$, $t = \pi$ and $\mathbf{r}'(\pi) = \langle -1, 1, -\pi \rangle$. Thus, parametric equations of the tangent line are $x = -\pi - t$, $y = \pi + t$, $z = -\pi t$.



31. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}_2'(0) = \langle 1, 2, 1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}} \langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

33.
$$\begin{aligned} \int_0^1 (16t^3\,\mathbf{i} - 9t^2\,\mathbf{j} + 25t^4\,\mathbf{k})\,dt &= \left(\int_0^1 16t^3\,dt\right)\mathbf{i} - \left(\int_0^1 9t^2\,dt\right)\mathbf{j} + \left(\int_0^1 25t^4\,dt\right)\mathbf{k} \\ &= \left[4t^4\right]_0^1\,\mathbf{i} - \left[3t^3\right]_0^1\,\mathbf{j} + \left[5t^5\right]_0^1\,\mathbf{k} = 4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k} \end{aligned}$$

35.
$$\begin{aligned} &\int_0^{\pi/2} (3\sin^2 t\,\cos t\,\mathbf{i} + 3\sin t\,\cos^2 t\,\mathbf{j} + 2\sin t\,\cos t\,\mathbf{k})\,dt \\ &\quad = \left(\int_0^{\pi/2} 3\sin^2 t\,\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/2} 3\sin t\,\cos^2 t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/2} 2\sin t\,\cos t\,dt\right)\mathbf{k} \\ &\quad = \left[\sin^3 t\right]_0^{\pi/2}\,\mathbf{i} + \left[-\cos^3 t\right]_0^{\pi/2}\,\mathbf{j} + \left[\sin^2 t\right]_0^{\pi/2}\,\mathbf{k} = (1 - 0)\,\mathbf{i} + (0 + 1)\,\mathbf{j} + (1 - 0)\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

37. $\int \left(e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}\right)\,dt = \left(\int e^t\,dt\right)\mathbf{i} + \left(\int 2t\,dt\right)\mathbf{j} + \left(\int \ln t\,dt\right)\mathbf{k}$

$$= e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}, \text{ where } \mathbf{C} \text{ is a vector constant of integration.}$$

39. $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + \sqrt{t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + \frac{2}{3}t^{3/2}\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.

But $\mathbf{i} + \mathbf{j} = \mathbf{r}\,(1) = \mathbf{i} + \mathbf{j} + \frac{2}{3}\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = -\frac{2}{3}\mathbf{k}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} + \left(\frac{2}{3}t^{3/2} - \frac{2}{3}\right)\mathbf{k}$.

For Exercises 41–44, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

41.
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] &= \frac{d}{dt}\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\rangle\\ &= \left\langle \frac{d}{dt}\left[u_1(t) + v_1\,(t)\right], \frac{d}{dt}\left[u_2(t) + v_2(t)\right], \frac{d}{dt}\left[u_3(t) + v_3(t)\right]\right\rangle\\ &= \langle u_1'(t) + v_1'(t), u_2'(t) + v_2'(t), u_3'(t) + v_3'(t)\rangle\\ &= \langle u_1'(t), u_2'\,(t)\,, u_3'(t)\rangle + \langle v_1'(t), v_2'(t), v_3'(t)\rangle = \mathbf{u}'(t) + \mathbf{v}'(t)\end{aligned}$$

43.
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] &= \frac{d}{dt}\langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t), u_1(t)v_2(t) - u_2(t)v_1(t)\rangle\\ &= \langle u_2'v_3(t) + u_2(t)v_3'(t) - u_3'(t)v_2(t) - u_3(t)v_2'(t),\\ &\qquad u_3'(t)v_1(t) + u_3(t)v_1'\,(t) - u_1'(t)v_3(t) - u_1(t)v_3'(t),\\ &\qquad\qquad u_1'(t)v_2(t) + u_1(t)v_2'(t) - u_2'(t)v_1(t) - u_2(t)v_1'(t)\rangle\\ &= \langle u_2'(t)v_3(t) - u_3'(t)v_2\,(t)\,, u_3'(t)v_1(t) - u_1'(t)v_3(t), u_1'(t)v_2(t) - u_2'(t)v_1(t)\rangle\\ &\qquad + \langle u_2(t)v_3'(t) - u_3(t)v_2'(t), u_3(t)v_1'\,(t) - u_1(t)v_3'(t), u_1(t)v_2'(t) - u_2(t)v_1'(t)\rangle\\ &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)\end{aligned}$$

Alternate solution: Let $\mathbf{r}(t) = \mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h) - \mathbf{r}(t) &= \left[\mathbf{u}(t+h)\times\mathbf{v}(t+h)\right] - \left[\mathbf{u}(t)\times\mathbf{v}(t)\right]\\ &= \left[\mathbf{u}(t+h)\times\mathbf{v}(t+h)\right] - \left[\mathbf{u}(t)\times\mathbf{v}(t)\right] + \left[\mathbf{u}(t+h)\times\mathbf{v}(t)\right] - \left[\mathbf{u}(t+h)\times\mathbf{v}(t)\right]\\ &= \mathbf{u}(t+h)\times\left[\mathbf{v}(t+h) - \mathbf{v}(t)\right] + \left[\mathbf{u}(t+h) - \mathbf{u}(t)\right]\times\mathbf{v}(t)\end{aligned}$$

(Be careful of the order of the cross product.) Dividing through by h and taking the limit as $h\to 0$ we have

$$\mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\left[\mathbf{v}(t+h) - \mathbf{v}(t)\right]}{h} + \lim_{h\to 0}\frac{\left[\mathbf{u}(t+h) - \mathbf{u}(t)\right]\times\mathbf{v}(t)}{h} = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$$

by Exercise 14.1.43(a) [ET 13.1.43(a)] and Definition 1.

45.
$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) \qquad \text{[by Formula 4 of Theorem 3]}\\ &= \langle\cos t, -\sin t, 1\rangle\cdot\langle t, \cos t, \sin t\rangle + \langle\sin t, \cos t, t\rangle\cdot\langle 1, -\sin t, \cos t\rangle\\ &= t\cos t - \cos t\,\sin t + \sin t + \sin t - \cos t\,\sin t + t\cos t\\ &= 2t\cos t + 2\sin t - 2\cos t\,\sin t\end{aligned}$$

47. $\dfrac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (by Example 2 in Section 13.4 [ET 12.4]). Thus, $\dfrac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

49. $\dfrac{d}{dt}|\mathbf{r}(t)| = \dfrac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{1/2} = \frac{1}{2}[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{-1/2}[2\mathbf{r}(t)\cdot\mathbf{r}'(t)] = \dfrac{1}{|\mathbf{r}(t)|}\mathbf{r}(t)\cdot\mathbf{r}'(t)$

51. Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)]$,

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{r}'(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)] + \mathbf{r}(t)\cdot\frac{d}{dt}[\mathbf{r}'(t)\times\mathbf{r}''(t)] \\ &= 0 + \mathbf{r}(t)\cdot[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)] && [\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)] \\ &= \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}'''(t)] && [\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}] \end{aligned}$$

14.3 Arc Length and Curvature ET 13.3

1. $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$.

Then using Formula 3, we have $L = \int_{-10}^{10}|\mathbf{r}'(t)|\,dt = \int_{-10}^{10}\sqrt{29}\,dt = \sqrt{29}\,t\Big]_{-10}^{10} = 20\sqrt{29}$.

3. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

5. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ [since $t \geq 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \frac{1}{18}\cdot\frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

7. $\mathbf{r}(t) = \langle\sqrt{t}, t, t^2\rangle \;\Rightarrow\; \mathbf{r}'(t) = \left\langle \dfrac{1}{2\sqrt{t}}, 1, 2t\right\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 1^2 + (2t)^2} = \sqrt{\frac{1}{4t} + 1 + 4t^2}$, so

$L = \int_1^4 |\mathbf{r}'(t)|\,dt = \int_1^4 \sqrt{\frac{1}{4t} + 1 + 4t^2}\,dt \approx 15.3841$.

9. $\mathbf{r}(t) = \langle\sin t, \cos t, \tan t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle\cos t, -\sin t, \sec^2 t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + (-\sin t)^2 + (\sec^2 t)^2} = \sqrt{1 + \sec^4 t}$ and $L = \int_0^{\pi/4}|\mathbf{r}'(t)|\,dt = \int_0^{\pi/4}\sqrt{1 + \sec^4 t}\,dt \approx 1.2780$.

11. The projection of the curve C onto the xy-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t \;\Rightarrow\; y = \frac{1}{2}t^2$. Since C also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for C are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\right\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$\begin{aligned} L &= \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\left\langle 1, t, \tfrac{1}{2}t^2\right\rangle\right|\,dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt \\ &= \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1 + \tfrac{1}{2}t^2)\,dt = \left[t + \tfrac{1}{6}t^3\right]_0^6 = 6 + 36 = 42 \end{aligned}$$

13. $\mathbf{r}(t) = 2t\,\mathbf{i} + (1-3t)\,\mathbf{j} + (5+4t)\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4+9+16} = \sqrt{29}$. Then $s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$.

15. Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0,0,3)$ corresponds to $t = 0$, so the arc length function beginning at $(0,0,3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $\quad s(t) = 5 \quad\Rightarrow\quad 5t = 5 \quad\Rightarrow\quad t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

17. (a) $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\sin t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \quad\Rightarrow\quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}$. Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle.$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/\sqrt{29}}{\sqrt{29}} = \dfrac{2}{29}$

19. (a) $\mathbf{r}(t) = \left\langle \sqrt{2}\,t, e^t, e^{-t}\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{e^t + e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle = \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle \quad \left[\text{after multiplying by } \frac{e^t}{e^t}\right] \quad \text{and}$$

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle \\ &= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle\right] = \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})} \\ &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1 + e^{2t}\right)^2} = \frac{\sqrt{2}e^t(1 + e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle \\ &= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle = \frac{1}{e^{2t}+1}\left\langle 1 - e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\right\rangle\end{aligned}$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\dfrac{1}{e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^t}{e^{3t} + 2e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^{2t}}{e^{4t} + 2e^{2t} + 1} = \dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$

21. $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{k},\quad \mathbf{r}''(t) = 2\,\mathbf{i},\quad |\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2+1},\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = 2\,\mathbf{j},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = 2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2}{\left(\sqrt{4t^2+1}\,\right)^3} = \dfrac{2}{(4t^2+1)^{3/2}}$.

23. $\mathbf{r}(t) = 3t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 4\cos t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 3\,\mathbf{i} + 4\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k},\quad \mathbf{r}''(t) = -4\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = \sqrt{9+16} = 5,\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = -16\,\mathbf{i} + 12\cos t\,\mathbf{j} - 12\sin t\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{20}{5^3} = \dfrac{4}{25}$.

25. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. The point $(1,1,1)$ corresponds to $t=1$, and $\mathbf{r}'(1) = \langle 1,2,3\rangle \quad\Rightarrow$

$|\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0,2,6t\rangle \quad\Rightarrow\quad \mathbf{r}''(1) = \langle 0,2,6\rangle$. $\mathbf{r}'(1)\times\mathbf{r}''(1) = \langle 6,-6,2\rangle$, so

$|\mathbf{r}'(1)\times\mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

27. $f(x) = 2x - x^2,\quad f'(x) = 2 - 2x,\quad f''(x) = -2,$

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{|-2|}{[1+(2-2x)^2]^{3/2}} = \frac{2}{(4x^2-8x+5)^{3/2}}$$

29. $f(x) = 4x^{5/2},\quad f'(x) = 10x^{3/2},\quad f''(x) = 15x^{1/2},$

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \frac{\left|15x^{1/2}\right|}{[1+(10x^{3/2})^2]^{3/2}} = \frac{15\sqrt{x}}{(1+100x^3)^{3/2}}$$

31. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1+(y'(x))^2]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}} = e^x(1+e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$\kappa'(x) = e^x(1+e^{2x})^{-3/2} + e^x\left(-\frac{3}{2}\right)(1+e^{2x})^{-5/2}(2e^{2x}) = e^x\dfrac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}} = e^x\dfrac{1-2e^{2x}}{(1+e^{2x})^{5/2}}$.

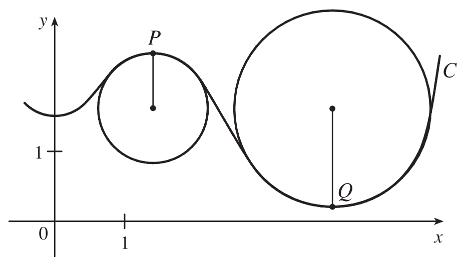
$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and

$1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} e^x(1+e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x\to\infty$.

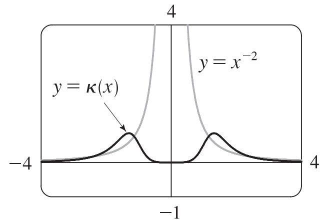
33. (a) C appears to be changing direction more quickly at P than Q, so we would expect the curvature to be greater at P.

(b) First we sketch approximate osculating circles at P and Q. Using the axes scale as a guide, we measure the radius of the osculating circle at P to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \quad\Rightarrow$ $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at Q to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.



35. $y = x^{-2} \;\Rightarrow\; y' = -2x^{-3},\;\; y'' = 6x^{-4}$, and

$$\kappa(x) = \frac{|y''|}{[1+(y')^2]^{3/2}} = \frac{|6x^{-4}|}{[1+(-2x^{-3})^2]^{3/2}} = \frac{6}{x^4(1+4x^{-6})^{3/2}}.$$

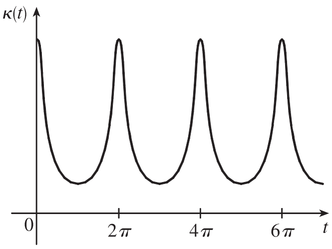


The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^{-2}$ increases asymptotically at the origin from both directions, and so its graph has very little bend there. (Note that $\kappa(0)$ is undefined.)

37. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y = f(x)$ rather than the graph of curvature, and b is the graph of $y = \kappa(x)$.

39. Using a CAS, we find (after simplifying)

$$\kappa(t) = \frac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}.$$ (To compute cross products in Maple, use the `VectorCalculus` package and the `CrossProduct(a,b)` command; in Mathematica, use `Cross[a,b]`.) Curvature is largest at integer multiples of 2π.



41. $x = e^t \cos t \;\Rightarrow\; \dot{x} = e^t(\cos t - \sin t) \;\Rightarrow\; \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t \sin t$,

$y = e^t \sin t \;\Rightarrow\; \dot{y} = e^t(\cos t + \sin t) \;\Rightarrow\; \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t \cos t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{\left|e^t(\cos t - \sin t)(2e^t\cos t) - e^t(\cos t + \sin t)(-2e^t \sin t)\right|}{([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2)^{3/2}}$$

$$= \frac{\left|2e^{2t}(\cos^2 t - \sin t\,\cos t + \sin t\,\cos t + \sin^2 t)\right|}{\left[e^{2t}(\cos^2 t - 2\cos t\,\sin t + \sin^2 t + \cos^2 t + 2\cos t\,\sin t + \sin^2 t)\right]^{3/2}} = \frac{\left|2e^{2t}(1)\right|}{[e^{2t}(1+1)]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}$$

43. $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$$\mathbf{T}'(t) = -4t(2t^2+1)^{-2}\left\langle 2t, 2t^2, 1\right\rangle + (2t^2+1)^{-1}\langle 2, 4t, 0\rangle \qquad \text{(by Formula 3 of Theorem 14.2 [ET 13.2])}$$

$$= (2t^2+1)^{-2}\left\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\right\rangle = 2(2t^2+1)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2+1)^{-2}\left\langle 1-2t^2, 2t, -2t\right\rangle}{2(2t^2+1)^{-2}\sqrt{(1-2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\left\langle 1-2t^2, 2t, -2t\right\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} = \frac{\left\langle 1-2t^2, 2t, -2t\right\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

45. $(0, \pi, -2)$ corresponds to $t = \pi$. $\mathbf{r}(t) = \langle 2\sin 3t, t, 2\cos 3t\rangle \;\Rightarrow$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 6\cos 3t, 1, -6\sin 3t\rangle}{\sqrt{36\cos^2 3t + 1 + 36\sin^2 3t}} = \frac{1}{\sqrt{37}}\langle 6\cos 3t, 1, -6\sin 3t\rangle.$$

$\mathbf{T}(\pi) = \frac{1}{\sqrt{37}}\langle -6, 1, 0\rangle$ is a normal vector for the normal plane, and so $\langle -6, 1, 0\rangle$ is also normal. Thus an equation for the

plane is $-6(x-0)+1(y-\pi)+0(z+2)=0$ or $y-6x=\pi$.

$\mathbf{T}'(t)=\frac{1}{\sqrt{37}}\langle -18\sin 3t, 0, -18\cos 3t\rangle \Rightarrow |\mathbf{T}'(t)|=\dfrac{\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}}{\sqrt{37}}=\dfrac{18}{\sqrt{37}} \Rightarrow$

$\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\langle -\sin 3t, 0, -\cos 3t\rangle$. So $\mathbf{N}(\pi)=\langle 0,0,1\rangle$ and $\mathbf{B}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle\times\langle 0,0,1\rangle=\frac{1}{\sqrt{37}}\langle 1,6,0\rangle$.

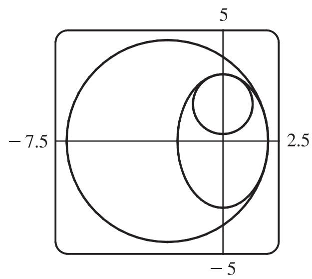
Since $\mathbf{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1,6,0\rangle$.

An equation for the plane is $1(x-0)+6(y-\pi)+0(z+2)=0$ or $x+6y=6\pi$.

47. The ellipse $9x^2+4y^2=36$ is given by the parametric equations $x=2\cos t$, $y=3\sin t$, so using the result from Exercise 40,

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{[\dot{x}^2+\dot{y}^2]^{3/2}}=\frac{|(-2\sin t)(-3\sin t)-(3\cos t)(-2\cos t)|}{(4\sin^2 t+9\cos^2 t)^{3/2}}=\frac{6}{(4\sin^2 t+9\cos^2 t)^{3/2}}.$$

At $(2,0)$, $t=0$. Now $\kappa(0)=\frac{6}{27}=\frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0)=\frac{9}{2}$ and its center is $\left(-\frac{5}{2},0\right)$. Its equation is therefore $\left(x+\frac{5}{2}\right)^2+y^2=\frac{81}{4}$. At $(0,3)$, $t=\frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right)=\frac{6}{8}=\frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0,\frac{5}{3}\right)$. Hence its equation is $x^2+\left(y-\frac{5}{3}\right)^2=\frac{16}{9}$.



49. The tangent vector is normal to the normal plane, and the vector $\langle 6,6,-8\rangle$ is normal to the given plane. But $\mathbf{T}(t)\parallel\mathbf{r}'(t)$ and $\langle 6,6,-8\rangle\parallel\langle 3,3,-4\rangle$, so we need to find t such that $\mathbf{r}'(t)\parallel\langle 3,3,-4\rangle$. $\mathbf{r}(t)=\langle t^3,3t,t^4\rangle \Rightarrow \mathbf{r}'(t)=\langle 3t^2,3,4t^3\rangle\parallel\langle 3,3,-4\rangle$ when $t=-1$. So the planes are parallel at the point $(-1,-3,1)$.

51. $\kappa=\left|\dfrac{d\mathbf{T}}{ds}\right|=\left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right|=\dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N}=\dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N}=\dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right|\dfrac{ds}{dt}}=\dfrac{d\mathbf{T}/dt}{ds/dt}=\dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

53. (a) $|\mathbf{B}|=1 \Rightarrow \mathbf{B}\cdot\mathbf{B}=1 \Rightarrow \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B})=0 \Rightarrow 2\,\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B}=0 \Rightarrow \dfrac{d\mathbf{B}}{ds}\perp\mathbf{B}$

(b) $\mathbf{B}=\mathbf{T}\times\mathbf{N} \Rightarrow$

$$\frac{d\mathbf{B}}{ds}=\frac{d}{ds}(\mathbf{T}\times\mathbf{N})=\frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{ds/dt}=\frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{|\mathbf{r}'(t)|}=[(\mathbf{T}'\times\mathbf{N})+(\mathbf{T}\times\mathbf{N}')]\frac{1}{|\mathbf{r}'(t)|}$$

$$=\left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right)+(\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|}=\frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|} \Rightarrow \frac{d\mathbf{B}}{ds}\perp\mathbf{T}$$

(c) $\mathbf{B}=\mathbf{T}\times\mathbf{N} \Rightarrow \mathbf{T}\perp\mathbf{N}$, $\mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. So $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds=-\tau(s)\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B}=\mathbf{T}\times\mathbf{N}$, $\mathbf{T}\perp\mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds=\mathbf{0}$, but $d\mathbf{B}/ds=-\tau(s)\mathbf{N}$ and $\mathbf{N}\neq\mathbf{0}$, so $\tau(s)=0$.

55. (a) $\mathbf{r}' = s'\,\mathbf{T} \quad\Rightarrow\quad \mathbf{r}'' = s''\,\mathbf{T} + s'\,\mathbf{T}' = s''\,\mathbf{T} + s'\,\dfrac{d\mathbf{T}}{ds}s' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= (s'\,\mathbf{T}) \times [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]\\ &= [(s'\,\mathbf{T}) \times (s''\,\mathbf{T})] + \big[(s'\mathbf{T}) \times (\kappa(s')^2\,\mathbf{N})\big] \qquad \text{(by Property 3 of Theorem 13.4.8 [ET 12.4.8])}\\ &= (s's'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3(\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\mathbf{r}''' &= [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]' = s'''\,\mathbf{T} + s''\,\mathbf{T}' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\mathbf{N}'\\ &= s'''\,\mathbf{T} + s''\frac{d\mathbf{T}}{ds}\,s' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\frac{d\mathbf{N}}{ds}\,s'\\ &= s'''\,\mathbf{T} + s''s'\kappa\,\mathbf{N} + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^3(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) \qquad \text{[by the second formula]}\\ &= [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B}\cdot\mathbf{T} = 0$, $\mathbf{B}\cdot\mathbf{N} = 0$, and $\mathbf{B}\cdot\mathbf{B} = 1$, we get

$$\frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\kappa(s')^3\,\mathbf{B}\cdot\big\{[s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\big\}}{|\kappa(s')^3\,\mathbf{B}|^2} = \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau.$$

57. $\mathbf{r} = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3\right\rangle \quad\Rightarrow\quad \mathbf{r}' = \left\langle 1, t, t^2\right\rangle,\ \mathbf{r}'' = \langle 0, 1, 2t\rangle,\ \mathbf{r}''' = \langle 0, 0, 2\rangle \quad\Rightarrow\quad \mathbf{r}' \times \mathbf{r}'' = \left\langle t^2, -2t, 1\right\rangle \quad\Rightarrow$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle t^2, -2t, 1\right\rangle\cdot\langle 0, 0, 2\rangle}{t^4 + 4t^2 + 1} = \frac{2}{t^4 + 4t^2 + 1}$$

59. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi)\rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$\begin{aligned}L &= \int_0^{2.9\times10^8\times2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2.9\times10^8\times2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\,dt = \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\,\Big]_0^{2.9\times10^8\times2\pi}\\ &= 2.9 \times 10^8 \times 2\pi\,\sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2} \approx 2.07 \times 10^{10}\ \text{Å} \text{ — more than two meters!}\end{aligned}$$

14.4 Motion in Space: Velocity and Acceleration ET 13.4

1. (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over the time interval $[0, 1]$ is

$\mathbf{v}_{\text{ave}} = \dfrac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \dfrac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$. Similarly, over the other intervals we have

$$[0.5, 1]:\quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$[1, 2]:\quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]:\quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$:

$\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is

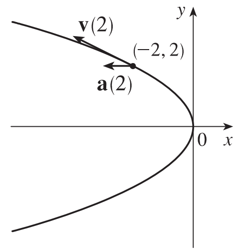
$|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

3. $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle \quad \Rightarrow$ At $t = 2$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -t, 1 \rangle$ $\qquad \mathbf{v}(2) = \langle -2, 1 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -1, 0 \rangle$ $\qquad \mathbf{a}(2) = \langle -1, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$



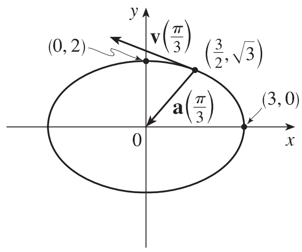
5. $(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} \quad \Rightarrow$ At $t = \pi/3$:

$\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ $\qquad \mathbf{v}\left(\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$ $\qquad \mathbf{a}\left(\frac{\pi}{3}\right) = -\frac{3}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{4 + 5\sin^2 t}$

Notice that $x^2/9 + y^2/4 = \sin^2 t + \cos^2 t = 1$, so the path is an ellipse.



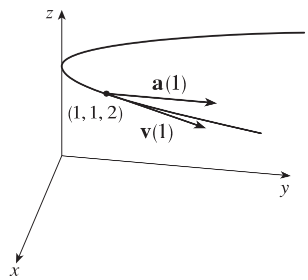
7. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k} \quad \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$ $\qquad \mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(t) = 2\,\mathbf{j}$ $\qquad \mathbf{a}(1) = 2\,\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$

Here $x = t$, $y = t^2 \quad \Rightarrow \quad y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.



9. $\mathbf{r}(t) = \left\langle t^2 + 1, t^3, t^2 - 1 \right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t, 3t^2, 2t \right\rangle, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 2 \rangle$,

$|\mathbf{v}(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 8t^2} = |t|\,\sqrt{9t^2 + 8}$.

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$,

$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

13. $\mathbf{r}(t) = e^t\langle \cos t, \sin t, t \rangle \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\langle \cos t, \sin t, t \rangle + e^t\,\langle -\sin t, \cos t, 1 \rangle = e^t\langle \cos t - \sin t, \sin t + \cos t, t + 1 \rangle$

$$\begin{aligned} \mathbf{a}(t) = \mathbf{v}'(t) &= e^t\langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1 \rangle \\ &= e^t\langle -2\sin t, 2\cos t, t + 2 \rangle \end{aligned}$$

$$\begin{aligned} |\mathbf{v}(t)| &= e^t\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1} \\ &= e^t\sqrt{t^2 + 2t + 3} \end{aligned}$$

15. $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (\mathbf{i} + 2\mathbf{j})\,dt = t\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{C}$ and $\mathbf{k} = \mathbf{v}(0) = \mathbf{C}$,

so $\mathbf{C} = \mathbf{k}$ and $\mathbf{v}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (t\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k})\,dt = \frac{1}{2}t^2\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k} + \mathbf{D}$.

But $\mathbf{i} = \mathbf{r}(0) = \mathbf{D}$, so $\mathbf{D} = \mathbf{i}$ and $\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$.

17. (a) $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}(t) = \int (2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt = t^2\,\mathbf{i} - \cos t\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k} + \mathbf{C}$

and $\mathbf{i} = \mathbf{v}(0) = -\mathbf{j} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} + \mathbf{j}$

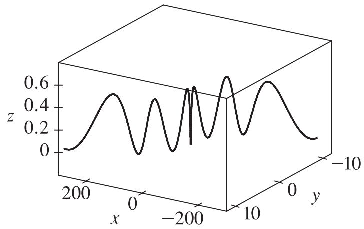
and $\mathbf{v}(t) = \left(t^2 + 1\right)\mathbf{i} + (1 - \cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}$.

$\mathbf{r}(t) = \int \left[\left(t^2 + 1\right)\mathbf{i} + (1 - \cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}\right]dt$

$= \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t)\,\mathbf{j} - \frac{1}{4}\cos 2t\,\mathbf{k} + \mathbf{D}$

But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{D}$, so $\mathbf{D} = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)



19. $\mathbf{r}(t) = \left\langle t^2, 5t, t^2 - 16t\right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$, $|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$

and $\dfrac{d}{dt}\,|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is,

$16t - 64 = 0$ or $t = 4$. Since $\dfrac{d}{dt}\,|\mathbf{v}(t)| < 0$ for $t < 4$ and $\dfrac{d}{dt}\,|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained

at $t = 4$ units of time.

21. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$.

Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the

speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$

and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

23. $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is 30°, the direction of the velocity is $\frac{1}{2}\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right)$. Thus

$\mathbf{v}(0) = 250\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right)$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the

only force is that due to gravity, so $\mathbf{F}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s^2. Thus $\mathbf{a}(t) = -g\,\mathbf{j}$ and $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{c}_1$. But

$250\left(\sqrt{3}\,\mathbf{i} + \mathbf{j}\right) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{c}_2$ where

$\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \dfrac{d}{dt}\left(250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s.

Thus, the maximum height is $(250)(250/g) - g(250/g)^2\,\frac{1}{2} = (250)^2/(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is

$\mathbf{v}(500/g) = 250\sqrt{3}\,\mathbf{i} + [250 - g(500/g)]\,\mathbf{j} = 250\sqrt{3}\,\mathbf{i} - 250\,\mathbf{j}$ and the speed is $|\mathbf{v}(500/g)| = 250\sqrt{3 + 1} = 500$ m/s.

25. As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. Then the ball lands at $t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = \frac{1}{2}v_0\,\sqrt{2}\,\dfrac{v_0\,\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

27. Let α be the angle of elevation. Then $v_0 = 150$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$. Thus $\dfrac{150^2 \sin 2\alpha}{g} = 800 \quad\Rightarrow\quad \sin 2\alpha = \dfrac{800g}{150^2} \approx 0.3484 \quad\Rightarrow\quad 2\alpha \approx 20.4°$ or $180 - 20.4 = 159.6°$. Two angles of elevation then are $\alpha \approx 10.2°$ and $\alpha \approx 79.8°$.

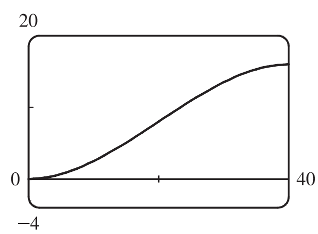
29. Place the catapult at the origin and assume the catapult is 100 meters from the city, so the city lies between $(100, 0)$ and $(600, 0)$. The initial speed is $v_0 = 80$ m/s and let θ be the angle the catapult is set at. As in Example 5, the trajectory of the catapulted rock is given by $\mathbf{r}\,(t) = (80\cos\theta)t\,\mathbf{i} + \left[(80\sin\theta)t - 4.9t^2\right]\mathbf{j}$. The top of the near city wall is at $(100, 15)$, which the rock will hit when $(80\cos\theta)\,t = 100 \quad\Rightarrow\quad t = \dfrac{5}{4\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 15 \quad\Rightarrow$

$80\sin\theta\cdot\dfrac{5}{4\cos\theta} - 4.9\left(\dfrac{5}{4\cos\theta}\right)^2 = 15 \quad\Rightarrow\quad 100\tan\theta - 7.65625\sec^2\theta = 15$. Replacing $\sec^2\theta$ with $\tan^2\theta + 1$ gives $7.65625\tan^2\theta - 100\tan\theta + 22.62625 = 0$. Using the quadratic formula, we have $\tan\theta \approx 0.230324, 12.8309 \quad\Rightarrow$ $\theta \approx 13.0°, 85.5°$. So for $13.0° < \theta < 85.5°$, the rock will land beyond the near city wall. The base of the far wall is located at $(600, 0)$ which the rock hits if $(80\cos\theta)t = 600 \quad\Rightarrow\quad t = \dfrac{15}{2\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 0 \quad\Rightarrow$

$80\sin\theta\cdot\dfrac{15}{2\cos\theta} - 4.9\left(\dfrac{15}{2\cos\theta}\right)^2 = 0 \quad\Rightarrow\quad 600\tan\theta - 275.625\sec^2\theta = 0 \quad\Rightarrow$

$275.625\tan^2\theta - 600\tan\theta + 275.625 = 0$. Solutions are $\tan\theta \approx 0.658678, 1.51819 \quad\Rightarrow\quad \theta \approx 33.4°, 56.6°$. Thus the rock lands beyond the enclosed city ground for $33.4° < \theta < 56.6°$, and the angles that allow the rock to land on city ground are $13.0° < \theta < 33.4°$, $56.6° < \theta < 85.5°$. If you consider that the rock can hit the far wall and bounce back into the city, we calculate the angles that cause the rock to hit the top of the wall at $(600, 15)$: $(80\cos\theta)t = 600 \quad\Rightarrow\quad t = \dfrac{15}{2\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 15 \quad\Rightarrow\quad 600\tan\theta - 275.625\sec^2\theta = 15 \quad\Rightarrow\quad 275.625\tan^2\theta - 600\tan\theta + 290.625 = 0$. Solutions are $\tan\theta \approx 0.727506, 1.44936 \quad\Rightarrow\quad \theta \approx 36.0°, 55.4°$, so the catapult should be set with angle θ where $13.0° < \theta < 36.0°$, $55.4° < \theta < 85.5°$.

31. (a) After t seconds, the boat will be $5t$ meters west of point A. The velocity of the water at that location is $\frac{3}{400}(5t)(40 - 5t)\,\mathbf{j}$. The velocity of the boat in still water is $5\,\mathbf{i}$, so the resultant velocity of the boat is $\mathbf{v}(t) = 5\,\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\,\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{2}t - \frac{3}{16}t^2\right)\mathbf{j}$. Integrating, we obtain $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$. If we place the origin at A (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \quad\Rightarrow\quad \mathbf{C} = \mathbf{0}$ and we have $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$. The boat



reaches the east bank after 8 s, and it is located at $\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\,\mathbf{i} + 16\,\mathbf{j}$. Thus the boat is 16 m downstream.

(b) Let α be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40 - 5(\cos\alpha)t]\;\mathbf{j}$. The resultant velocity of the boat is given by

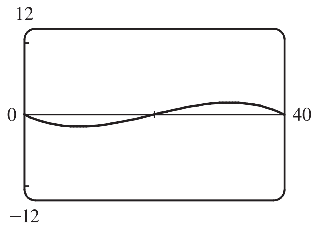
$\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + \frac{3}{400}(5t\cos\alpha)(40 - 5t\cos\alpha)\right]\mathbf{j} = (5\cos\alpha)\,\mathbf{i} + \left(5\sin\alpha + \frac{3}{2}t\cos\alpha - \frac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}$.

Integrating, $\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left(5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed the origin at A). The boat will reach the east bank when $5t\cos\alpha = 40 \;\Rightarrow\; t = \dfrac{40}{5\cos\alpha} = \dfrac{8}{\cos\alpha}$.

In order to land at point $B(40, 0)$ we need $5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha = 0 \;\Rightarrow$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha + \frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha - \frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha = 0 \;\Rightarrow\; \frac{1}{\cos\alpha}(40\sin\alpha + 48 - 32) = 0 \;\Rightarrow$$

$40\sin\alpha + 16 = 0 \;\Rightarrow\; \sin\alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6^\circ$, so the boat should head 23.6° south of east (upstream). The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point B.



33. $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 3t^2\,\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = (3 - 3t^2)\,\mathbf{i} + 6t\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(3 - 3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3 - 3t^2)^2} = 3 + 3t^2$,

$\mathbf{r}''(t) = -6t\,\mathbf{i} + 6\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = (18 + 18t^2)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(3 - 3t^2)(-6t) + (6t)(6)}{3 + 3t^2} = \frac{18t + 18t^3}{3 + 3t^2} = \frac{18t(1 + t^2)}{3(1 + t^2)} = 6t \quad \Big[\text{or by Equation 8,}$$

$$a_T = v' = \frac{d}{dt}\left[3 + 3t^2\right] = 6t\Big] \quad \text{and Equation 10 gives } a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{18 + 18t^2}{3 + 3t^2} = \frac{18(1 + t^2)}{3(1 + t^2)} = 6.$$

35. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,

$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

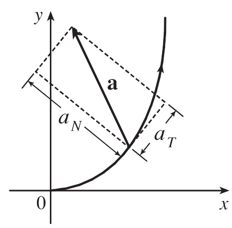
Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\,\cos t - \sin t\,\cos t}{\sqrt{2}} = 0$ and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$.

37. $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + e^{-t}\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = e^t\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $|\mathbf{r}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$,

$\mathbf{r}''(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{k}$. Then $a_T = \dfrac{e^{2t} - e^{-2t}}{e^t + e^{-t}} = \dfrac{(e^t + e^{-t})(e^t - e^{-t})}{e^t + e^{-t}} = e^t - e^{-t} = 2\sinh t$

and $a_N = \dfrac{\left|\sqrt{2}e^{-t}\,\mathbf{i} - 2\,\mathbf{j} - \sqrt{2}e^t\,\mathbf{k}\right|}{e^t + e^{-t}} = \dfrac{\sqrt{2(e^{-2t} + 2 + e^{2t})}}{e^t + e^{-t}} = \sqrt{2}\,\dfrac{e^t + e^{-t}}{e^t + e^{-t}} = \sqrt{2}$.

39. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5\text{ cm/s}^2$ and $a_N \approx 9.0\text{ cm/s}^2$.



41. If the engines are turned off at time t, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a t such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\mathbf{k} \;\Rightarrow$

$$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \frac{s}{t}, 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2}\right\rangle \;\Rightarrow\; 3 + t + s = 6 \;\Rightarrow\; s = 3 - t,$$

so $7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \;\Leftrightarrow\; \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \;\Leftrightarrow\; t^4 + 8t^2 - 12t + 3 = 0.$

It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

14 Review ET 13

CONCEPT CHECK

1. A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.

2. The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.

3. The tangent vector to a smooth curve at a point P with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at P is the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

4. (a)–(f) See Theorem 14.2.3 [ET 13.2.3].

5. Use Formula 14.3.2 [ET 13.3.2], or equivalently, 14.3.3 [ET 13.3.3].

6. (a) The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

(b) $\kappa(t) = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$ (c) $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

7. (a) The unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$.

(b) See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

8. (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(b) $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.

9. See the statement of Kepler's Laws on page 880 [ET page 844].

TRUE-FALSE QUIZ

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

3. False. By Formula 5 of Theorem 14.2.3[ET 13.2.3], $\frac{d}{dt}\,[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

5. False. κ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length s, not with respect to t.

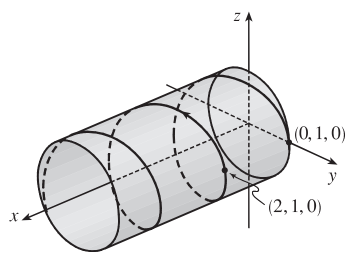
7. True. At an inflection point where f is twice continuously differentiable we must have $f''(x) = 0$, and by Equation 14.3.11 [ET 13.3.11], the curvature is 0 there.

9. False. If $\mathbf{r}(t)$ is the position of a moving particle at time t and $|\mathbf{r}(t)| = 1$ then the particle lies on the unit circle or the unit sphere, but this does not mean that the speed $|\mathbf{r}'(t)|$ must be constant. As a counterexample, let $\mathbf{r}(t) = \left\langle t, \sqrt{1-t^2}\right\rangle$, then $\mathbf{r}'(t) = \left\langle 1, -t/\sqrt{1-t^2}\right\rangle$ and $|\mathbf{r}(t)| = \sqrt{t^2 + 1 - t^2} = 1$ but $|\mathbf{r}'(t)| = \sqrt{1 + t^2/(1-t^2)} = 1/\sqrt{1-t^2}$ which is not constant.

11. True. See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

EXERCISES

1. (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos\,\pi t$, $z = \sin\,\pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the x-axis. Since $x = t$, the curve is a helix.



(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos\,\pi t\,\mathbf{j} + \sin\,\pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin\,\pi t\,\mathbf{j} + \pi \cos\,\pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}''(t) = -\pi^2 \cos\,\pi t\,\mathbf{j} - \pi^2 \sin\,\pi t\,\mathbf{k}$

3. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 16$, $z = 0$. So we can write $x = 4\cos t,\ y = 4\sin t,\ 0 \le t \le 2\pi$. From the equation of the plane, we have $z = 5 - x = 5 - 4\cos t$, so parametric equations for C are $x = 4\cos t,\ y = 4\sin t,\ z = 5 - 4\cos t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

5. $\int_0^1 (t^2\,\mathbf{i} + t\cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k})\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 t\cos \pi t\,dt\right)\mathbf{j} + \left(\int_0^1 \sin \pi t\,dt\right)\mathbf{k}$

$$= \left[\tfrac{1}{3}t^3\right]_0^1 \mathbf{i} + \left(\tfrac{t}{\pi}\sin \pi t\Big]_0^1 - \int_0^1 \tfrac{1}{\pi}\sin \pi t\,dt\right)\mathbf{j} + \left[-\tfrac{1}{\pi}\cos \pi t\right]_0^1 \mathbf{k}$$

$$= \tfrac{1}{3}\,\mathbf{i} + \left[\tfrac{1}{\pi^2}\cos \pi t\right]_0^1 \mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k} = \tfrac{1}{3}\,\mathbf{i} - \tfrac{2}{\pi^2}\,\mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k}$$

where we integrated by parts in the y-component.

7. $\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4 + 16t^6}$ and

$L = \int_0^3 |\mathbf{r}'(t)|\,dt = \int_0^3 \sqrt{4t^2 + 9t^4 + 16t^6}\,dt$. Using Simpson's Rule with $f(t) = \sqrt{4t^2 + 9t^4 + 16t^6}$ and $n = 6$ we have $\Delta t = \frac{3-0}{6} = \frac{1}{2}$ and

$$\begin{aligned} L &\approx \tfrac{\Delta t}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right] \\ &= \tfrac{1}{6}\Bigg[\sqrt{0+0+0} + 4\cdot\sqrt{4\left(\tfrac{1}{2}\right)^2 + 9\left(\tfrac{1}{2}\right)^4 + 16\left(\tfrac{1}{2}\right)^6} + 2\cdot\sqrt{4(1)^2 + 9(1)^4 + 16(1)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{3}{2}\right)^2 + 9\left(\tfrac{3}{2}\right)^4 + 16\left(\tfrac{3}{2}\right)^6} + 2\cdot\sqrt{4(2)^2 + 9(2)^4 + 16(2)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{5}{2}\right)^2 + 9\left(\tfrac{5}{2}\right)^4 + 16\left(\tfrac{5}{2}\right)^6} + \sqrt{4(3)^2 + 9(3)^4 + 16(3)^6}\,\Bigg] \\ &\approx 86.631 \end{aligned}$$

9. The angle of intersection of the two curves, θ, is the angle between their respective tangents at the point of intersection. For both curves the point $(1, 0, 0)$ occurs when $t = 0$.

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}_1'(0) = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}_2'(0) = \mathbf{i}$.

$\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0) = (\mathbf{j} + \mathbf{k})\cdot\mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

11. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle t^2, t, 1\rangle}{|\langle t^2, t, 1\rangle|} = \dfrac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$

(b) $$\begin{aligned} \mathbf{T}'(t) &= -\tfrac{1}{2}(t^4 + t^2 + 1)^{-3/2}(4t^3 + 2t)\left\langle t^2, t, 1\right\rangle + (t^4 + t^2 + 1)^{-1/2}\langle 2t, 1, 0\rangle \\ &= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}}\left\langle t^2, t, 1\right\rangle + \frac{1}{(t^4 + t^2 + 1)^{1/2}}\langle 2t, 1, 0\rangle \\ &= \frac{\left\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t\right\rangle + \left\langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0\right\rangle}{(t^4 + t^2 + 1)^{3/2}} = \frac{\left\langle 2t, -t^4 + 1, -2t^3 - t\right\rangle}{(t^4 + t^2 + 1)^{3/2}} \end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{(t^4 + t^2 + 1)^{3/2}} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^{3/2}} \quad\text{and}\quad \mathbf{N}(t) = \frac{\left\langle 2t, 1 - t^4, -2t^3 - t\right\rangle}{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}.$$

(c) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^2}$

13. $y' = 4x^3$, $y'' = 12x^2$ and $\kappa(x) = \dfrac{|y''|}{[1 + (y')^2]^{3/2}} = \dfrac{|12x^2|}{(1 + 16x^6)^{3/2}}$, so $\kappa(1) = \dfrac{12}{17^{3/2}}$.

15. $\mathbf{r}(t) = \langle \sin 2t, t, \cos 2t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2\cos 2t, 1, -2\sin 2t\rangle \;\Rightarrow\; \mathbf{T}(t) = \frac{1}{\sqrt{5}}\langle 2\cos 2t, 1, -2\sin 2t\rangle \;\Rightarrow$

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -4\sin 2t, 0, -4\cos 2t\rangle \;\Rightarrow\; \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t\rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1\rangle$ and

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}}\langle -1, 2, 0\rangle$. So a normal to the osculating plane is $\langle -1, 2, 0\rangle$ and an equation is $-1(x-0) + 2(y-\pi) + 0(z-1) = 0$ or $x - 2y + 2\pi = 0$.

17. $\mathbf{r}(t) = t\ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $\quad \mathbf{v}(t) = \mathbf{r}'(t) = (1 + \ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k}$,

$|\mathbf{v}(t)| = \sqrt{(1+\ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\quad \mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t}\,\mathbf{i} + e^{-t}\,\mathbf{k}$

19. We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a 45° angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \;\Rightarrow\; \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 14.4.5 [ET 13.4.5] we have $\mathbf{a} = -g\,\mathbf{j}$ where here $g \approx 32$ ft/s^2. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \;\Rightarrow\; \mathbf{v}(t) = \frac{43}{\sqrt{2}}\,\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\,\mathbf{j} \;\Rightarrow\; \mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\,\mathbf{i} + 3.8\,\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \;\Rightarrow$

$t = \dfrac{43}{\sqrt{2}\,g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\,\mathbf{i} + 21.4\,\mathbf{j}$, so the maximum height is approximately 21.4 ft.

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \;\Rightarrow$

$-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \;\Rightarrow\; t \approx 2.11$ s. $\quad \mathbf{r}(2.11) \approx 64.2\,\mathbf{i} - 0.08\,\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

21. (a) Instead of proceeding directly, we use Formula 3 of Theorem 14.2.3 [ET 13.2.3]: $\mathbf{r}(t) = t\,\mathbf{R}(t) \;\Rightarrow$

$\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have

$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] \;\Rightarrow$

$$\begin{aligned}\mathbf{a} = \mathbf{v}' &= e^{-t}[\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t}[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}\end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

PROBLEMS PLUS

1. (a) $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{r}'(t) = -\omega R\sin\omega t\,\mathbf{i} + \omega R\cos\omega t\,\mathbf{j}$, so $\mathbf{r} = R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$ and $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$. $\mathbf{v}\cdot\mathbf{r} = \omega R^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$, so $\mathbf{v}\perp\mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v}\perp\mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.

(b) In (a), we wrote $\mathbf{v}$ in the form $\omega R\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j}$. Clearly $|\mathbf{v}| = \omega R\,|\mathbf{u}| = \omega R$. At speed ωR, the particle completes one revolution, a distance $2\pi R$, in time $T = \dfrac{2\pi R}{\omega R} = \dfrac{2\pi}{\omega}$.

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 R\cos\omega t\,\mathbf{i} - \omega^2 R\sin\omega t\,\mathbf{j} = -\omega^2 R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$, so $\mathbf{a} = -\omega^2\mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2\,|\mathbf{r}| = \omega^2 R$.

(d) By Newton's Second Law (see Section 14.4 [ET 13.4]), $\mathbf{F} = m\mathbf{a}$, so $|\mathbf{F}| = m\,|\mathbf{a}| = mR\omega^2 = \dfrac{m\,(\omega R)^2}{R} = \dfrac{m\,|\mathbf{v}|^2}{R}$.

3. (a) The projectile reaches maximum height when $0 = \dfrac{dy}{dt} = \dfrac{d}{dt}\left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right] = v_0\sin\alpha - gt$; that is, when $t = \dfrac{v_0\sin\alpha}{g}$ and $y = (v_0\sin\alpha)\left(\dfrac{v_0\sin\alpha}{g}\right) - \dfrac{1}{2}g\left(\dfrac{v_0\sin\alpha}{g}\right)^2 = \dfrac{v_0^2\sin^2\alpha}{2g}$. This is the maximum height attained when the projectile is fired with an angle of elevation α. This maximum height is largest when $\alpha = \frac{\pi}{2}$. In that case, $\sin\alpha = 1$ and the maximum height is $\dfrac{v_0^2}{2g}$.

(b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. The points on or inside this parabola are those for which $-R \le x \le R$ and $0 \le y \le \dfrac{-1}{2R}x^2 + \dfrac{R}{2}$. When the projectile is fired at angle of elevation α, the points (x, y) along its path satisfy the relations $x = (v_0\cos\alpha)\,t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, where $0 \le t \le (2v_0\sin\alpha)/g$ (as in Example 14.4.5 [ET 13.4.5]). Thus

$$|x| \le \left|v_0\cos\alpha\left(\frac{2v_0\sin\alpha}{g}\right)\right| = \left|\frac{v_0^2}{g}\sin 2\alpha\right| \le \left|\frac{v_0^2}{g}\right| = |R|.$$ This shows that $-R \le x \le R$.

For t in the specified range, we also have $y = t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = \frac{1}{2}gt\left(\dfrac{2v_0\sin\alpha}{g} - t\right) \ge 0$ and

$$y = (v_0\sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \frac{g}{2v_0^2\cos^2\alpha}x^2 = -\frac{1}{2R\cos^2\alpha}x^2 + (\tan\alpha)\,x.$$ Thus

$$\begin{aligned}
y - \left(\frac{-1}{2R}x^2 + \frac{R}{2}\right) &= \frac{-1}{2R\cos^2\alpha}x^2 + \frac{1}{2R}x^2 + (\tan\alpha)\,x - \frac{R}{2}\\
&= \frac{x^2}{2R}\left(1 - \frac{1}{\cos^2\alpha}\right) + (\tan\alpha)\,x - \frac{R}{2} = \frac{x^2(1-\sec^2\alpha) + 2R\,(\tan\alpha)\,x - R^2}{2R}\\
&= \frac{-(\tan^2\alpha)\,x^2 + 2R\,(\tan\alpha)\,x - R^2}{2R} = \frac{-\left[(\tan\alpha)\,x - R\right]^2}{2R} \le 0
\end{aligned}$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$.

Now let (a,b) be any point on or inside the parabola $y = -\frac{1}{2R}x^2 + \frac{R}{2}$. Then $-R \le a \le R$ and $0 \le b \le -\frac{1}{2R}a^2 + \frac{R}{2}$.

We seek an angle α such that (a,b) lies in the path of the projectile; that is, we wish to find an angle α such that

$b = -\frac{1}{2R\cos^2\alpha}a^2 + (\tan\alpha)\,a$ or equivalently $b = \frac{-1}{2R}(\tan^2\alpha + 1)a^2 + (\tan\alpha)\,a$. Rearranging this equation we get

$\frac{a^2}{2R}\tan^2\alpha - a\tan\alpha + \left(\frac{a^2}{2R} + b\right) = 0$ or $a^2(\tan\alpha)^2 - 2aR(\tan\alpha) + (a^2 + 2bR) = 0 \;\; (*)$. This quadratic equation

for $\tan\alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2 - 4AC \ge 0 \;\; \Leftrightarrow$

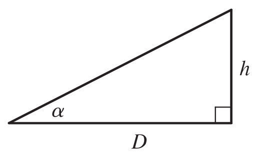
$(-2aR)^2 - 4a^2(a^2 + 2bR) \ge 0 \;\; \Leftrightarrow \;\; 4a^2(R^2 - a^2 - 2bR) \ge 0 \;\; \Leftrightarrow \;\; -a^2 - 2bR + R^2 \ge 0 \;\; \Leftrightarrow$

$b \le \frac{1}{2R}(R^2 - a^2) \;\; \Leftrightarrow \;\; b \le \frac{-1}{2R}a^2 + \frac{R}{2}$. This condition is satisfied since (a,b) is on or inside the parabola

$y = -\frac{1}{2R}x^2 + \frac{R}{2}$. It follows that (a,b) lies in the path of the projectile when $\tan\alpha$ satisfies $(*)$, that is, when

$$\tan\alpha = \frac{2aR \pm \sqrt{4a^2(R^2 - a^2 - 2bR)}}{2a^2} = \frac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}.$$

(c)



If the gun is pointed at a target with height h at a distance D downrange, then $\tan\alpha = h/D$. When the projectile reaches a distance D downrange (remember we are assuming that it doesn't hit the ground first), we have $D = x = (v_0\cos\alpha)t$,

so $t = \frac{D}{v_0\cos\alpha}$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 = D\tan\alpha - \frac{gD^2}{2v_0^2\cos^2\alpha}$.

Meanwhile, the target, whose x-coordinate is also D, has fallen from height h to height

$h - \frac{1}{2}gt^2 = D\tan\alpha - \frac{gD^2}{2v_0^2\cos^2\alpha}$. Thus the projectile hits the target.

5. (a) $\mathbf{a} = -g\,\mathbf{j} \;\; \Rightarrow \;\; \mathbf{v} = \mathbf{v}_0 - gt\,\mathbf{j} = 2\,\mathbf{i} - gt\,\mathbf{j} \;\; \Rightarrow \;\; \mathbf{s} = \mathbf{s}_0 + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} = 3.5\,\mathbf{j} + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} \;\; \Rightarrow$

$\mathbf{s} = 2t\,\mathbf{i} + \left(3.5 - \frac{1}{2}gt^2\right)\mathbf{j}$. Therefore $y = 0$ when $t = \sqrt{7/g}$ seconds. At that instant, the ball is $2\sqrt{7/g} \approx 0.94$ ft to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are $(0.94, 0)$. At impact, the velocity is $\mathbf{v} = 2\,\mathbf{i} - \sqrt{7g}\,\mathbf{j}$, so the speed is $|\mathbf{v}| = \sqrt{4 + 7g} \approx 15$ ft/s.

(b) The slope of the curve when $t = \sqrt{\frac{7}{g}}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt}{2} = \frac{-g\sqrt{7/g}}{2} = \frac{-\sqrt{7g}}{2}$. Thus $\cot\theta = \frac{\sqrt{7g}}{2}$ and $\theta \approx 7.6^\circ$.

(c) From (a), $|\mathbf{v}| = \sqrt{4 + 7g}$. So the ball rebounds with speed $0.8\sqrt{4 + 7g} \approx 12.08$ ft/s at angle of inclination $90^\circ - \theta \approx 82.3886^\circ$. By Example 14.4.5 [ET 13.4.5], the horizontal distance traveled between bounces is

$d = \frac{v_0^2\sin 2\alpha}{g}$, where $v_0 \approx 12.08$ ft/s and $\alpha \approx 82.3886^\circ$. Therefore, $d \approx 1.197$ ft. So the ball strikes the floor at

about $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table's edge.

7. The trajectory of the projectile is given by $\mathbf{r}(t) = (v\cos\alpha)t\,\mathbf{i} + \left[(v\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$, so

$\mathbf{v}(t) = \mathbf{r}'(t) = v\cos\alpha\,\mathbf{i} + (v\sin\alpha - gt)\,\mathbf{j}$ and

$$|\mathbf{v}(t)| = \sqrt{(v\cos\alpha)^2 + (v\sin\alpha - gt)^2} = \sqrt{v^2 - (2vg\sin\alpha)\,t + g^2t^2} = \sqrt{g^2\left(t^2 - \frac{2v}{g}(\sin\alpha)\,t + \frac{v^2}{g^2}\right)}$$

$$= g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2} - \frac{v^2}{g^2}\sin^2\alpha} = g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha}$$

The projectile hits the ground when $(v\sin\alpha)t - \frac{1}{2}gt^2 = 0 \quad\Rightarrow\quad t = \frac{2v}{g}\sin\alpha$, so the distance traveled by the projectile is

$$L(\alpha) = \int_0^{(2v/g)\sin\alpha} |\mathbf{v}(t)|\,dt = \int_0^{(2v/g)\sin\alpha} g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha}\,dt$$

$$= g\left[\frac{t - (v/g)\sin\alpha}{2}\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right.$$

$$\left. + \frac{[(v/g)\cos\alpha]^2}{2}\ln\left(t - \frac{v}{g}\sin\alpha + \sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]_0^{(2v/g)\sin\alpha}$$

[using Formula 21 in the Table of Integrals]

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} + \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right.$$

$$\left. + \frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} - \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(-\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]$$

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\cdot\frac{v}{g} + \frac{v^2}{g^2}\cos^2\alpha\ln\left(\frac{v}{g}\sin\alpha + \frac{v}{g}\right) + \frac{v}{g}\sin\alpha\cdot\frac{v}{g} - \frac{v^2}{g^2}\cos^2\alpha\ln\left(-\frac{v}{g}\sin\alpha + \frac{v}{g}\right)\right]$$

$$= \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{(v/g)\sin\alpha + v/g}{-(v/g)\sin\alpha + v/g}\right) = \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

We want to maximize $L(\alpha)$ for $0 \le \alpha \le \pi/2$.

$$L'(\alpha) = \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{1-\sin\alpha}{1+\sin\alpha}\cdot\frac{2\cos\alpha}{(1-\sin\alpha)^2} - 2\cos\alpha\,\sin\alpha\,\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{2}{\cos\alpha} - 2\cos\alpha\,\sin\alpha\,\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{g}\cos\alpha\left[1 - \sin\alpha\,\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right] = \frac{v^2}{g}\cos\alpha\left[2 - \sin\alpha\,\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$L(\alpha)$ has critical points for $0 < \alpha < \pi/2$ when $L'(\alpha) = 0 \quad\Rightarrow\quad 2 - \sin\alpha\ln\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right) = 0$ (since $\cos\alpha \ne 0$).

Solving by graphing (or using a CAS) gives $\alpha \approx 0.9855$. Compare values at the critical point and the endpoints: $L(0) = 0$, $L(\pi/2) = v^2/g$, and $L(0.9855) \approx 1.20v^2/g$. Thus the distance traveled by the projectile is maximized for $\alpha \approx 0.9855$ or $\approx 56°$.

15 □ PARTIAL DERIVATIVES □ ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°\text{C}$ and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°\text{C}$ without wind.

(b) The question is asking: when the temperature is $-20°\text{C}$, what wind speed gives a wind-chill index of $-30°\text{C}$? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°\text{C}$? From Table 1, the temperature is $-35°\text{C}$.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°\text{C}$. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25} = 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^{\alpha}K^{1-\alpha}$:
$P(2L, 2K) = b(2L)^{\alpha}(2K)^{1-\alpha} = b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha} = 2P(L, K)$.

5. (a) According to Table 4, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

7. (a) $f(2, 0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y. Thus the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x, y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0, \infty)$, the range of $e^{g(x,y)} = e^{3xy}$ is $(0, \infty)$.

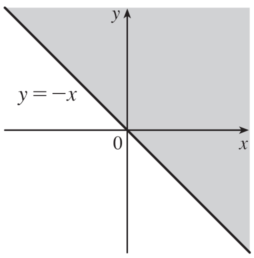
The range of x^2 is $[0, \infty)$, so the range of the product $x^2 e^{3xy}$ is $[0, \infty)$.

9. (a) $f(2, -1, 6) = e^{\sqrt{6-2^2-(-1)^2}} = e^{\sqrt{1}} = e.$

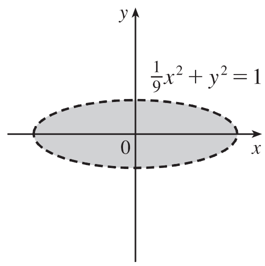
(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z - x^2 - y^2 \geq 0 \;\;\Rightarrow\;\; z \geq x^2 + y^2$. Thus the domain of f is $\{(x, y, z) \mid z \geq x^2 + y^2\}$.

(c) Since $\sqrt{z - x^2 - y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1, \infty)$.

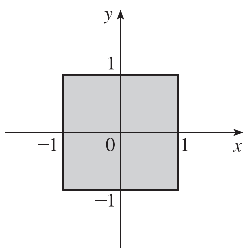
11. $\sqrt{x + y}$ is defined only when $x + y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x, y) \mid y \geq -x\}$.



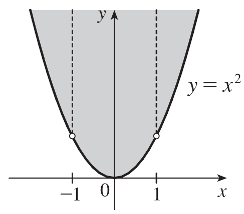
13. $\ln(9 - x^2 - 9y^2)$ is defined only when $9 - x^2 - 9y^2 > 0$, or $\frac{1}{9}x^2 + y^2 < 1$. So the domain of f is $\left\{(x, y) \mid \frac{1}{9}x^2 + y^2 < 1\right\}$, the interior of an ellipse.



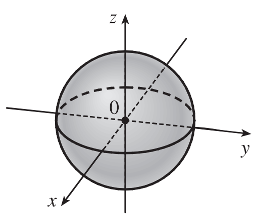
15. $\sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0$, or $x^2 \leq 1 \;\;\Leftrightarrow\;\; -1 \leq x \leq 1$, and $\sqrt{1 - y^2}$ is defined only when $1 - y^2 \geq 0$, or $y^2 \leq 1 \;\;\Leftrightarrow\;\; -1 \leq y \leq 1$. Thus the domain of f is $\{(x, y) \mid -1 \leq x \leq 1, \; -1 \leq y \leq 1\}$.



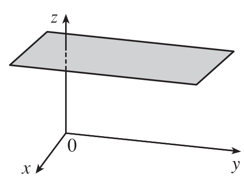
17. $\sqrt{y - x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0 \;\;\Rightarrow\;\; x = \pm 1$. Thus the domain of f is $\left\{(x, y) \mid y \geq x^2, \; x \neq \pm 1\right\}$.



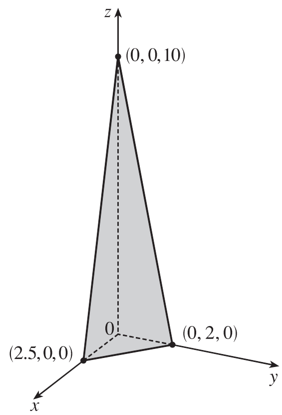
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\right\}$ (the points inside or on the sphere of radius 1, center the origin).



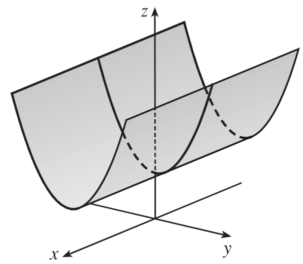
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



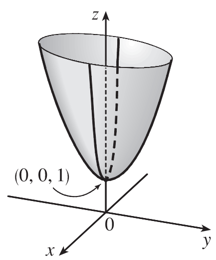
23. $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



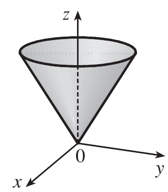
25. $z = y^2 + 1$, a parabolic cylinder



27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



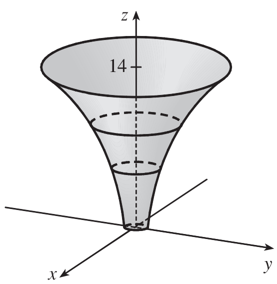
29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



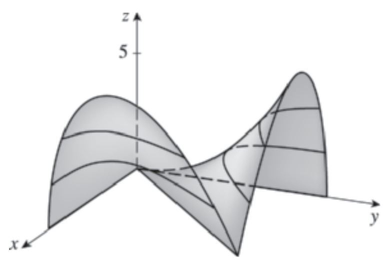
31. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

33. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

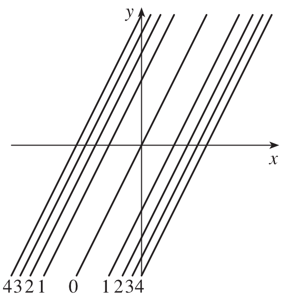
35.



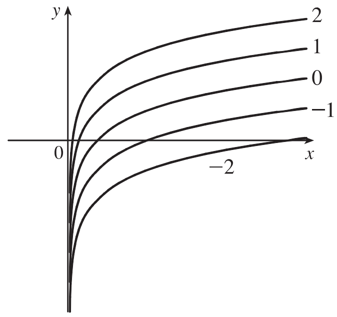
37.



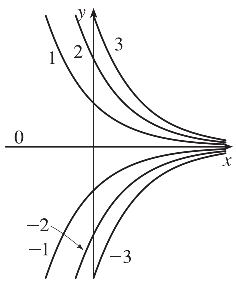
39. The level curves are $(y-2x)^2 = k$ or $y = 2x \pm \sqrt{k}$, $k \geq 0$, a family of pairs of parallel lines.



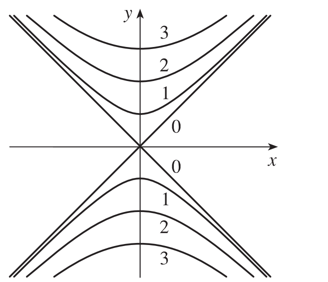
41. The level curves are $y - \ln x = k$ or $y = \ln x + k$.



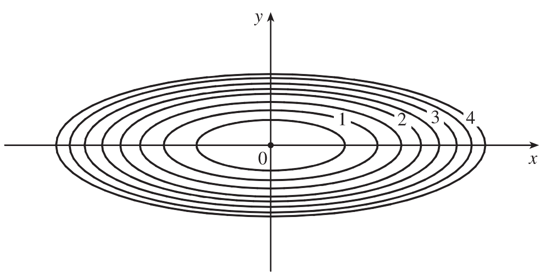
43. The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.

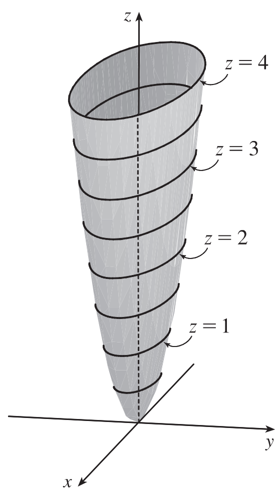


45. The level curves are $\sqrt{y^2 - x^2} = k$ or $y^2 - x^2 = k^2$, $k \geq 0$. When $k = 0$ the level curve is the pair of lines $y = \pm x$. For $k > 0$, the level curves are hyperbolas with axis the y-axis.



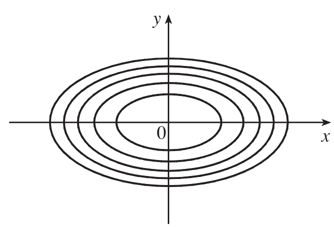
47. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)
The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.



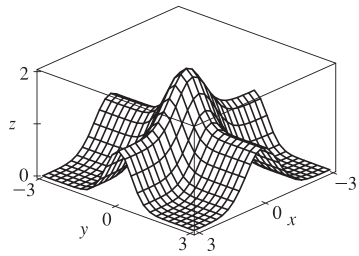


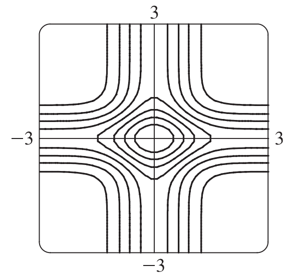
If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

49. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $[0 < k \leq 100]$, a family of ellipses.

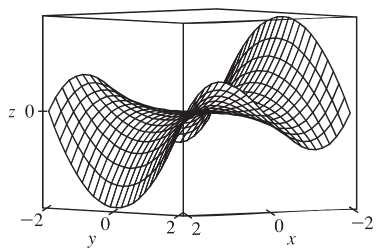


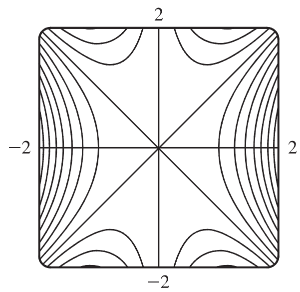
51. $f(x,y) = e^{-x^2} + e^{-2y^2}$





53. $f(x,y) = xy^2 - x^3$





The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

55. (a) C (b) II

Reasons: This function is periodic in both x and y, and the function is the same when x is interchanged with y, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the x- and y-axes. These conditions are satisfied only by C and II.

57. (a) F (b) I

Reasons: This function is periodic in both x and y but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

59. (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the xz-plane is the parabola $z = 1 - x^2$ and the trace in the yz-plane is the parabola $z = 1 - y^2$, so the graph is B.

61. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

63. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

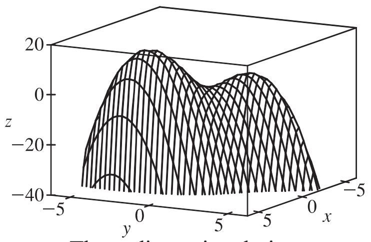
65. (a) The graph of g is the graph of f shifted upward 2 units.

(b) The graph of g is the graph of f stretched vertically by a factor of 2.

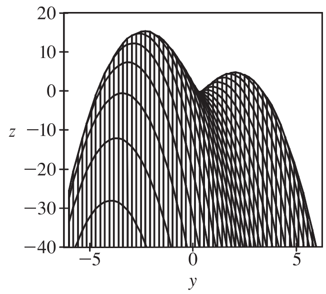
(c) The graph of g is the graph of f reflected about the xy-plane.

(d) The graph of $g(x,y) = -f(x,y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

67. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



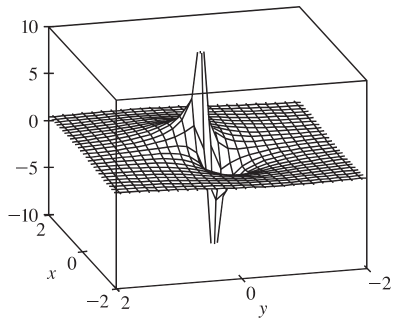
Three-dimensional view



Front view

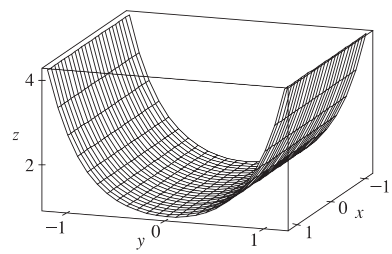
It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

69.



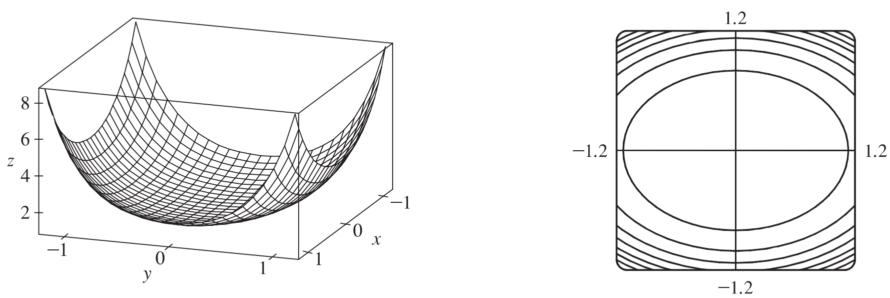
$f(x, y) = \dfrac{x + y}{x^2 + y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x, y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x, y) \to \infty$, while in others $f(x, y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x, y)$ approaches 0 along the line $y = -x$.

71. $f(x, y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



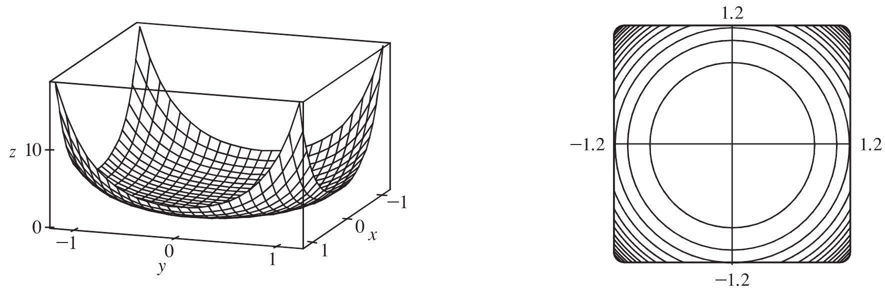
$c = 0$

For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.



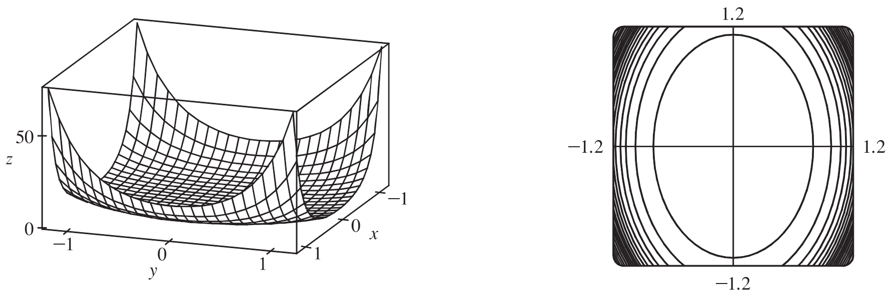
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



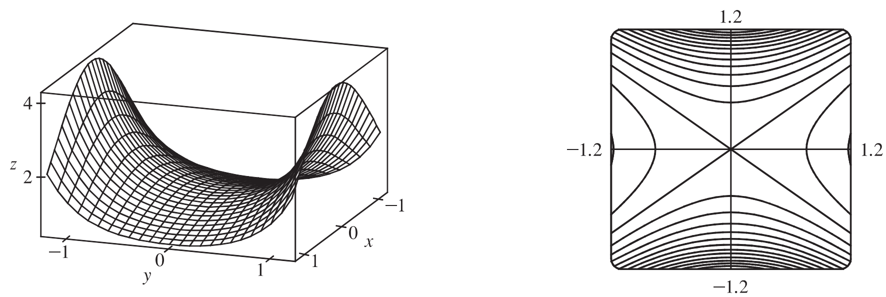
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

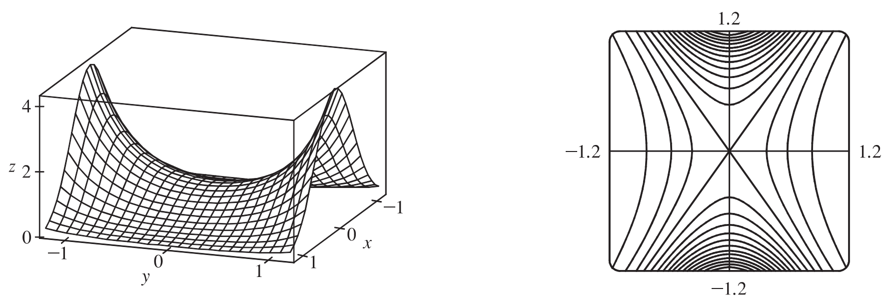


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.



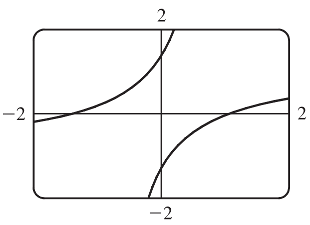
$c = -0.5$ (level curves in increments of 0.25)



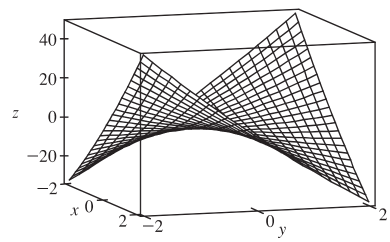
$c = -2$ (level curves in increments of 0.25)

73. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

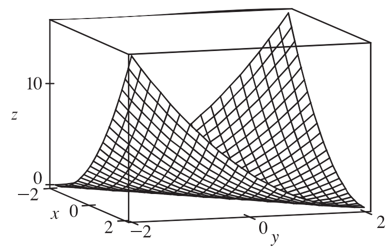
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5,\ z = 2$



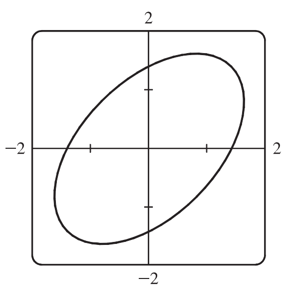
$c = -10$



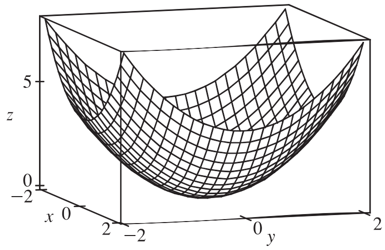
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

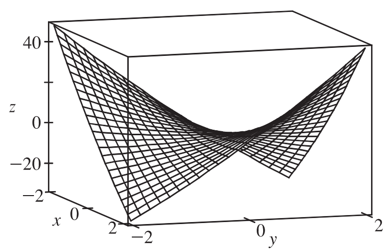
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1,\ z = 2$



$c = 0$



$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

75. (a) $P = bL^\alpha K^{1-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = bL^\alpha K^{-\alpha} \quad \Rightarrow \quad \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \quad \Rightarrow \quad \ln \dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \quad \Rightarrow$

$$\ln \frac{P}{K} = \ln b + \alpha \ln\left(\frac{L}{K}\right)$$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	-0.02	-0.06
1901	-0.04	-0.02
1902	-0.04	0
1903	-0.07	-0.05
1904	-0.13	-0.12
1905	-0.18	-0.04
1906	-0.20	-0.07
1907	-0.23	-0.15
1908	-0.41	-0.38
1909	-0.33	-0.24
1910	-0.35	-0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	-0.38	-0.34
1912	-0.38	-0.24
1913	-0.41	-0.25
1914	-0.47	-0.37
1915	-0.53	-0.34
1916	-0.49	-0.28
1917	-0.53	-0.39
1918	-0.60	-0.50
1919	-0.68	-0.57
1920	-0.74	-0.57
1921	-1.05	-0.85
1922	-0.98	-0.59

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \quad \Rightarrow \quad b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^\alpha K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

15.2 Limits and Continuity ET 14.2

1. In general, we can't say anything about $f(3,1)$! $\lim\limits_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, $(3,1)$. If f is continuous, we know that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim\limits_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

3. We make a table of values of $f(x,y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.2	-0.1	-0.05	0	0.05	0.1	0.2
-0.2	-2.551	-2.525	-2.513	-2.500	-2.488	-2.475	-2.451
-0.1	-2.525	-2.513	-2.506	-2.500	-2.494	-2.488	-2.475
-0.05	-2.513	-2.506	-2.503	-2.500	-2.497	-2.494	-2.488
0	-2.500	-2.500	-2.500		-2.500	-2.500	-2.500
0.05	-2.488	-2.494	-2.497	-2.500	-2.503	-2.506	-2.513
0.1	-2.475	-2.488	-2.494	-2.500	-2.506	-2.513	-2.525
0.2	-2.451	-2.475	-2.488	-2.500	-2.513	-2.525	-2.551

As the table shows, the values of $f(x,y)$ seem to approach -2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = -2.5$. Since f is a rational function, it is continuous on its domain. f is defined at $(0,0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{0^2 0^3 + 0^3 0^2 - 5}{2 - 0 \cdot 0} = -\dfrac{5}{2}$, verifying our guess.

5. $f(x,y) = 5x^3 - x^2y^2$ is a polynomial, and hence continuous, so $\lim\limits_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 5(1)^3 - (1)^2(2)^2 = 1$.

7. $f(x,y) = \dfrac{4-xy}{x^2+3y^2}$ is a rational function and hence continuous on its domain.

$(2,1)$ is in the domain of f, so f is continuous there and $\lim\limits_{(x,y)\to(2,1)} f(x,y) = f(2,1) = \dfrac{4-(2)(1)}{(2)^2+3(1)^2} = \dfrac{2}{7}$.

9. $f(x,y) = y^4/(x^4+3y^4)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^4 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the y-axis. Then for $y \neq 0$, $f(0,y) = y^4/3y^4 = 1/3$, so $f(x,y) \to 1/3$. Since f has two different limits along two different lines, the limit does not exist.

11. $f(x,y) = (xy\cos y)/(3x^2+y^2)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = (x^2\cos x)/4x^2 = \frac{1}{4}\cos x$ for $x \neq 0$, so $f(x,y) \to \frac{1}{4}$ along this line. Thus the limit does not exist.

13. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$ since $|y| \leq \sqrt{x^2+y^2}$, and $|x| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

15. Let $f(x,y) = \dfrac{x^2ye^y}{x^4+4y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y = x$ also gives a limit of 0. But $f(x,x^2) = \dfrac{x^2x^2\,e^{x^2}}{x^4+4(x^2)^2} = \dfrac{x^4e^{x^2}}{5x^4} = \dfrac{e^{x^2}}{5}$ for $x \neq 0$, so $f(x,y) \to e^0/5 = \frac{1}{5}$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

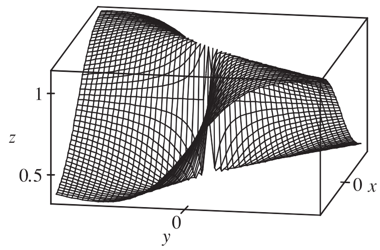
17.
$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\ &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2\end{aligned}$$

19. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z) = e^{-xy}\sin(\pi z/2)$ is a continuous function. Then

$$\lim_{(x,y,z)\to(3,0,1)} f(x,y,z) = f(3,0,1) = e^{-(3)(0)}\sin(\pi\cdot 1/2) = 1.$$

21. $f(x,y,z) = \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

23.

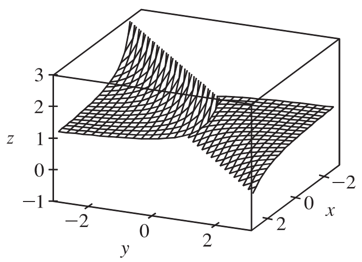


From the ridges on the graph, we see that as $(x,y) \to (0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

25. $h(x,y) = g(f(x,y)) = (2x+3y-6)^2 + \sqrt{2x+3y-6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain.

$D = \{(x,y) \mid 2x+3y-6 \geq 0\} = \left\{(x,y) \mid y \geq -\frac{2}{3}x + 2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

27.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x,y) = e^{1/(x-y)}$ as a composition of functions, $g(x,y) = 1/(x-y)$ is a rational function and therefore continuous except where $x - y = 0 \;\Rightarrow\; y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x,y)) = e^{1/(x-y)} = f(x,y)$ is continuous except along the line $y = x$, as we suspected.

29. The functions $\sin(xy)$ and $e^x - y^2$ are continuous everywhere, so $F(x,y) = \dfrac{\sin(xy)}{e^x - y^2}$ is continuous except where $e^x - y^2 = 0 \;\Rightarrow\; y^2 = e^x \;\Rightarrow\; y = \pm\sqrt{e^x} = \pm e^{\frac{1}{2}x}$. Thus F is continuous on its domain $\left\{(x,y) \mid y \neq \pm e^{x/2}\right\}$.

31. $F(x,y) = \arctan\left(x + \sqrt{y}\right) = g(f(x,y))$ where $f(x,y) = x + \sqrt{y}$, continuous on its domain $\{(x,y) \mid y \geq 0\}$, and $g(t) = \arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x,y) \mid y \geq 0\}$.

33. $G(x,y) = \ln\left(x^2 + y^2 - 4\right) = g(f(x,y))$ where $f(x,y) = x^2 + y^2 - 4$, continuous on $\mathbb{R}^2$, and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus G is continuous on its domain $\left\{(x,y) \mid x^2 + y^2 - 4 > 0\right\} = \left\{(x,y) \mid x^2 + y^2 > 4\right\}$, the exterior of the circle $x^2 + y^2 = 4$.

35. $\sqrt{y}$ is continuous on its domain $\{y \mid y \geq 0\}$ and $x^2 - y^2 + z^2$ is continuous everywhere, so $f(x,y,z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$ is continuous for $y \geq 0$ and $x^2 - y^2 + z^2 \neq 0 \;\Rightarrow\; y^2 \neq x^2 + z^2$, that is, $\left\{(x,y,z) \mid y \geq 0, y \neq \sqrt{x^2 + z^2}\right\}$.

37. $f(x,y) = \begin{cases} \dfrac{x^2 y^3}{2x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $\left|x^2 y^3/(2x^2 + y^2)\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y^3}{2x^2 + y^2} = 0$.

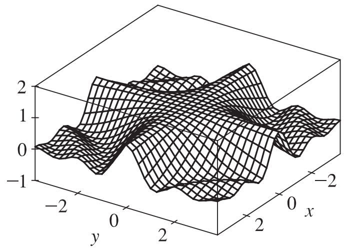
But $f(0,0) = 1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

39. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2} = \lim_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

41. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{e^{-r^2}(-2r)}{2r}$ [using l'Hospital's Rule]

$$= \lim_{r\to 0^+} -e^{-r^2} = -e^0 = -1$$

43. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$



From the graph, it appears that f is continuous everywhere. We know xy is continuous on $\mathbb{R}^2$ and $\sin t$ is continuous everywhere, so $\sin(xy)$ is continuous on $\mathbb{R}^2$ and $\dfrac{\sin(xy)}{xy}$ is continuous on $\mathbb{R}^2$ except possibly where $xy=0$. To show that f is continuous at those points, consider any point (a,b) in $\mathbb{R}^2$ where $ab=0$. Because xy is continuous, $xy\to ab=0$ as $(x,y)\to(a,b)$. If we let $t=xy$, then $t\to 0$ as $(x,y)\to(a,b)$ and $\displaystyle\lim_{(x,y)\to(a,b)} \frac{\sin(xy)}{xy} = \lim_{t\to 0}\frac{\sin(t)}{t} = 1$ by Equation 3.4.2 [ET 3.3.2]. Thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and f is continuous on $\mathbb{R}^2$.

45. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2+|\mathbf{a}|^2 - 2|\mathbf{x}|\,|\mathbf{a}| = (|\mathbf{x}|-|\mathbf{a}|)^2$, we have $\big||\mathbf{x}|-|\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon>0$ be given and set $\delta=\epsilon$. Then if $0<|\mathbf{x}-\mathbf{a}|<\delta$, $\big||\mathbf{x}|-|\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}}|\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives — ET 14.3

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158,21,9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 AM when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158,21,9)$ to be positive. $f_y(158,21,9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158,21,9)$ to be negative. $f_t(158,21,9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air

temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

3. (a) By Definition 4, $f_T(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15 + h, 30) - f(-15, 30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15, 30) \approx \dfrac{f(-10, 30) - f(-15, 30)}{5} = \dfrac{-20 - (-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15, 30) \approx \dfrac{f(-20, 30) - f(-15, 30)}{-5} = \dfrac{-33 - (-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15, 30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°\text{C}$ and the wind speed is 30 km/h, the apparent temperature rises by about $1.3°\text{C}$ for every degree that the actual temperature rises.

Similarly, $f_v(-15, 30) = \lim\limits_{h \to 0} \dfrac{f(-15, 30 + h) - f(-15, 30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15, 30) \approx \dfrac{f(-15, 40) - f(-15, 30)}{10} = \dfrac{-27 - (-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15, 30) \approx \dfrac{f(-15, 20) - f(-15, 30)}{-10} = \dfrac{-24 - (-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15, 30)$ to be approximately -0.15. Thus, when the actual temperature is $-15°\text{C}$ and the wind speed is 30 km/h, the apparent temperature decreases by about $0.15°\text{C}$ for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T, v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v \to \infty} (\partial W/\partial v) = 0$.

5. (a) If we start at $(1, 2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1, 2)$ is positive.

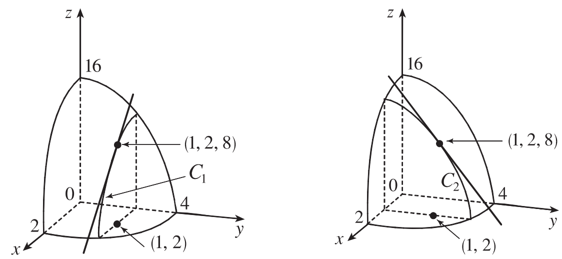
(b) If we start at $(1, 2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1, 2)$ is negative.

7. (a) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1, 2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1, 2)$ is positive.

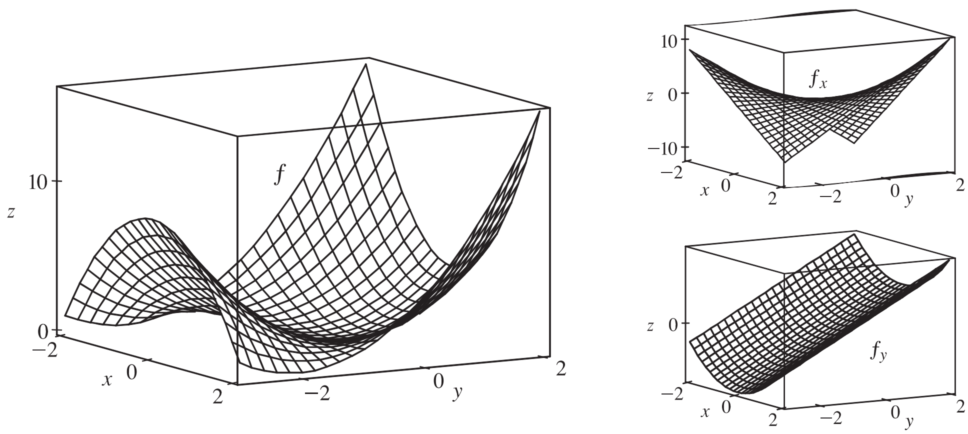
(b) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1, 2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1, 2)$ is negative.

9. First of all, if we start at the point $(3, -3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3, -1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3, -1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

11. $f(x, y) = 16 - 4x^2 - y^2 \quad \Rightarrow \quad f_x(x, y) = -8x$ and $f_y(x, y) = -2y \quad \Rightarrow \quad f_x(1, 2) = -8$ and $f_y(1, 2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.



13. $f(x, y) = x^2 + y^2 + x^2y \quad \Rightarrow \quad f_x = 2x + 2xy, \quad f_y = 2y + x^2$



Note that the traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz-plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

15. $f(x, y) = y^5 - 3xy \quad \Rightarrow \quad f_x(x, y) = 0 - 3y = -3y$, $f_y(x, y) = 5y^4 - 3x$

17. $f(x, t) = e^{-t} \cos \pi x \quad \Rightarrow \quad f_x(x, t) = e^{-t}\,(-\sin \pi x)\,(\pi) = -\pi e^{-t} \sin \pi x$, $f_t(x, t) = e^{-t}(-1) \cos \pi x = -e^{-t} \cos \pi x$

19. $z = (2x + 3y)^{10} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = 10(2x + 3y)^9 \cdot 2 = 20(2x + 3y)^9$, $\dfrac{\partial z}{\partial y} = 10(2x + 3y)^9 \cdot 3 = 30(2x + 3y)^9$

21. $f(x, y) = \dfrac{x - y}{x + y} \quad \Rightarrow \quad f_x(x, y) = \dfrac{(1)(x + y) - (x - y)(1)}{(x + y)^2} = \dfrac{2y}{(x + y)^2}$,

$f_y(x, y) = \dfrac{(-1)(x + y) - (x - y)(1)}{(x + y)^2} = -\dfrac{2x}{(x + y)^2}$

23. $w = \sin \alpha \, \cos \beta \quad \Rightarrow \quad \dfrac{\partial w}{\partial \alpha} = \cos \alpha \, \cos \beta$, $\dfrac{\partial w}{\partial \beta} = -\sin \alpha \, \sin \beta$

25. $f(r, s) = r \ln(r^2 + s^2) \quad \Rightarrow \quad f_r(r, s) = r \cdot \dfrac{2r}{r^2 + s^2} + \ln(r^2 + s^2) \cdot 1 = \dfrac{2r^2}{r^2 + s^2} + \ln(r^2 + s^2)$,

$f_s(r, s) = r \cdot \dfrac{2s}{r^2 + s^2} + 0 = \dfrac{2rs}{r^2 + s^2}$

27. $u = te^{w/t} \Rightarrow \dfrac{\partial u}{\partial t} = t \cdot e^{w/t}(-wt^{-2}) + e^{w/t} \cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1 - \dfrac{w}{t}\right),\ \dfrac{\partial u}{\partial w} = te^{w/t} \cdot \dfrac{1}{t} = e^{w/t}$

29. $f(x,y,z) = xz - 5x^2y^3z^4 \Rightarrow f_x(x,y,z) = z - 10xy^3z^4,\ f_y(x,y,z) = -15x^2y^2z^4,\ f_z(x,y,z) = x - 20x^2y^3z^3$

31. $w = \ln(x + 2y + 3z) \Rightarrow \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z},\ \dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z},\ \dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

33. $u = xy\sin^{-1}(yz) \Rightarrow \dfrac{\partial u}{\partial x} = y\sin^{-1}(yz),\ \dfrac{\partial u}{\partial y} = xy \cdot \dfrac{1}{\sqrt{1-(yz)^2}}\,(z) + \sin^{-1}(yz) \cdot x = \dfrac{xyz}{\sqrt{1-y^2z^2}} + x\sin^{-1}(yz),$

$\dfrac{\partial u}{\partial z} = xy \cdot \dfrac{1}{\sqrt{1-(yz)^2}}\,(y) = \dfrac{xy^2}{\sqrt{1-y^2z^2}}$

35. $f(x,y,z,t) = xyz^2\tan(yt) \Rightarrow f_x(x,y,z,t) = yz^2\tan(yt),$

$f_y(x,y,z,t) = xyz^2 \cdot \sec^2(yt) \cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt),$

$f_z(x,y,z,t) = 2xyz\tan(yt),\ f_t(x,y,z,t) = xyz^2\sec^2(yt) \cdot y = xy^2z^2\sec^2(yt)$

37. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For each $i = 1, \ldots, n$, $u_{x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$.

39. $f(x,y) = \ln\left(x + \sqrt{x^2+y^2}\right) \Rightarrow$

$f_x(x,y) = \dfrac{1}{x+\sqrt{x^2+y^2}}\left[1 + \frac{1}{2}(x^2+y^2)^{-1/2}(2x)\right] = \dfrac{1}{x+\sqrt{x^2+y^2}}\left(1 + \dfrac{x}{\sqrt{x^2+y^2}}\right),$

so $f_x(3,4) = \dfrac{1}{3+\sqrt{3^2+4^2}}\left(1 + \dfrac{3}{\sqrt{3^2+4^2}}\right) = \frac{1}{8}\left(1 + \frac{3}{5}\right) = \frac{1}{5}$.

41. $f(x,y,z) = \dfrac{y}{x+y+z} \Rightarrow f_y(x,y,z) = \dfrac{1(x+y+z) - y(1)}{(x+y+z)^2} = \dfrac{x+z}{(x+y+z)^2},$

so $f_y(2,1,-1) = \dfrac{2+(-1)}{(2+1+(-1))^2} = \dfrac{1}{4}$.

43. $f(x,y) = xy^2 - x^3y \Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)y^2 - (x+h)^3y - (xy^2 - x^3y)}{h}$$

$$= \lim_{h\to 0}\frac{h(y^2 - 3x^2y - 3xyh - yh^2)}{h} = \lim_{h\to 0}(y^2 - 3x^2y - 3xyh - yh^2) = y^2 - 3x^2y$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0}\frac{x(y+h)^2 - x^3(y+h) - (xy^2 - x^3y)}{h} = \lim_{h\to 0}\frac{h(2xy + xh - x^3)}{h}$$

$$= \lim_{h\to 0}(2xy + xh - x^3) = 2xy - x^3$$

45. $x^2 + y^2 + z^2 = 3xyz \Rightarrow \dfrac{\partial}{\partial x}(x^2+y^2+z^2) = \dfrac{\partial}{\partial x}(3xyz) \Rightarrow 2x + 0 + 2z\dfrac{\partial z}{\partial x} = 3y\left(x\dfrac{\partial z}{\partial x} + z \cdot 1\right) \Leftrightarrow$

$2z\dfrac{\partial z}{\partial x} - 3xy\dfrac{\partial z}{\partial x} = 3yz - 2x \Leftrightarrow (2z - 3xy)\dfrac{\partial z}{\partial x} = 3yz - 2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz - 2x}{2z - 3xy}$.

$\dfrac{\partial}{\partial y}(x^2+y^2+z^2) = \dfrac{\partial}{\partial y}(3xyz) \Rightarrow 0 + 2y + 2z\dfrac{\partial z}{\partial y} = 3x\left(y\dfrac{\partial z}{\partial y} + z \cdot 1\right) \Leftrightarrow 2z\dfrac{\partial z}{\partial y} - 3xy\dfrac{\partial z}{\partial y} = 3xz - 2y \Leftrightarrow$

$(2z - 3xy)\dfrac{\partial z}{\partial y} = 3xz - 2y$, so $\dfrac{\partial z}{\partial y} = \dfrac{3xz - 2y}{2z - 3xy}$.

47. $x - z = \arctan(yz) \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(x-z) = \dfrac{\partial}{\partial x}(\arctan(yz)) \quad \Rightarrow \quad 1 - \dfrac{\partial z}{\partial x} = \dfrac{1}{1+(yz)^2} \cdot y\,\dfrac{\partial z}{\partial x} \quad \Leftrightarrow$

$1 = \left(\dfrac{y}{1+y^2z^2} + 1\right)\dfrac{\partial z}{\partial x} \quad \Leftrightarrow \quad 1 = \left(\dfrac{y+1+y^2z^2}{1+y^2z^2}\right)\dfrac{\partial z}{\partial x}$, so $\dfrac{\partial z}{\partial x} = \dfrac{1+y^2z^2}{1+y+y^2z^2}$.

$\dfrac{\partial}{\partial y}(x-z) = \dfrac{\partial}{\partial y}(\arctan(yz)) \quad \Rightarrow \quad 0 - \dfrac{\partial z}{\partial y} = \dfrac{1}{1+(yz)^2} \cdot \left(y\,\dfrac{\partial z}{\partial y} + z \cdot 1\right) \quad \Leftrightarrow$

$-\dfrac{z}{1+y^2z^2} = \left(\dfrac{y}{1+y^2z^2} + 1\right)\dfrac{\partial z}{\partial y} \quad \Leftrightarrow \quad -\dfrac{z}{1+y^2z^2} = \left(\dfrac{y+1+y^2z^2}{1+y^2z^2}\right)\dfrac{\partial z}{\partial y} \quad \Leftrightarrow \quad \dfrac{\partial z}{\partial y} = -\dfrac{z}{1+y+y^2z^2}$.

49. (a) $z = f(x) + g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x), \quad \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x+y)$. Let $u = x + y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$,

$\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$.

51. $f(x,y) = x^3y^5 + 2x^4y \quad \Rightarrow \quad f_x(x,y) = 3x^2y^5 + 8x^3y$, $f_y(x,y) = 5x^3y^4 + 2x^4$. Then $f_{xx}(x,y) = 6xy^5 + 24x^2y$, $f_{xy}(x,y) = 15x^2y^4 + 8x^3$, $f_{yx}(x,y) = 15x^2y^4 + 8x^3$, and $f_{yy}(x,y) = 20x^3y^3$.

53. $w = \sqrt{u^2+v^2} \quad \Rightarrow \quad w_u = \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2u = \dfrac{u}{\sqrt{u^2+v^2}}$, $w_v = \frac{1}{2}(u^2+v^2)^{-1/2} \cdot 2v = \dfrac{v}{\sqrt{u^2+v^2}}$. Then

$$w_{uu} = \frac{1 \cdot \sqrt{u^2+v^2} - u \cdot \frac{1}{2}(u^2+v^2)^{-1/2}(2u)}{\left(\sqrt{u^2+v^2}\right)^2} = \frac{\sqrt{u^2+v^2} - u^2/\sqrt{u^2+v^2}}{u^2+v^2} = \frac{u^2+v^2-u^2}{(u^2+v^2)^{3/2}} = \frac{v^2}{(u^2+v^2)^{3/2}},$$

$$w_{uv} = u\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2v) = -\frac{uv}{(u^2+v^2)^{3/2}},\ w_{vu} = v\left(-\tfrac{1}{2}\right)\left(u^2+v^2\right)^{-3/2}(2u) = -\frac{uv}{(u^2+v^2)^{3/2}},$$

$$w_{vv} = \frac{1 \cdot \sqrt{u^2+v^2} - v \cdot \frac{1}{2}(u^2+v^2)^{-1/2}(2v)}{\left(\sqrt{u^2+v^2}\right)^2} = \frac{\sqrt{u^2+v^2} - v^2/\sqrt{u^2+v^2}}{u^2+v^2} = \frac{u^2+v^2-v^2}{(u^2+v^2)^{3/2}} = \frac{u^2}{(u^2+v^2)^{3/2}}.$$

55. $z = \arctan\dfrac{x+y}{1-xy} \quad \Rightarrow$

$$z_x = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2} = \frac{1+y^2}{1+x^2+y^2+x^2y^2}$$

$$= \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2},$$

$$z_y = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2+(x+y)^2} = \frac{1+x^2}{(1+x^2)(1+y^2)} = \frac{1}{1+y^2}.$$

Then $z_{xx} = -(1+x^2)^{-2} \cdot 2x = -\dfrac{2x}{(1+x^2)^2}$, $z_{xy} = 0$, $z_{yx} = 0$, $z_{yy} = -(1+y^2)^{-2} \cdot 2y = -\dfrac{2y}{(1+y^2)^2}$.

57. $u = x\sin(x+2y) \quad \Rightarrow \quad u_x = x \cdot \cos(x+2y)(1) + \sin(x+2y) \cdot 1 = x\cos(x+2y) + \sin(x+2y)$,

$u_{xy} = x(-\sin(x+2y)(2)) + \cos(x+2y)(2) = 2\cos(x+2y) - 2x\sin(x+2y)$,

$u_y = x\cos(x+2y)(2) = 2x\cos(x+2y)$,

$u_{yx} = 2x \cdot (-\sin(x+2y)(1)) + \cos(x+2y) \cdot 2 = 2\cos(x+2y) - 2x\sin(x+2y)$. Thus $u_{xy} = u_{yx}$.

59. $u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2) \quad\Rightarrow\quad u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,

$u_{xy} = x(-1)(x^2+y^2)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,

$u_{yx} = y(-1)(x^2+y^2)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

61. $f(x,y) = 3xy^4 + x^3y^2 \quad\Rightarrow\quad f_x = 3y^4 + 3x^2y^2,\; f_{xx} = 6xy^2,\; f_{xxy} = 12xy$ and

$f_y = 12xy^3 + 2x^3y,\; f_{yy} = 36xy^2 + 2x^3,\; f_{yyy} = 72xy$.

63. $f(x,y,z) = \cos(4x+3y+2z) \quad\Rightarrow$

$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z),\; f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z)$,

$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and

$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z)$,

$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z),\quad f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z)$.

65. $u = e^{r\theta}\sin\theta \quad\Rightarrow\quad \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta\cdot e^{r\theta}\,(r) = e^{r\theta}\,(\cos\theta + r\sin\theta)$,

$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}\,(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}\,(\theta) = e^{r\theta}\,(\sin\theta + \theta\cos\theta + r\theta\sin\theta)$,

$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}\,(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta)\cdot e^{r\theta}\;(\theta) = \theta e^{r\theta}\,(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$.

67. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \quad\Rightarrow\quad \dfrac{\partial w}{\partial x} = (y+2z)^{-1},\quad \dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2}$,

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and $\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}$,

$\dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2},\quad \dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0$.

69. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$:

$f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$, $f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0}\dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$, $f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$.

Averaging these values, we get $f_x(3,1.8) \approx 7.5$. Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of two

variables, so Definition 4 says that $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[f_x(x,y)\right] = \displaystyle\lim_{h\to 0}\frac{f_x(x,y+h)-f_x(x,y)}{h} \Rightarrow$

$f_{xy}(3,2) = \displaystyle\lim_{h\to 0}\frac{f_x(3,2+h)-f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$$f_{xy}(3,2) \approx \frac{f_x(3,2.2)-f_x(3,2)}{0.2} = \frac{16.8-12.2}{0.2} = 23,\ f_{xy}(3,2) \approx \frac{f_x(3,1.8)-f_x(3,2)}{-0.2} = \frac{7.5-12.2}{-0.2} = 23.5.$$

Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

71. $u = e^{-\alpha^2k^2t}\sin kx \quad\Rightarrow\quad u_x = ke^{-\alpha^2k^2t}\cos kx$, $u_{xx} = -k^2e^{-\alpha^2k^2t}\sin kx$, and $u_t = -\alpha^2k^2e^{-\alpha^2k^2t}\sin kx$.

Thus $\alpha^2 u_{xx} = u_t$.

73. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad\Rightarrow\quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx}+u_{yy}+u_{zz} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

75. Let $v = x+at$, $\ w = x-at$. Then $u_t = \dfrac{\partial[f(v)+g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and

$u_{tt} = \dfrac{\partial[af'(v)-ag'(w)]}{\partial t} = a[af''(v)+ag''(w)] = a^2[f''(v)+g''(w)]$. Similarly, by using the Chain Rule we have

$u_x = f'(v)+g'(w)$ and $u_{xx} = f''(v)+g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

77. $z = \ln(e^x+e^y) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^x+e^y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{e^y}{e^x+e^y}$, so $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y} = \dfrac{e^x}{e^x+e^y}+\dfrac{e^y}{e^x+e^y} = \dfrac{e^x+e^y}{e^x+e^y} = 1$.

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{e^x(e^x+e^y)-e^x(e^x)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$, $\quad\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{0-e^y(e^x)}{(e^x+e^y)^2} = -\dfrac{e^{x+y}}{(e^x+e^y)^2}$, and

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{e^y(e^x+e^y)-e^y(e^y)}{(e^x+e^y)^2} = \dfrac{e^{x+y}}{(e^x+e^y)^2}$. Thus

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = \frac{e^{x+y}}{(e^x+e^y)^2}\cdot\frac{e^{x+y}}{(e^x+e^y)^2} - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 = \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0$$

79. If we fix $K = K_0$, $P(L,K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then

$\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad\Rightarrow\quad \displaystyle\int\frac{dP}{P} = \int\alpha\,\frac{dL}{L} \quad\Rightarrow\quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then

$|P| = e^{\alpha\ln|L|+C(K_0)}$, and since $P>0$ and $L>0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where

$C_1(K_0) = e^{C(K_0)}$.

81. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$\dfrac{\partial R^{-1}}{\partial R}\dfrac{\partial R}{\partial R_1} = \dfrac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1}$ or $-R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2}$. Thus $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$.

83. By Exercise 82, $PV = mRT \Rightarrow P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \Rightarrow V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$.

Since $T = \dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR} \cdot \dfrac{mR}{V} \cdot \dfrac{mR}{P} = mR$.

85. $\dfrac{\partial K}{\partial m} = \frac{1}{2}v^2$, $\dfrac{\partial K}{\partial v} = mv$, $\dfrac{\partial^2 K}{\partial v^2} = m$. Thus $\dfrac{\partial K}{\partial m} \cdot \dfrac{\partial^2 K}{\partial v^2} = \frac{1}{2}v^2 m = K$.

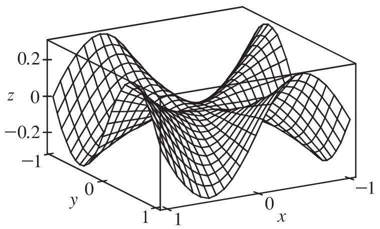
87. $f_x(x,y) = x + 4y \Rightarrow f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \Rightarrow f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

89. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \Rightarrow \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

91. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

93. Let $g(x) = f(x,0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

95. (a)



(b) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(0/h^2) - 0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$.

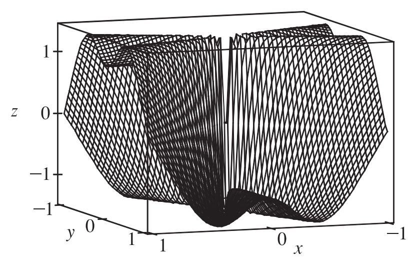
(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(-h^5 - 0)/h^4}{h} = -1$ while by (2),

$$f_{yx}(0,0) = \frac{\partial f_y}{\partial x} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h^5/h^4}{h} = 1.$$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to -1$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



15.4 Tangent Planes and Linear Approximations ET 14.4

1. $z = f(x, y) = 4x^2 - y^2 + 2y \quad \Rightarrow \quad f_x(x, y) = 8x$, $f_y(x, y) = -2y + 2$, so $f_x(-1, 2) = -8$, $f_y(-1, 2) = -2$.

By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1, 2)[x - (-1)] + f_y(-1, 2)(y - 2) \quad \Rightarrow$

$z - 4 = -8(x + 1) - 2(y - 2)$ or $z = -8x - 2y$.

3. $z = f(x, y) = \sqrt{xy} \quad \Rightarrow \quad f_x(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{1}{2}\sqrt{y/x}$, $f_y(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{x/y}$, so $f_x(1, 1) = \frac{1}{2}$

and $f_y(1, 1) = \frac{1}{2}$. Thus an equation of the tangent plane is $z - 1 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \quad \Rightarrow$

$z - 1 = \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1)$ or $x + y - 2z = 0$.

5. $z = f(x, y) = y\cos(x - y) \quad \Rightarrow \quad f_x = y(-\sin(x - y)(1)) = -y\sin(x - y)$,

$f_y = y(-\sin(x - y)(-1)) + \cos(x - y) = y\sin(x - y) + \cos(x - y)$, so $f_x(2, 2) = -2\sin(0) = 0$,

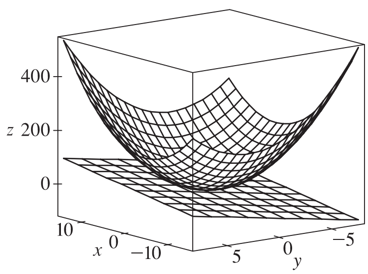
$f_y(2, 2) = 2\sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x - 2) + 1(y - 2)$ or $z = y$.

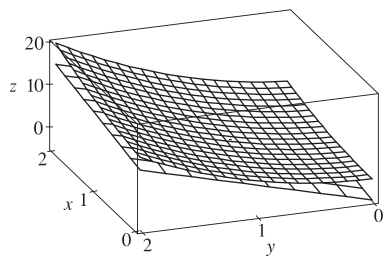
6. $z = f(x, y) = e^{x^2 - y^2} \quad \Rightarrow \quad f_x(x, y) = 2xe^{x^2 - y^2}$, $f_y(x, y) = -2ye^{x^2 - y^2}$, so $f_x(1, -1) = 2$, $f_y(1, -1) = 2$.

By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)[y - (-1)] \quad \Rightarrow$

$z - 1 = 2(x - 1) + 2(y + 1)$ or $z = 2x + 2y + 1$.

7. $z = f(x, y) = x^2 + xy + 3y^2$, so $f_x(x, y) = 2x + y \quad \Rightarrow \quad f_x(1, 1) = 3$, $f_y(x, y) = x + 6y \quad \Rightarrow \quad f_y(1, 1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x - 1) + 7(y - 1)$ or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



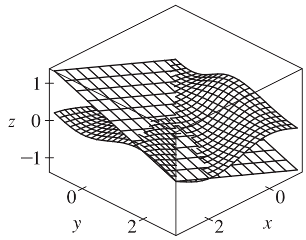


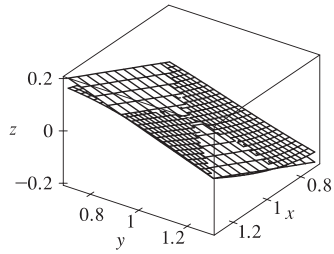
9. $f(x, y) = \dfrac{xy\sin(x - y)}{1 + x^2 + y^2}$. A CAS gives $f_x(x, y) = \dfrac{y\sin(x - y) + xy\cos(x - y)}{1 + x^2 + y^2} - \dfrac{2x^2y\sin(x - y)}{(1 + x^2 + y^2)^2}$ and

$f_y(x, y) = \dfrac{x\sin(x - y) - xy\cos(x - y)}{1 + x^2 + y^2} - \dfrac{2xy^2\sin(x - y)}{(1 + x^2 + y^2)^2}$. We use the CAS to evaluate these at $(1, 1)$, and then

substitute the results into Equation 2 to compute an equation of the tangent plane: $z = \frac{1}{3}x - \frac{1}{3}y$. The surface and tangent plane are shown in the first graph below. After zooming in, the surface and the tangent plane become almost indistinguishable,

as shown in the second graph. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y) = x\sqrt{y}$. The partial derivatives are $f_x(x,y) = \sqrt{y}$ and $f_y(x,y) = \dfrac{x}{2\sqrt{y}}$, so $f_x(1,4) = 2$ and $f_y(1,4) = \frac{1}{4}$. Both f_x and f_y are continuous functions for $y > 0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by $L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4) = 2 + 2(x-1) + \frac{1}{4}(y-4) = 2x + \frac{1}{4}y - 1$.

13. $f(x,y) = \dfrac{x}{x+y}$. The partial derivatives are $f_x(x,y) = \dfrac{1(x+y) - x(1)}{(x+y)^2} = y/(x+y)^2$ and $f_y(x,y) = x(-1)(x+y)^{-2} \cdot 1 = -x/(x+y)^2$, so $f_x(2,1) = \frac{1}{9}$ and $f_y(2,1) = -\frac{2}{9}$. Both f_x and f_y are continuous functions for $y \neq -x$, so f is differentiable at $(2,1)$ by Theorem 8. The linearization of f at $(2,1)$ is given by $L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = \frac{2}{3} + \frac{1}{9}(x-2) - \frac{2}{9}(y-1) = \frac{1}{9}x - \frac{2}{9}y + \frac{2}{3}$.

15. $f(x,y) = e^{-xy}\cos y$. The partial derivatives are $f_x(x,y) = e^{-xy}(-y)\cos y = -ye^{-xy}\cos y$ and $f_y(x,y) = e^{-xy}(-\sin y) + (\cos y)e^{-xy}(-x) = -e^{-xy}(\sin y + x\cos y)$, so $f_x(\pi,0) = 0$ and $f_y(\pi,0) = -\pi$. Both f_x and f_y are continuous functions, so f is differentiable at $(\pi,0)$, and the linearization of f at $(\pi,0)$ is $L(x,y) = f(\pi,0) + f_x(\pi,0)(x-\pi) + f_y(\pi,0)(y-0) = 1 + 0(x-\pi) - \pi(y-0) = 1 - \pi y$.

17. Let $f(x,y) = \dfrac{2x+3}{4y+1}$. Then $f_x(x,y) = \dfrac{2}{4y+1}$ and $f_y(x,y) = (2x+3)(-1)(4y+1)^{-2}(4) = \dfrac{-8x-12}{(4y+1)^2}$. Both f_x and f_y are continuous functions for $y \neq -\frac{1}{4}$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0) = 2$, $f_y(0,0) = -12$ and the linear approximation of f at $(0,0)$ is $f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 3 + 2x - 12y$.

19. $f(x,y) = \sqrt{20 - x^2 - 7y^2} \;\Rightarrow\; f_x(x,y) = -\dfrac{x}{\sqrt{20 - x^2 - 7y^2}}$ and $f_y(x,y) = -\dfrac{7y}{\sqrt{20 - x^2 - 7y^2}}$, so $f_x(2,1) = -\frac{2}{3}$ and $f_y(2,1) = -\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by $f(x,y) \approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1) = -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}$. Thus $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.84\overline{6}$.

21. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \;\Rightarrow\; f_x(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and $f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6) = \frac{3}{7}$, $f_y(3,2,6) = \frac{2}{7}$, $f_z(3,2,6) = \frac{6}{7}$. Then the linear approximation of f at

$(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

23. From the table, $f(94,80) = 127$. To estimate $f_T(94,80)$ and $f_H(94,80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94,80) = \lim\limits_{h\to 0} \dfrac{f(94+h,80) - f(94,80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table:

$$f_T(94,80) \approx \frac{f(96,80) - f(94,80)}{2} = \frac{135-127}{2} = 4, \qquad f_T(94,80) \approx \frac{f(92,80) - f(94,80)}{-2} = \frac{119-127}{-2} = 4$$

Averaging these values gives $f_T(94,80) \approx 4$. Similarly, $f_H(94,80) = \lim\limits_{h\to 0} \dfrac{f(94,80+h) - f(94,80)}{h}$, so we use $h = \pm 5$:

$$f_H(94,80) \approx \frac{f(94,85) - f(94,80)}{5} = \frac{132-127}{5} = 1, \qquad f_H(94,80) \approx \frac{f(94,75) - f(94,80)}{-5} = \frac{122-127}{-5} = 1$$

Averaging these values gives $f_H(94,80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T,H) &\approx f(94,80) + f_T(94,80)(T-94) + f_H(94,80)(H-80) \\ &\approx 127 + 4(T-94) + 1(H-80) \qquad [\text{or } 4T + H - 329] \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95,78) \approx 127 + 4(95-94) + 1(78-80) = 129$, so we estimate the heat index to be approximately 129°F.

25. $z = x^3 \ln(y^2) \quad\Rightarrow\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = 3x^2 \ln(y^2)\, dx + x^3 \cdot \dfrac{1}{y^2}(2y)\, dy = 3x^2 \ln(y^2)\, dx + \dfrac{2x^3}{y} dy$

27. $m = p^5 q^3 \quad\Rightarrow\quad dm = \dfrac{\partial m}{\partial p}\, dp + \dfrac{\partial m}{\partial q}\, dq = 5p^4 q^3\, dp + 3p^5 q^2\, dq$

29. $R = \alpha\beta^2 \cos\gamma \quad\Rightarrow\quad dR = \dfrac{\partial R}{\partial \alpha}\, d\alpha + \dfrac{\partial R}{\partial \beta}\, d\beta + \dfrac{\partial R}{\partial \gamma}\, d\gamma = \beta^2 \cos\gamma\, d\alpha + 2\alpha\beta\, \cos\gamma\, d\beta - \alpha\beta^2 \sin\gamma\, d\gamma$

31. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$,
$dz = z_x(1,2)\, dx + z_y(1,2)\, dy = (10)(0.05) + (4)(0.1) = 0.9$ while
$\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

33. $dA = \dfrac{\partial A}{\partial x}\, dx + \dfrac{\partial A}{\partial y}\, dy = y\, dx + x\, dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\ \text{cm}^2$.

35. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi r h\, dr + \pi r^2\, dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\ \text{cm}^3$. Thus the amount of tin is about $16\ \text{cm}^3$.

37. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150\text{ ft}^2$. Thus there are approximately 150 ft^2 of paint in the stripe.

39. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$\dfrac{\partial}{\partial R_1}\left(\dfrac{1}{R}\right) = \dfrac{\partial\left[(1/R_1) + (1/R_2) + (1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\dfrac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. Then by symmetry,

$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \quad\Leftrightarrow\quad R = \frac{200}{17}\ \Omega$.

Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$\Delta R \approx dR = \dfrac{\partial R}{\partial R_1}\,\Delta R_1 + \dfrac{\partial R}{\partial R_2}\,\Delta R_2 + \dfrac{\partial R}{\partial R_3}\,\Delta R_3 = (0.005)R^2\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}\right) = (0.005)R = \frac{1}{17} \approx 0.059\ \Omega$.

41. The errors in measurement are at most 2%, so $\left|\dfrac{\Delta w}{w}\right| \le 0.02$ and $\left|\dfrac{\Delta h}{h}\right| \le 0.02$. The relative error in the calculated surface area is

$$\frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{0.1091(0.425w^{0.425-1})h^{0.725}\,dw + 0.1091w^{0.425}(0.725h^{0.725-1})\,dh}{0.1091w^{0.425}h^{0.725}} = 0.425\frac{dw}{w} + 0.725\frac{dh}{h}$$

To estimate the maximum relative error, we use $\dfrac{dw}{w} = \left|\dfrac{\Delta w}{w}\right| = 0.02$ and $\dfrac{dh}{h} = \left|\dfrac{\Delta h}{h}\right| = 0.02 \quad\Rightarrow$

$\dfrac{dS}{S} = 0.425\,(0.02) + 0.725\,(0.02) = 0.023$. Thus the maximum percentage error is approximately 2.3%.

43. $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$

$\quad = a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$

But $f_x(a, b) = 2a$ and $f_y(a, b) = 2b$ and so $\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

45. To show that f is continuous at (a, b) we need to show that $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ or equivalently $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Since f is differentiable at (a, b),

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0, 0)$ gives $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$. Thus f is continuous at (a, b).

15.5 The Chain Rule — ET 14.5

1. $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + y)\cos t + (2y + x)e^t$

3. $z = \sqrt{1 + x^2 + y^2},\ \ x = \ln t,\ \ y = \cos t \quad\Rightarrow$

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \frac{1}{2}(1 + x^2 + y^2)^{-1/2}(2x)\cdot\dfrac{1}{t} + \frac{1}{2}(1 + x^2 + y^2)^{-1/2}(2y)(-\sin t) = \dfrac{1}{\sqrt{1 + x^2 + y^2}}\left(\dfrac{x}{t} - y\sin t\right)$

5. $w = xe^{y/z},\ x = t^2,\ y = 1 - t,\ z = 1 + 2t \quad \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z} \cdot 2t + xe^{y/z}\left(\frac{1}{z}\right) \cdot (-1) + xe^{y/z}\left(-\frac{y}{z^2}\right) \cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

7. $z = x^2y^3,\ x = s\cos t,\ y = s\sin t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos t + 3x^2y^2\sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2sxy^3\sin t + 3sx^2y^2\cos t$$

9. $z = \sin\theta\cos\phi,\ \ \theta = st^2,\ \ \phi = s^2t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s} = (\cos\theta\,\cos\phi)(t^2) + (-\sin\theta\,\sin\phi)(2st) = t^2\cos\theta\,\cos\phi - 2st\sin\theta\,\sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t} = (\cos\theta\,\cos\phi)(2st) + (-\sin\theta\,\sin\phi)(s^2) = 2st\cos\theta\,\cos\phi - s^2\sin\theta\,\sin\phi$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2 + t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta \cdot t + e^r(-\sin\theta) \cdot \tfrac{1}{2}(s^2 + t^2)^{-1/2}(2s) = te^r\cos\theta - e^r\sin\theta \cdot \frac{s}{\sqrt{s^2 + t^2}}$$

$$= e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta \cdot s + e^r(-\sin\theta) \cdot \tfrac{1}{2}(s^2 + t^2)^{-1/2}(2t) = se^r\cos\theta - e^r\sin\theta \cdot \frac{t}{\sqrt{s^2 + t^2}}$$

$$= e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v,\ y = e^u + \cos v \quad \Rightarrow$

$\dfrac{\partial x}{\partial u} = e^u,\ \dfrac{\partial x}{\partial v} = \cos v,\ \dfrac{\partial y}{\partial u} = e^u,\ \dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

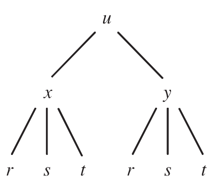
$$g_u(0,0) = f_x(x(0,0), y(0,0))\,x_u(0,0) + f_y(x(0,0), y(0,0))\,y_u(0,0) = f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$g_v(0,0) = f_x(x(0,0), y(0,0))\,x_v(0,0) + f_y(x(0,0), y(0,0))\,y_v(0,0) = f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0)$$

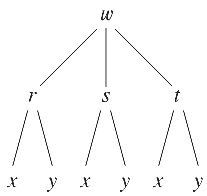
$$= 2(1) + 5(0) = 2$$

17.



$u = f(x,y),\ x = x(r,s,t),\ y = y(r,s,t) \quad \Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

19. 

$w = f(r, s, t),\ \ r = r(x, y),\ \ s = s(x, y),\ t = t(x, y) \quad \Rightarrow$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x},\quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$

21. $z = x^2 + xy^3,\ x = uv^2 + w^3,\ y = u + ve^w \quad \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85$, $\dfrac{\partial z}{\partial v} = (31)\,(4) + (54)(1) = 178$, $\dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54$.

23. $R = \ln(u^2 + v^2 + w^2),\ u = x + 2y,\ v = 2x - y,\ w = 2xy \quad \Rightarrow$

$$\begin{aligned}\frac{\partial R}{\partial x} &= \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2+v^2+w^2}\,(1) + \frac{2v}{u^2+v^2+w^2}\,(2) + \frac{2w}{u^2+v^2+w^2}\,(2y)\\ &= \frac{2u + 4v + 4wy}{u^2+v^2+w^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial y} &= \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{2u}{u^2+v^2+w^2}\,(2) + \frac{2v}{u^2+v^2+w^2}\,(-1) + \frac{2w}{u^2+v^2+w^2}\,(2x)\\ &= \frac{4u - 2v + 4wx}{u^2+v^2+w^2}.\end{aligned}$$

When $x = y = 1$ we have $u = 3$, $v = 1$, and $w = 2$, so $\dfrac{\partial R}{\partial x} = \dfrac{9}{7}$ and $\dfrac{\partial R}{\partial y} = \dfrac{9}{7}$.

25. $u = x^2 + yz,\ x = pr\cos\theta,\ y = pr\sin\theta,\ z = p + r \quad \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p} = (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1) = 2xr\cos\theta + zr\sin\theta + y,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1) = 2xp\cos\theta + zp\sin\theta + y,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = (2x)(-pr\sin\theta) + (z)(pr\cos\theta) + (y)(0) = -2xpr\sin\theta + zpr\cos\theta.$$

When $p = 2$, $r = 3$, and $\theta = 0$ we have $x = 6$, $y = 0$, and $z = 5$, so $\dfrac{\partial u}{\partial p} = 36$, $\dfrac{\partial u}{\partial r} = 24$, and $\dfrac{\partial u}{\partial \theta} = 30$.

27. $\sqrt{xy} = 1 + x^2y$, so let $F(x, y) = (xy)^{1/2} - 1 - x^2y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-1/2}(y) - 2xy}{\frac{1}{2}(xy)^{-1/2}(x) - x^2} = -\frac{y - 4xy\sqrt{xy}}{x - 2x^2\sqrt{xy}} = \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}.$$

29. $\cos(x - y) = xe^y$, so let $F(x, y) = \cos(x - y) - xe^y = 0$.

Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{-\sin(x-y) - e^y}{-\sin(x-y)(-1) - xe^y} = \dfrac{\sin(x-y) + e^y}{\sin(x-y) - xe^y}$.

31. $x^2 + y^2 + z^2 = 3xyz$, so let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

33. $x - z = \arctan(yz)$, so let $F(x, y, z) = x - z - \arctan(yz) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = \frac{1 + y^2z^2}{1 + y + y^2z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\dfrac{1}{1 + (yz)^2}(z)}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = -\frac{\dfrac{z}{1 + y^2z^2}}{\dfrac{1 + y^2z^2 + y}{1 + y^2z^2}} = -\frac{z}{1 + y + y^2z^2}$$

35. Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$. After 3 seconds, $x = \sqrt{1 + t} = \sqrt{1 + 3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1 + t}} = \dfrac{1}{2\sqrt{1 + 3}} = \dfrac{1}{4}$, and $\dfrac{dy}{dt} = \dfrac{1}{3}$.

Then $\dfrac{dT}{dt} = T_x(2, 3)\,\dfrac{dx}{dt} + T_y(2, 3)\,\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$.

According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate $\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule, $\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$.

Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\,\frac{d\ell}{dt} + \ell h\,\frac{dw}{dt} + \ell w\,\frac{dh}{dt} = 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3) = 6\ \text{m}^3/\text{s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned}\frac{dS}{dt} &= \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w + h)\,\frac{d\ell}{dt} + 2(\ell + h)\,\frac{dw}{dt} + 2(\ell + w)\,\frac{dh}{dt} \\ &= 2(2 + 2)2 + 2(1 + 2)2 + 2(1 + 2)(-3) = 10\ \text{m}^2/\text{s}\end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \quad\Rightarrow\quad 2L\,\dfrac{dL}{dt} = 2\ell\,\dfrac{d\ell}{dt} + 2w\,\dfrac{dw}{dt} + 2h\,\dfrac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \quad\Rightarrow$

$dL/dt = 0$ m/s.

41. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$,

$$\frac{dV}{dt} = 8.31\left[\frac{0.15}{20} - \frac{(0.05)(320)}{400}\right] \approx -0.27\ \text{L/s}.$$

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by $A = \frac{1}{2}xy\sin\theta$ where θ is the angle between the two sides. Thus A is a function of x, y, and θ, and x, y, and θ are each in

turn functions of time t. We are given that $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = -2$, and because A is constant, $\dfrac{dA}{dt} = 0$. By the Chain Rule,

$\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}y\sin\theta\cdot\dfrac{dx}{dt} + \frac{1}{2}x\sin\theta\cdot\dfrac{dy}{dt} + \frac{1}{2}xy\cos\theta\cdot\dfrac{d\theta}{dt}$. When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned} 0 &= \tfrac{1}{2}(30)\left(\sin\tfrac{\pi}{6}\right)(3) + \tfrac{1}{2}(20)\left(\sin\tfrac{\pi}{6}\right)(-2) + \tfrac{1}{2}(20)(30)\left(\cos\tfrac{\pi}{6}\right)\frac{d\theta}{dt} \\ &= 45\cdot\tfrac{1}{2} - 20\cdot\tfrac{1}{2} + 300\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt} = \tfrac{25}{2} + 150\sqrt{3}\,\frac{d\theta}{dt} \end{aligned}$$

Solving for $\dfrac{d\theta}{dt}$ gives $\dfrac{d\theta}{dt} = \dfrac{-25/2}{150\sqrt{3}} = -\dfrac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of $1/\left(12\sqrt{3}\right) \approx 0.048\ \text{rad/s}$.

45. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

47. Let $u = x - y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}(-1)$. Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

49. Let $u = x + at$, $v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$\dfrac{\partial^2 z}{\partial t^2} = a\dfrac{\partial}{\partial t}\left[f'(u) - g'(v)\right] = a\left(\dfrac{df'(u)}{du}\dfrac{\partial u}{\partial t} - \dfrac{dg'(v)}{dv}\dfrac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v)$.

Similarly $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$.

51. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}2s + \dfrac{\partial z}{\partial y}2r$. Then

$$\begin{aligned} \frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right) \\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}2r + \frac{\partial z}{\partial y}2 \\ &= 4rs\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}4s^2 + 0 + 4rs\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}4r^2 + 2\frac{\partial z}{\partial y} \end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\dfrac{\partial^2 z}{\partial x^2} + 4rs\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

53. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}r\sin\theta + \dfrac{\partial z}{\partial y}r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\ &= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}r\cos\theta\right)\\ &\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\ &= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\,\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2}\\ &\qquad -\frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x} - \frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

55. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$f(tx,ty) = (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3 = t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x,y)$.

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial t} f(tx,ty) = \dfrac{\partial}{\partial t}[t^n f(x,y)] \quad\Leftrightarrow$

$\dfrac{\partial}{\partial(tx)} f(tx,ty)\cdot\dfrac{\partial(tx)}{\partial t} + \dfrac{\partial}{\partial(ty)} f(tx,ty)\cdot\dfrac{\partial(ty)}{\partial t} = x\,\dfrac{\partial}{\partial(tx)} f(tx,ty) + y\,\dfrac{\partial}{\partial(ty)} f(tx,ty) = nt^{n-1}f(x,y)$.

Setting $t=1$: $x\,\dfrac{\partial}{\partial x} f(x,y) + y\,\dfrac{\partial}{\partial y} f(x,y) = nf(x,y)$.

57. Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to x using the Chain Rule, we get

$\dfrac{\partial}{\partial x} f(tx,ty) = \dfrac{\partial}{\partial x}[t^n f(x,y)] \quad\Leftrightarrow$

$\dfrac{\partial}{\partial\,(tx)} f(tx,ty)\cdot\dfrac{\partial\,(tx)}{\partial x} + \dfrac{\partial}{\partial\,(ty)} f(tx,ty)\cdot\dfrac{\partial\,(ty)}{\partial x} = t^n\,\dfrac{\partial}{\partial x} f(x,y) \quad\Leftrightarrow\quad tf_x(tx,ty) = t^n f_x(x,y)$.

Thus $f_x(tx,ty) = t^{n-1}f_x(x,y)$.

15.6 Directional Derivatives and the Gradient Vector — ET 14.6

1. We can approximate the directional derivative of the pressure function at K in the direction of S by the average rate of change of pressure between the points where the red line intersects the contour lines closest to K (extend the red line slightly at the left). In the direction of S, the pressure changes from 1000 millibars to 996 millibars and we estimate the distance between these two points to be approximately 50 km (using the fact that the distance from K to S is 300 km). Then the rate of change of pressure in the direction given is approximately $\frac{996-1000}{50} = -0.08$ millibar/km.

3. $D_{\mathbf{u}}\, f(-20, 30) = \nabla f\,(-20, 30) \cdot \mathbf{u} = f_T(-20, 30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20, 30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20 + h, 30) - f(-20, 30)}{h}$, so we can approximate $f_T(-20, 30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20, 30) \approx \dfrac{f(-15, 30) - f(-20, 30)}{5} = \dfrac{-26 - (-33)}{5} = 1.4$,

$f_T(-20, 30) \approx \dfrac{f(-25, 30) - f(-20, 30)}{-5} = \dfrac{-39 - (-33)}{-5} = 1.2$. Averaging these values gives $f_T(-20, 30) \approx 1.3$.

Similarly, $f_v(-20, 30) = \lim\limits_{h \to 0} \dfrac{f(-20, 30 + h) - f(-20, 30)}{h}$, so we can approximate $f_v(-20, 30)$ with $h = \pm 10$:

$f_v(-20, 30) \approx \dfrac{f(-20, 40) - f(-20, 30)}{10} = \dfrac{-34 - (-33)}{10} = -0.1$,

$f_v(-20, 30) \approx \dfrac{f(-20, 20) - f(-20, 30)}{-10} = \dfrac{-30 - (-33)}{-10} = -0.3$. Averaging these values gives $f_v(-20, 30) \approx -0.2$.

Then $D_{\mathbf{u}} f(-20, 30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

5. $f(x, y) = ye^{-x} \quad \Rightarrow \quad f_x(x, y) = -ye^{-x}$ and $f_y(x, y) = e^{-x}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = 2\pi/3$, then from Equation 6, $D_{\mathbf{u}}\, f(0, 4) = f_x(0, 4) \cos\left(\frac{2\pi}{3}\right) + f_y(0, 4) \sin\left(\frac{2\pi}{3}\right) = -4 \cdot \left(-\frac{1}{2}\right) + 1 \cdot \frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}$.

7. $f(x, y) = \sin(2x + 3y)$

(a) $\nabla f(x, y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = [\cos(2x + 3y) \cdot 2]\,\mathbf{i} + [\cos(2x + 3y) \cdot 3]\,\mathbf{j} = 2\cos(2x + 3y)\,\mathbf{i} + 3\cos(2x + 3y)\,\mathbf{j}$

(b) $\nabla f(-6, 4) = (2\cos 0)\mathbf{i} + (3\cos 0)\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(-6, 4) = \nabla f(-6, 4) \cdot \mathbf{u} = (2\,\mathbf{i} + 3\,\mathbf{j}) \cdot \frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right) = \frac{1}{2}\left(2\sqrt{3} - 3\right) = \sqrt{3} - \frac{3}{2}$.

9. $f(x, y, z) = xe^{2yz}$

(a) $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \left\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\right\rangle$

(b) $\nabla f(3, 0, 2) = \langle 1, 12, 0\rangle$

(c) By Equation 14, $D_{\mathbf{u}} f(3, 0, 2) = \nabla f(3, 0, 2) \cdot \mathbf{u} = \langle 1, 12, 0\rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

11. $f(x, y) = 1 + 2x\sqrt{y} \quad \Rightarrow \quad \nabla f(x, y) = \left\langle 2\sqrt{y}, 2x \cdot \frac{1}{2}y^{-1/2}\right\rangle = \left\langle 2\sqrt{y}, x/\sqrt{y}\right\rangle$, $\nabla f(3, 4) = \left\langle 4, \frac{3}{2}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{1}{\sqrt{4^2 + (-3)^2}}\,\langle 4, -3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$, so $D_{\mathbf{u}}\, f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u} = \left\langle 4, \frac{3}{2}\right\rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{23}{10}$.

13. $g(p, q) = p^4 - p^2q^3 \quad \Rightarrow \quad \nabla g(p, q) = \left(4p^3 - 2pq^3\right)\mathbf{i} + \left(-3p^2q^2\right)\mathbf{j}$, $\nabla g(2, 1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2 + 3^2}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j})$, so

$D_{\mathbf{u}}\, g(2, 1) = \nabla g(2, 1) \cdot \mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j}) \cdot \frac{1}{\sqrt{10}}(\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}}\,(28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

15. $f(x, y, z) = xe^y + ye^z + ze^x \quad \Rightarrow \quad \nabla f(x, y, z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x\rangle$, $\nabla f(0, 0, 0) = \langle 1, 1, 1\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{25 + 1 + 4}}\langle 5, 1, -2\rangle = \frac{1}{\sqrt{30}}\,\langle 5, 1, -2\rangle$, so

$D_{\mathbf{u}}\, f(0, 0, 0) = \nabla f(0, 0, 0) \cdot \mathbf{u} = \langle 1, 1, 1\rangle \cdot \frac{1}{\sqrt{30}}\,\langle 5, 1, -2\rangle = \frac{4}{\sqrt{30}}$.

17. $g(x,y,z) = (x+2y+3z)^{3/2} \quad \Rightarrow$

$$\nabla g(x,y,z) = \left\langle \tfrac{3}{2}(x+2y+3z)^{1/2}(1), \tfrac{3}{2}(x+2y+3z)^{1/2}(2), \tfrac{3}{2}(x+2y+3z)^{1/2}(3) \right\rangle$$
$$= \left\langle \tfrac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \tfrac{9}{2}\sqrt{x+2y+3z} \right\rangle, \nabla g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle,$$

and a unit vector in the direction of $\mathbf{v} = 2\,\mathbf{j} - \mathbf{k}$ is $\mathbf{u} = \frac{2}{\sqrt{5}}\,\mathbf{j} - \frac{1}{\sqrt{5}}\,\mathbf{k}$, so

$$D_{\mathbf{u}}\, g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle \cdot \left\langle 0, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}} \right\rangle = \tfrac{18}{\sqrt{5}} - \tfrac{27}{2\sqrt{5}} = \tfrac{9}{2\sqrt{5}}.$$

19. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so

$D_{\mathbf{u}}\, f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

21. $f(x,y) = y^2/x = y^2 x^{-1} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle -y^2 x^{-2}, 2yx^{-1} \right\rangle = \left\langle -y^2/x^2, 2y/x \right\rangle$.

$\nabla f(2,4) = \langle -4, 4 \rangle$, or equivalently $\langle -1, 1 \rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)| = \sqrt{16+16} = 4\sqrt{2}$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

25. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y, \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2z \right\rangle$$
$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle,$$

$\nabla f(3,6,-2) = \left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}} \right\rangle = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$. Thus the maximum rate of change is

$|\nabla f(3,6,-2)| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = \sqrt{\frac{9+36+4}{49}} = 1$ in the direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$ or equivalently $\langle 3, 6, -2 \rangle$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}}\, f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}}\, f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4 y - x^2 y^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \right\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i} + \mathbf{j}) \quad \Leftrightarrow \quad k = 2x-2$ and $k = 2y-4$. Then $2x-2 = 2y-4 \quad \Rightarrow$ $y = x+1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x+1$.

31. $T = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1,-1,1\rangle}{\sqrt{3}}$,

$D_{\mathbf{u}}T(1,2,2) = \nabla T(1,2,2)\cdot\mathbf{u} = \left[-360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle\right]_{(1,2,2)}\cdot\mathbf{u} = -\frac{40}{3}\langle 1,2,2\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle = -\frac{40}{3\sqrt{3}}$

(b) From (a), $\nabla T = -360\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle$, and since $\langle x,y,z\rangle$ is the position vector of the point (x,y,z), the vector $-\langle x,y,z\rangle$, and thus ∇T, always points toward the origin.

33. $\nabla V(x,y,z) = \langle 10x-3y+yz, xz-3x, xy\rangle$, $\nabla V(3,4,5) = \langle 38,6,12\rangle$

(a) $D_{\mathbf{u}}V(3,4,5) = \langle 38,6,12\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$, or equivalently, $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}\,f(1,3) = f_x(1,3) = 3$ and $D_{\overrightarrow{AC}}\,f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle = \langle 3,26\rangle$, and by definition, $D_{\overrightarrow{AD}}\,f(1,3) = \nabla f\cdot\mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle\frac{5}{13},\frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}\,f(1,3) = \langle 3,26\rangle\cdot\left\langle\frac{5}{13},\frac{12}{13}\right\rangle = 3\cdot\frac{5}{13}+26\cdot\frac{12}{13} = \frac{327}{13}$.

37. (a) $\nabla(au+bv) = \left\langle\dfrac{\partial(au+bv)}{\partial x},\dfrac{\partial(au+bv)}{\partial y}\right\rangle = \left\langle a\dfrac{\partial u}{\partial x}+b\dfrac{\partial v}{\partial x}, a\dfrac{\partial u}{\partial y}+b\dfrac{\partial v}{\partial y}\right\rangle = a\left\langle\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\right\rangle + b\left\langle\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}\right\rangle$

$= a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = \left\langle v\dfrac{\partial u}{\partial x}+u\dfrac{\partial v}{\partial x}, v\dfrac{\partial u}{\partial y}+u\dfrac{\partial v}{\partial y}\right\rangle = v\left\langle\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\right\rangle + u\left\langle\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\dfrac{u}{v}\right) = \left\langle\dfrac{v\dfrac{\partial u}{\partial x}-u\dfrac{\partial v}{\partial x}}{v^2},\dfrac{v\dfrac{\partial u}{\partial y}-u\dfrac{\partial v}{\partial y}}{v^2}\right\rangle = \dfrac{v\left\langle\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}\right\rangle - u\left\langle\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}\right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle\dfrac{\partial(u^n)}{\partial x},\dfrac{\partial(u^n)}{\partial y}\right\rangle = \left\langle nu^{n-1}\dfrac{\partial u}{\partial x}, nu^{n-1}\dfrac{\partial u}{\partial y}\right\rangle = nu^{n-1}\,\nabla u$

39. Let $F(x,y,z) = 2(x-2)^2+(y-1)^2+(z-3)^2$. Then $2(x-2)^2+(y-1)^2+(z-3)^2 = 10$ is a level surface of F. $F_x(x,y,z) = 4(x-2) \;\Rightarrow\; F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \;\Rightarrow\; F_y(3,3,5) = 4$, and $F_z(x,y,z) = 2(z-3) \;\Rightarrow\; F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3)+4(y-3)+4(z-5) = 0 \;\Leftrightarrow\; 4x+4y+4z = 44$ or equivalently $x+y+z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x-3 = y-3 = z-5$. Corresponding parametric equations are $x = 3+t$, $y = 3+t$, $z = 5+t$.

41. Let $F(x,y,z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x,y,z) = \langle 2x, -4y + z, 2z + y \rangle$.

(a) $\nabla F(2,1,-1) = \langle 4,-5,-1 \rangle$ is a normal vector for the tangent plane at $(2,1,-1)$, so an equation of the tangent plane is $4(x-2) - 5(y-1) - 1(z+1) = 0$ or $4x - 5y - z = 4$.

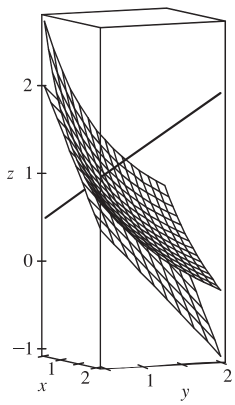
(b) The normal line has direction $\langle 4,-5,-1 \rangle$, so parametric equations are $x = 2 + 4t$, $y = 1 - 5t$, $z = -1 - t$, and symmetric equations are $\dfrac{x-2}{4} = \dfrac{y-1}{-5} = \dfrac{z+1}{-1}$.

43. $F(x,y,z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x,y,z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$ and $\nabla F(1,0,0) = \langle 1,1,-1 \rangle$.

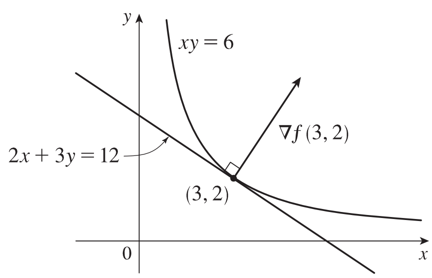
(a) $1(x-1) + 1(y-0) - 1(z-0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$

45. $F(x,y,z) = xy + yz + zx$, $\nabla F(x,y,z) = \langle y+z, x+z, y+x \rangle$, $\nabla F(1,1,1) = \langle 2,2,2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$. To graph the surface we solve for z: $z = \dfrac{3 - xy}{x + y}$.



47. $f(x,y) = xy \quad \Rightarrow \quad \nabla f(x,y) = \langle y, x \rangle$, $\nabla f(3,2) = \langle 2,3 \rangle$. $\nabla f(3,2)$ is perpendicular to the tangent line, so the tangent line has equation $\nabla f(3,2) \cdot \langle x-3, y-2 \rangle = 0 \quad \Rightarrow \quad \langle 2,3 \rangle \cdot \langle x-3, x-2 \rangle = 0 \quad \Rightarrow$ $2(x-3) + 3(y-2) = 0$ or $2x + 3y = 12$.



49. $\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2} \right\rangle$. Thus an equation of the tangent plane at (x_0,y_0,z_0) is

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y + \dfrac{2z_0}{c^2}\,z = 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0,y_0,z_0) is a point on the ellipsoid. Hence $\dfrac{x_0}{a^2}\,x + \dfrac{y_0}{b^2}\,y + \dfrac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

51. $\nabla F(x_0, y_0, z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c} \right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y - \dfrac{1}{c}\,z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$

or $\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$, so the equation can be written as

$\dfrac{2x_0}{a^2}\,x + \dfrac{2y_0}{b^2}\,y = \dfrac{z + z_0}{c}$.

53. The hyperboloid $x^2 - y^2 - z^2 = 1$ is a level surface of $F(x, y, z) = x^2 - y^2 - z^2$ and $\nabla F\,(x, y, z) = \langle 2x, -2y, -2z\rangle$ is a normal vector to the surface and hence a normal vector for the tangent plane at (x, y, z). The tangent plane is parallel to the plane $z = x + y$ or $x + y - z = 0$ if and only if the corresponding normal vectors are parallel, so we need a point (x_0, y_0, z_0) on the hyperboloid where $\langle 2x_0, -2y_0, -2z_0\rangle = c\,\langle 1, 1, -1\rangle$ or equivalently $\langle x_0, -y_0, -z_0\rangle = k\,\langle 1, 1, -1\rangle$ for some $k \neq 0$. Then we must have $x_0 = k$, $y_0 = -k$, $z_0 = k$ and substituting into the equation of the hyperboloid gives $k^2 - (-k)^2 - k^2 = 1 \quad\Leftrightarrow\quad -k^2 = 1$, an impossibility. Thus there is no such point on the hyperboloid.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2\left(x_0^2 + y_0^2 - z_0^2\right)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is

$\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

59. If $f(x, y, z) = z - x^2 - y^2$ and $g(x, y, z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1, 1, 2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line.

We have $\nabla f(x, y, z) = \langle -2x, -2y, 1\rangle \quad\Rightarrow\quad \nabla f(-1, 1, 2) = \langle 2, -2, 1\rangle$, and $\nabla g(x, y, z) = \langle 8x, 2y, 2z\rangle \quad\Rightarrow$

$\nabla g(-1, 1, 2) = \langle -8, 2, 4\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$.

Parametric equations are: $x = -1 - 10t,\ \ y = 1 - 16t,\ \ z = 2 - 12t$.

61. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at $P \quad\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z\rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z\rangle = 0$ at $P \quad\Leftrightarrow\quad F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z\rangle \cdot \langle 2x, 2y, 2z\rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point (x, y, z) lies on the graph of

$F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

63. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

15.7 Maximum and Minimum Values ET 14.7

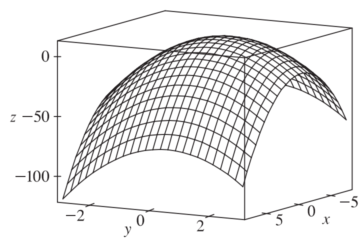
1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

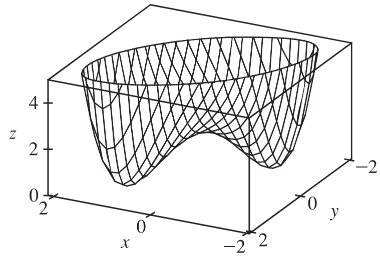
3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

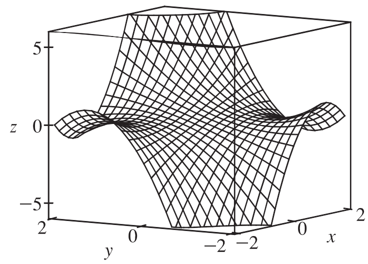
5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.



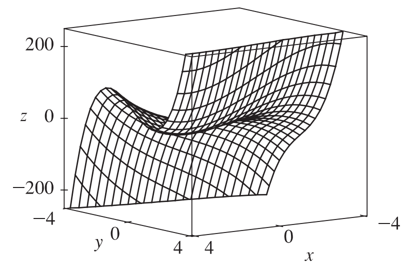
7. $f(x,y) = x^4 + y^4 - 4xy + 2 \quad\Rightarrow\quad f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = x^3$, and substitution into $f_y = 0 \quad\Rightarrow\quad x = y^3$ gives $x^9 - x = 0 \quad\Rightarrow$ $x(x^8 - 1) = 0 \quad\Rightarrow\quad x = 0$ or $x = \pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Now $D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0$, so $(0,0)$ is a saddle point. $D(1,1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(1,1) = 12 > 0$, so $f(1,1) = 0$ is a local minimum. $D(-1,-1) = (12)(12) - (-4)^2 > 0$ and $f_{xx} = (-1,-1) = 12 > 0$, so $f(-1,-1) = 0$ is also a local minimum.



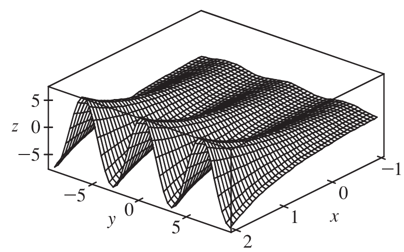
9. $f(x,y) = (1 + xy)(x + y) = x + y + x^2 y + xy^2 \quad\Rightarrow$ $f_x = 1 + 2xy + y^2$, $f_y = 1 + x^2 + 2xy$, $f_{xx} = 2y$, $f_{xy} = 2x + 2y$, $f_{yy} = 2x$. Then $f_x = 0$ implies $1 + 2xy + y^2 = 0$ and $f_y = 0$ implies $1 + x^2 + 2xy = 0$. Subtracting the second equation from the first gives $y^2 - x^2 = 0 \quad\Rightarrow\quad y = \pm x$, but if $y = x$ then $1 + 2xy + y^2 = 0 \quad\Rightarrow$ $1 + 3x^2 = 0$ which has no real solution. If $y = -x$ then $1 + 2xy + y^2 = 0 \quad\Rightarrow\quad 1 - x^2 = 0 \quad\Rightarrow\quad x = \pm 1$, so critical points are $(1,-1)$ and $(-1,1)$. $D(1,-1) = (-2)(2) - 0 < 0$ and $D(-1,1) = (2)(-2) - 0 < 0$, so $(-1,1)$ and $(1,-1)$ are saddle points.



11. $f(x,y) = x^3 - 12xy + 8y^3 \quad\Rightarrow\quad f_x = 3x^2 - 12y$, $f_y = -12x + 24y^2$, $f_{xx} = 6x$, $f_{xy} = -12$, $f_{yy} = 48y$. Then $f_x = 0$ implies $x^2 = 4y$ and $f_y = 0$ implies $x = 2y^2$. Substituting the second equation into the first gives $(2y^2)^2 = 4y \quad\Rightarrow\quad 4y^4 = 4y \quad\Rightarrow\quad 4y(y^3 - 1) = 0 \quad\Rightarrow\quad y = 0$ or $y = 1$. If $y = 0$ then $x = 0$ and if $y = 1$ then $x = 2$, so the critical points are $(0,0)$ and $(2,1)$. $D(0,0) = (0)(0) - (-12)^2 = -144 < 0$, so $(0,0)$ is a saddle point. $D(2,1) = (12)(48) - (-12)^2 = 432 > 0$ and $f_{xx}(2,1) = 12 > 0$ so $f(2,1) = -8$ is a local minimum.



13. $f(x,y) = e^x \cos y \quad\Rightarrow\quad f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



15. $f(x, y) = (x^2 + y^2)e^{y^2 - x^2} \quad \Rightarrow$

$f_x = (x^2 + y^2)e^{y^2 - x^2}(-2x) + 2xe^{y^2 - x^2} = 2xe^{y^2 - x^2}(1 - x^2 - y^2)$,

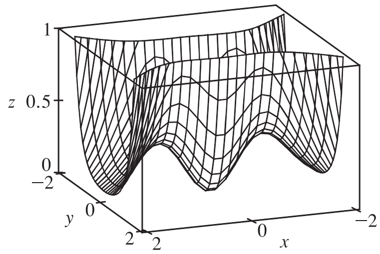
$f_y = (x^2 + y^2)e^{y^2 - x^2}(2y) + 2ye^{y^2 - x^2} = 2ye^{y^2 - x^2}(1 + x^2 + y^2)$,

$f_{xx} = 2xe^{y^2 - x^2}(-2x) + (1 - x^2 - y^2)\left(2x\left(-2xe^{y^2 - x^2}\right) + 2e^{y^2 - x^2}\right) = 2e^{y^2 - x^2}((1 - x^2 - y^2)(1 - 2x^2) - 2x^2)$,

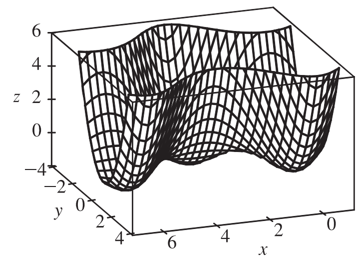
$f_{xy} = 2xe^{y^2 - x^2}(-2y) + 2x(2y)e^{y^2 - x^2}(1 - x^2 - y^2) = -4xye^{y^2 - x^2}(x^2 + y^2)$,

$f_{yy} = 2ye^{y^2 - x^2}(2y) + (1 + x^2 + y^2)\left(2y\left(2ye^{y^2 - x^2}\right) + 2e^{y^2 - x^2}\right) = 2e^{y^2 - x^2}((1 + x^2 + y^2)(1 + 2y^2) + 2y^2)$.

$f_y = 0$ implies $y = 0$, and substituting into $f_x = 0$ gives $2xe^{-x^2}(1 - x^2) = 0 \quad \Rightarrow \quad x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$ and $(\pm 1, 0)$. Now $D(0, 0) = (2)(2) - 0 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $f(0, 0) = 0$ is a local minimum. $D(\pm 1, 0) = (-4e^{-1})(4e^{-1}) - 0 < 0$ so $(\pm 1, 0)$ are saddle points.



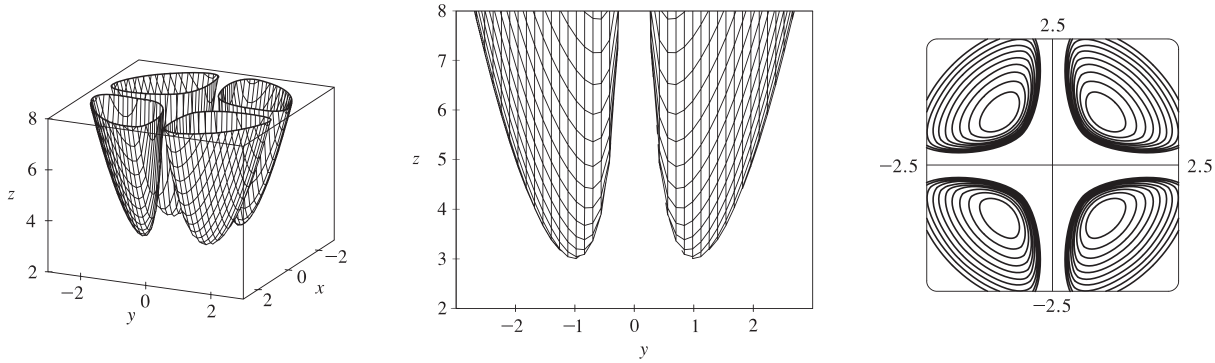
17. $f(x, y) = y^2 - 2y\cos x \quad \Rightarrow \quad f_x = 2y\sin x$, $f_y = 2y - 2\cos x$, $f_{xx} = 2y\cos x$, $f_{xy} = 2\sin x$, $f_{yy} = 2$. Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad \Rightarrow \quad x = 0, \pi$, or 2π for $-1 \le x \le 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.



$D\left(\frac{\pi}{2}, 0\right) = D\left(\frac{3\pi}{2}, 0\right) = -4 < 0$ so $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are saddle points. $D(0, 1) = D(\pi, -1) = D(2\pi, 1) = 4 > 0$ and $f_{xx}(0, 1) = f_{xx}(\pi, -1) = f_{xx}(2\pi, 1) = 2 > 0$, so $f(0, 1) = f(\pi, -1) = f(2\pi, 1) = -1$ are local minima.

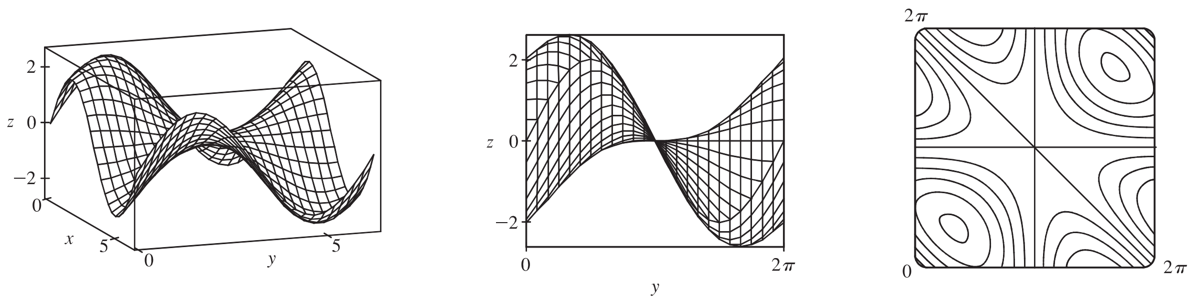
19. $f(x, y) = x^2 + 4y^2 - 4xy + 2 \quad \Rightarrow \quad f_x = 2x - 4y$, $f_y = 8y - 4x$, $f_{xx} = 2$, $f_{xy} = -4$, $f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0, \frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x, y) = x^2 + 4y^2 - 4xy + 2 = (x - 2y)^2 + 2 \ge 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0, \frac{1}{2}x_0\right) = 2$ are all local (and absolute) minima.

21. $f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$



From the graphs, there appear to be local minima of about $f(1,\pm 1) = f(-1,\pm 1) \approx 3$ (and no local maxima or saddle points). $f_x = 2x - 2x^{-3}y^{-2}$, $f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{xy} = 4x^{-3}y^{-3}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$,$(-1,1)$ and $(-1,-1)$. Now $D(1,\pm 1) = D(-1,\pm 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(1,\pm 1) = f(-1,\pm 1) = 3$ are local minima.

23. $f(x,y) = \sin x + \sin y + \sin(x+y),\ 0 \le x \le 2\pi,\ 0 \le y \le 2\pi$

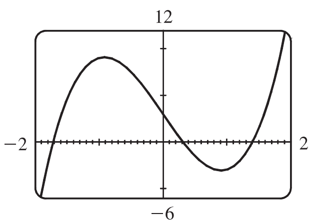


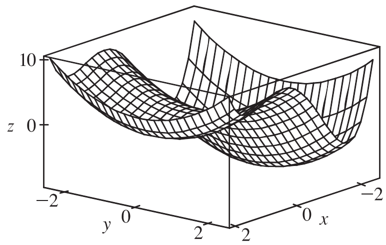
From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6, a local minimum at about $(5,5)$ with value approximately -2.6, and a saddle point at about $(3,3)$.

$f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$, $f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$, $f_{xy} = -\sin(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π,π), $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π,π). Now $D(x,y) = \sin x\ \sin y + \sin x\ \sin(x+y) + \sin y\ \sin(x+y)$. So $D(\pi,\pi) = 0$ and the Second Derivatives Test doesn't apply. However, along the line $y = x$ we have $f(x,x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\ \cos x = 2\sin x(1+\cos x)$, and $f(x,x) > 0$ for $0 < x < \pi$ while $f(x,x) < 0$ for $\pi < x < 2\pi$. Thus every disk with center (π,π) contains points where f is

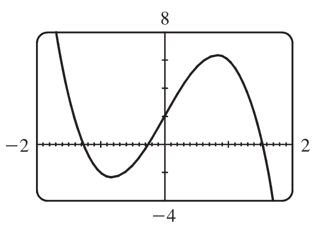
positive as well as points where f is negative, so the graph crosses its tangent plane ($z = 0$) there and (π, π) is a saddle point. $D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

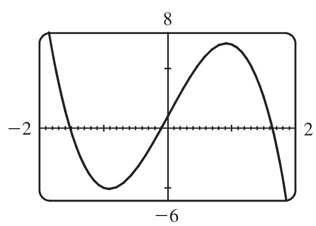
25. $f(x, y) = x^4 - 5x^2 + y^2 + 3x + 2 \quad \Rightarrow \quad f_x(x, y) = 4x^3 - 10x + 3$ and $f_y(x, y) = 2y$. $f_y = 0 \quad \Rightarrow \quad y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714, 0.312$ and 1.402. (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714, 0) > 0$, $f_{xx}(-1.714, 0) > 0$, $D(1.402, 0) > 0$, $f_{xx}(1.402, 0) > 0$, and $D(0.312, 0) < 0$. Therefore $f(-1.714, 0) \approx -9.200$ and $f(1.402, 0) \approx 0.242$ are local minima, and $(0.312, 0)$ is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.

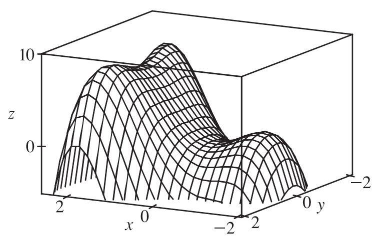




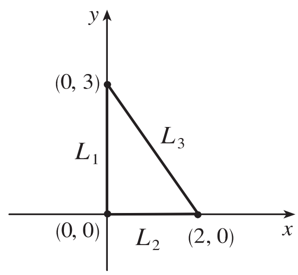
27. $f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \quad \Rightarrow \quad f_x(x, y) = 2 + 8x + 2y^2 - 4x^3$, $f_y(x, y) = -2y + 4xy - 4y^3$. Now $f_y = 0 \quad \Leftrightarrow \quad 2y(2y^2 - 2x + 1) = 0 \quad \Leftrightarrow \quad y = 0$ or $y^2 = x - \frac{1}{2}$. The first of these implies that $f_x = -4x^3 + 8x + 2$, and the second implies that $f_x = 2 + 8x + 2\left(x - \frac{1}{2}\right) - 4x^3 = -4x^3 + 10x + 1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267, -0.259, and 1.526, and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2 = x - \frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267, 0)$, $(-0.259, 0)$, $(1.526, 0)$, and $(1.629, \pm 1.063)$. Now since $f_{xx} = 8 - 12x^2$, $f_{xy} = 4y$, $f_{yy} = 4x - 12y^2$, and $D = (8 - 12x^2)(4x - 12y^2) - 16y^2$, we have $D(-1.267, 0) > 0$, $f_{xx}(-1.267, 0) > 0$, $D(-0.259, 0) < 0$, $D(1.526, 0) < 0$, $D(1.629, \pm 1.063) > 0$, and $f_{xx}(1.629, \pm 1.063) < 0$. Therefore, to three decimal places, $f(-1.267, 0) \approx 1.310$ and $f(1.629, \pm 1.063) \approx 8.105$ are local maxima, and $(-0.259, 0)$ and $(1.526, 0)$ are saddle points. The highest points on the graph are approximately $(1.629, \pm 1.063, 8.105)$.



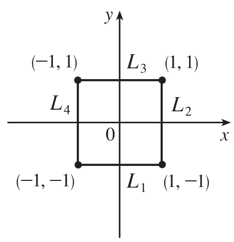




29. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 4$, $f_y = -5$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1: $x = 0$ and $f(0, y) = 1 - 5y$ for $0 \le y \le 3$, a decreasing function in y, so the maximum value is $f(0, 0) = 1$ and the minimum value is $f(0, 3) = -14$. Along L_2: $y = 0$ and $f(x, 0) = 1 + 4x$ for $0 \le x \le 2$, an increasing function in x, so the minimum value is $f(0, 0) = 1$ and the maximum value is $f(2, 0) = 9$. Along L_3: $y = -\frac{3}{2}x + 3$ and $f\left(x, -\frac{3}{2}x + 3\right) = \frac{23}{2}x - 14$ for $0 \le x \le 2$, an increasing function in x, so the minimum value is $f(0, 3) = -14$ and the maximum value is $f(2, 0) = 9$. Thus the absolute maximum of f on D is $f(2, 0) = 9$ and the absolute minimum is $f(0, 3) = -14$.



31. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.



On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

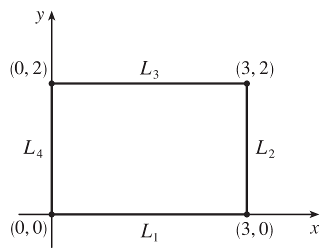
On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

33. $f(x, y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. In Exercise 7, we found the critical points of f; only $(1, 1)$ with $f(1, 1) = 0$ is inside D. On L_1: $y = 0$, $f(x, 0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3, 0) = 83$, and its minimum at $x = 0$, $f(0, 0) = 2$.



On L_2: $x = 3$, $f(3, y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f\left(3, \sqrt[3]{3}\right) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3, 0) = 83$.

On L_3: $y = 2$, $f(x, 2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f\left(\sqrt[3]{2}, 2\right) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3, 2) = 75$. On L_4: $x = 0$, $f(0, y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0, 2) = 18$, and its minimum at $y = 0$, $f(0, 0) = 2$. Thus the absolute maximum of f on D is $f(3, 0) = 83$ and the absolute minimum is $f(1, 1) = 0$.

35. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1-x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0$, -2, or $\frac{1}{2}$. $f(0,\pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

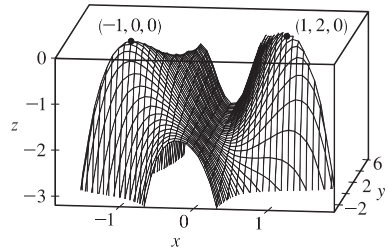
37. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 \quad\Rightarrow\quad f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1)$ and $f_y(x,y) = -2(x^2y-x-1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$.

There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \quad\Leftrightarrow\quad y = \dfrac{x+1}{x^2} \quad [x \ne 0]$,

so $f_x\left(x, \dfrac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\dfrac{x+1}{x^2} - x - 1\right)\left(2x\dfrac{x+1}{x^2} - 1\right) = -4x(x^2-1)$. Therefore $f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate

$f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$D(-1,0) = f_{xx}(-1,0)\, f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 16 > 0$, $f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



39. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x + y - z = 1$, so $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$ where $z = x + y - 1$, and we minimize $d^2 = f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$. Then $f_x(x,y) = 2(x-2) + 2(x+y) = 4x + 2y - 4$, $f_y(x,y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$. Solving $4x + 2y - 4 = 0$ and $2x + 4y - 2 = 0$ simultaneously gives $x = 1$, $y = 0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = 1$, $y = 0$ for which $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

41. Let d be the distance from the point $(4,2,0)$ to any point (x,y,z) on the cone, so $d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$ where $z^2 = x^2 + y^2$, and we minimize $d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2 = f(x,y)$. Then $f_x(x,y) = 2(x-4) + 2x = 4x - 8$, $f_y(x,y) = 2(y-2) + 2y = 4y - 4$, and the critical points occur when $f_x = 0 \Rightarrow x = 2$, $f_y = 0 \Rightarrow y = 1$. Thus the only critical point is $(2,1)$. An absolute minimum exists (since there is a minimum distance from the cone to the point) which must occur at a critical point, so the points on the cone closest to $(4,2,0)$ are $\left(2, 1, \pm\sqrt{5}\right)$.

43. $x + y + z = 100$, so maximize $f(x, y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$.

Thus the critical points are $(0, 0)$, $(100, 0)$, $(0, 100)$ and $\left(\frac{100}{3}, \frac{100}{3}\right)$.

$D(0, 0) = D(100, 0) = D(0, 100) = -10{,}000$ while $D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0, 0)$, $(100, 0)$ and $(0, 100)$ are saddle points whereas $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

45. Center the sphere at the origin so that its equation is $x^2 + y^2 + z^2 = r^2$, and orient the inscribed rectangular box so that its edges are parallel to the coordinate axes. Any vertex of the box satisfies $x^2 + y^2 + z^2 = r^2$, so take (x, y, z) to be the vertex in the first octant. Then the box has length $2x$, width $2y$, and height $2z = 2\sqrt{r^2 - x^2 - y^2}$ with volume given by

$V(x, y) = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$ for $0 < x < r$, $0 < y < r$. Then

$$V_x = (8xy) \cdot \tfrac{1}{2}(r^2 - x^2 - y^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2 - y^2} \cdot 8y = \frac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}} \text{ and } V_y = \frac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}}.$$

Setting $V_x = 0$ gives $y = 0$ or $2x^2 + y^2 = r^2$, but $y > 0$ so only the latter solution applies. Similarly, $V_y = 0$ with $x > 0$ implies $x^2 + 2y^2 = r^2$. Substituting, we have $2x^2 + y^2 = x^2 + 2y^2 \Rightarrow x^2 = y^2 \Rightarrow y = x$. Then $x^2 + 2y^2 = r^2 \Rightarrow 3x^2 = r^2 \Rightarrow x = \sqrt{r^2/3} = r/\sqrt{3} = y$. Thus the only critical point is $\left(r/\sqrt{3}, r/\sqrt{3}\right)$. There must be a maximum volume and here it must occur at a critical point, so the maximum volume occurs when $x = y = r/\sqrt{3}$ and the maximum volume is $V\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right) = 8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \dfrac{8}{3\sqrt{3}}\, r^3$.

47. Maximize $f(x, y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x\,(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

49. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

51. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ cm}^3$. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

53. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and

$L = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad L^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad z = \sqrt{L^2 - x^2 - y^2}$.

Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$, $(x, y > 0)$.

$V_x = xy \cdot \frac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \dfrac{x^2 y}{\sqrt{L^2 - x^2 - y^2}}$,

$V_y = x\sqrt{L^2 - x^2 - y^2} - \dfrac{xy^2}{\sqrt{L^2 - x^2 - y^2}}$. $V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \quad \Rightarrow \quad y(L^2 - 2x^2 - y^2) = 0 \quad \Rightarrow$ $2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \quad \Rightarrow \quad x(L^2 - x^2 - 2y^2) = 0 \quad \Rightarrow$ $x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \quad \Rightarrow$ $3x^2 = L^2 \quad \Rightarrow \quad x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$. So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2 \sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

55. Note that here the variables are m and b, and $f(m, b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then $f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} \left(x_i y_i - m x_i^2 - b x_i\right) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$ and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations.

Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m, b) > 0$ always and

$D(m, b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

15.8 Lagrange Multipliers — ET 14.8

1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

3. $f(x, y) = x^2 + y^2$, $g(x, y) = xy = 1$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2x, 2y \rangle = \langle \lambda y, \lambda x \rangle$, so $2x = \lambda y$, $2y = \lambda x$, and $xy = 1$. From the last equation, $x \neq 0$ and $y \neq 0$, so $2x = \lambda y \quad \Rightarrow \quad \lambda = 2x/y$. Substituting, we have $2y = (2x/y)\,x \quad \Rightarrow$ $y^2 = x^2 \quad \Rightarrow \quad y = \pm x$. But $xy = 1$, so $x = y = \pm 1$ and the possible points for the extreme values of f are $(1, 1)$ and $(-1, -1)$. Here there is no maximum value, since the constraint $xy = 1$ allows x or y to become arbitrarily large, and hence $f(x, y) = x^2 + y^2$ can be made arbitrarily large. The minimum value is $f(1, 1) = f(-1, -1) = 2$.

5. $f(x,y) = x^2y,\ g(x,y) = x^2 + 2y^2 = 6 \quad\Rightarrow\quad \nabla f = \langle 2xy, x^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y\rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x,y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x,y) = 6 \quad\Rightarrow\quad y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x,y) = 6 \quad\Rightarrow$ $4y^2 + 2y^2 = 6 \quad\Rightarrow\quad y^2 = 1 \quad\Rightarrow\quad y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

7. $f(x,y,z) = 2x + 6y + 10z,\ g(x,y,z) = x^2 + y^2 + z^2 = 35 \quad\Rightarrow\quad \nabla f = \langle 2, 6, 10\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{3}{\lambda}$, and $z = \dfrac{5}{\lambda}$. But $35 = x^2 + y^2 + z^2 = \left(\dfrac{1}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 + \left(\dfrac{5}{\lambda}\right)^2 \quad\Rightarrow$ $35 = \dfrac{35}{\lambda^2} \quad\Rightarrow\quad \lambda = \pm 1$, so f has possible extreme values at the points $(1, 3, 5)$, $(-1, -3, -5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1, 3, 5) = 70$, and the minimum is $f(-1, -3, -5) = -70$.

9. $f(x,y,z) = xyz,\ g(x,y,z) = x^2 + 2y^2 + 3z^2 = 6 \quad\Rightarrow\quad \nabla f = \langle yz, xz, xy\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z\rangle$. If $\lambda = 0$ then at least one of the coordinates is 0, in which case $f(x,y,z) = 0$. (None of these ends up giving a maximum or minimum.) If $\lambda \neq 0$, then $\nabla f = \lambda\nabla g$ implies $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$ or $x^2 = 2y^2$ and $z^2 = \frac{2}{3}y^2$. Thus $x^2 + 2y^2 + 3z^2 = 6$ implies $6y^2 = 6$ or $y = \pm 1$. Thus the possible remaining points are $\left(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

11. $f(x,y,z) = x^2 + y^2 + z^2,\ g(x,y,z) = x^4 + y^4 + z^4 = 1 \quad\Rightarrow\quad \nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$, then $\nabla f = \lambda\nabla g$ implies $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$ or $x^2 = y^2 = z^2$ and $3x^4 = 1$ or $x = \pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4 + y^4 + z^4 = 1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

13. $f(x,y,z,t) = x + y + z + t,\ g(x,y,z,t) = x^2 + y^2 + z^2 + t^2 = 1 \quad\Rightarrow\quad \langle 1, 1, 1, 1\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t\rangle$, so $\lambda = 1/(2x) = 1/(2y) = 1/(2z) = 1/(2t)$ and $x = y = z = t$. But $x^2 + y^2 + z^2 + t^2 = 1$, so the possible points are $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ and the minimum value is $f\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -2$.

15. $f(x,y,z) = x + 2y,\ g(x,y,z) = x + y + z = 1,\ h(x,y,z) = y^2 + z^2 = 4 \quad \Rightarrow \quad \nabla f = \langle 1, 2, 0\rangle, \lambda \nabla g = \langle \lambda, \lambda, \lambda\rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\,\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\,\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\,\right) = 1 - 2\sqrt{2}$.

17. $f(x,y,z) = yz + xy,\ g(x,y,z) = xy = 1,\ h(x,y,z) = y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle y, x + z, y\rangle, \lambda \nabla g = \langle \lambda y, \lambda x, 0\rangle$, $\mu \nabla h = \langle 0, 2\mu y, 2\mu z\rangle$. Then $y = \lambda y$ implies $\lambda = 1$ $[y \neq 0$ since $g(x,y,z) = 1]$, $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

19. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \quad \Rightarrow$ $\lambda \nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \quad \Rightarrow \quad x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

21. (a) $f(x,y) = x,\quad g(x,y) = y^2 + x^4 - x^3 = 0 \quad \Rightarrow \quad \nabla f = \langle 1, 0\rangle = \lambda \nabla g = \lambda \left\langle 4x^3 - 3x^2, 2y\right\rangle$. Then $1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \neq 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \quad \Rightarrow \quad x^3(x - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1,0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \quad \Leftrightarrow \quad y^2 = x^3 - x^4$. The left side is non-negative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore the minimum possible value for $f(x,y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0,0) = \lambda \langle 0 - 0, 0\rangle = \langle 0, 0\rangle$ and $\nabla f(0,0) = \langle 1, 0\rangle$, so $\nabla f(0,0) \neq \lambda \nabla g(0,0)$ for all values of λ.

(c) Here $\nabla g(0,0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

23. $P(L,K) = bL^\alpha K^{1-\alpha},\ g(L,K) = mL + nK = p \quad \Rightarrow \quad \nabla P = \left\langle \alpha b L^{\alpha-1} K^{1-\alpha}, (1-\alpha) b L^\alpha K^{-\alpha}\right\rangle,\ \lambda \nabla g = \langle \lambda m, \lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha} = \lambda m$ and $(1-\alpha) b(L/K)^\alpha = \lambda n$ and $mL + nK = p$, so $\alpha b(K/L)^{1-\alpha}/m = (1-\alpha) b(L/K)^\alpha/n$ or $n\alpha/[m(1-\alpha)] = (L/K)^\alpha (L/K)^{1-\alpha}$ or $L = Kn\alpha/[m(1-\alpha)]$. Substituting into $mL + nK = p$ gives $K = (1-\alpha)p/n$ and $L = \alpha p/m$ for the maximum production.

25. Let the sides of the rectangle be x and y. Then $f(x,y) = xy$, $g(x,y) = 2x + 2y = p \Rightarrow \nabla f(x,y) = \langle y, x \rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

27. Let $f(x,y,z) = d^2 = (x-2)^2 + (y-1)^2 + (z+1)^2$, then we want to minimize f subject to the constraint $g(x,y,z) = x + y - z = 1$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-2), 2(y-1), 2(z+1) \rangle = \lambda \langle 1, 1, -1 \rangle$, so $x = (\lambda + 4)/2$, $y = (\lambda + 2)/2$, $z = -(\lambda + 2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda + 4}{2} + \dfrac{\lambda + 2}{2} + \dfrac{\lambda + 2}{2} = 1 \Rightarrow 3\lambda + 8 = 2 \Rightarrow \lambda = -2$, so $x = 1$, $y = 0$, and $z = 0$. This must correspond to a minimum, so the shortest distance is $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$.

29. Let $f(x,y,z) = d^2 = (x-4)^2 + (y-2)^2 + z^2$. Then we want to minimize f subject to the constraint $g(x,y,z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-4), 2(y-2), 2z \rangle = \langle 2\lambda x, 2\lambda y, -2\lambda z \rangle$, so $x - 4 = \lambda x$, $y - 2 = \lambda y$, and $z = -\lambda z$. From the last equation we have $z + \lambda z = 0 \Rightarrow z(1 + \lambda) = 0$, so either $z = 0$ or $\lambda = -1$. But from the constraint equation we have $z = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$ which is not possible from the first two equations. So $\lambda = -1$ and $x - 4 = \lambda x \Rightarrow x = 2$, $y - 2 = \lambda y \Rightarrow y = 1$, and $x^2 + y^2 - z^2 = 0 \Rightarrow 4 + 1 - z^2 = 0 \Rightarrow z = \pm\sqrt{5}$. This must correspond to a minimum, so the points on the cone closest to $(4,2,0)$ are $(2, 1, \pm\sqrt{5})$.

31. $f(x,y,z) = xyz$, $g(x,y,z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

33. If the dimensions are $2x$, $2y$, and $2z$, then maximize $f(x,y,z) = (2x)(2y)(2z) = 8xyz$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = r^2$ $(x > 0, y > 0, z > 0)$. Then $\nabla f = \lambda \nabla g \Rightarrow \langle 8yz, 8xz, 8xy \rangle = \lambda \langle 2x, 2y, 2z \rangle \Rightarrow$ $8yz = 2\lambda x$, $8xz = 2\lambda y$, and $8xy = 2\lambda z$, so $\lambda = \dfrac{4yz}{x} = \dfrac{4xz}{y} = \dfrac{4xy}{z}$. This gives $x^2 z = y^2 z \Rightarrow x^2 = y^2$ (since $z \neq 0$) and $xy^2 = xz^2 \Rightarrow z^2 = y^2$, so $x^2 = y^2 = z^2 \Rightarrow x = y = z$, and substituting into the constraint equation gives $3x^2 = r^2 \Rightarrow x = r/\sqrt{3} = y = z$. Thus the largest volume of such a box is

$$f\left(\tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}, \tfrac{r}{\sqrt{3}}\right) = 8\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right)\left(\tfrac{r}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}r^3.$$

35. $f(x,y,z) = xyz$, $g(x,y,z) = x + 2y + 3z = 6 \Rightarrow \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda \rangle$. Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies $x = 2y$, $z = \frac{2}{3}y$. But $2y + 2y + 2y = 6$ so $y = 1$, $x = 2$, $z = \frac{2}{3}$ and the volume is $V = \frac{4}{3}$.

37. $f(x,y,z) = xyz$, $g(x,y,z) = 4(x + y + z) = c \Rightarrow \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 4\lambda, 4\lambda, 4\lambda \rangle$. Thus $4\lambda = yz = xz = xy$ or $x = y = z = \frac{1}{12}c$ are the dimensions giving the maximum volume.

39. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x,y,z) = xyz$ $[x, y, z > 0]$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x,y,z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \Rightarrow$ $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$, so $yz = 2\lambda x \Rightarrow \lambda = \dfrac{yz}{2x}$, $xz = 2\lambda y \Rightarrow \lambda = \dfrac{xz}{2y}$, and $xy = 2\lambda z \Rightarrow \lambda = \dfrac{xy}{2z}$. Thus

$\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} \quad\Rightarrow\quad x^2 = y^2$ [since $z \neq 0$] $\Rightarrow$ $x = y$ and $\lambda = \dfrac{yz}{2x} = \dfrac{xy}{2z} \quad\Rightarrow\quad x = z$ [since $y \neq 0$].

Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \quad\Rightarrow\quad x^2 = L^2/3 \quad\Rightarrow\quad x = L/\sqrt{3} = y = z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3 = L^3/\left(3\sqrt{3}\right)$.

41. We need to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x,y,z) = x + y + 2z = 2$ and $h(x,y,z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, 2\lambda\rangle$ and $\mu\nabla h = \langle 2\mu x, 2\mu y, -\mu\rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**.
From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

43. $f(x,y,z) = ye^{x-z}$, $g(x,y,z) = 9x^2 + 4y^2 + 36z^2 = 36$, $h(x,y,z) = xy + yz = 1$. $\nabla f = \lambda\nabla g + \mu\nabla h \quad\Rightarrow$ $\left\langle ye^{x-z}, e^{x-z}, -ye^{x-z}\right\rangle = \lambda\langle 18x, 8y, 72z\rangle + \mu\langle y, x + z, y\rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x + z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$\begin{array}{lllll}
x \approx 0.222444, & y \approx -2.157012, & z \approx -0.686049, & \lambda \approx -0.200401, & \mu \approx 2.108584 \\
x \approx -1.951921, & y \approx -0.545867, & z \approx 0.119973, & \lambda \approx 0.003141, & \mu \approx -0.076238 \\
x \approx 0.155142, & y \approx 0.904622, & z \approx 0.950293, & \lambda \approx -0.012447, & \mu \approx 0.489938 \\
x \approx 1.138731, & y \approx 1.768057, & z \approx -0.573138, & \lambda \approx 0.317141, & \mu \approx 1.862675
\end{array}$$

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$, $f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$, $f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the mininum is approximately -5.3506.

45. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$\nabla f = \left\langle \frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_2\cdots x_n), \frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1x_3\cdots x_n), \ldots, \frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1\cdots x_{n-1})\right\rangle$

and $\lambda\nabla g = \langle \lambda, \lambda, \ldots, \lambda\rangle$, so we need to solve the system of equations

$$\begin{aligned}
\tfrac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_2\cdots x_n) = \lambda \quad&\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_1 \\
\tfrac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1x_3\cdots x_n) = \lambda \quad&\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_2 \\
&\;\vdots \\
\tfrac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1\cdots x_{n-1}) = \lambda \quad&\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_n
\end{aligned}$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \quad \Rightarrow \quad nx_1 = c \quad \Rightarrow \quad x_1 = \dfrac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is

$$f\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right) = \sqrt[n]{\frac{c}{n} \cdot \frac{c}{n} \cdot \cdots \cdot \frac{c}{n}} = \frac{c}{n}.$$

(b) From part (a), $\dfrac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\dfrac{c}{n}$, but this can occur only at the point $\left(\dfrac{c}{n}, \dfrac{c}{n}, \ldots, \dfrac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \dfrac{c}{n}$.

15 Review — ET 14

CONCEPT CHECK

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

(b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as Figure 1 in Section 15.1 [ET 14.1].

2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) See Definition 15.2.4 [ET 14.2.4].

(b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) See (2) and (3) in Section 15.3 [ET 14.3].

(b) See "Interpretations of Partial Derivatives" on page 917 [ET 881].

(c) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. See the statement of Clairaut's Theorem on page 921 [ET 885].

7. (a) See (2) in Section 15.4 [ET 14.4]

(b) See (19) and the preceding discussion in Section 15.6 [ET 14.6].

8. See (3) and (4) and the accompanying discussion in Section 15.4 [ET 14.4]. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b). Thus it is the linear function which best approximates f near (a, b).

9. (a) See Definition 15.4.7 [ET 14.4.7].

(b) Use Theorem 15.4.8 [ET 14.4.8].

10. See (10) and the associated discussion in Section 15.4 [ET 14.4].

11. See (2) and (3) in Section 15.5 [ET 14.5].

12. See (7) and the preceding discussion in Section 15.5 [ET 14.5].

13. (a) See Definition 15.6.2 [ET 14.6.2]. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P. (See Figure 5 in Section 15.6 [ET 14.6].)

(b) See Theorem 15.6.3 [ET 14.6.3].

14. (a) See (8) and (13) in Section 15.6 [ET 14.6].

(b) $D_{\mathbf{u}}\, f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}}\, f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

(c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

15. (a) f has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b).

(b) f has an absolute maximum at (a, b) if $f(x, y) \le f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b) if $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b) if $f(x, y) \ge f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) By Theorem 15.7.2 [ET 14.7.2], if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. See (3) in Section 15.7 [ET 14.7]

18. (a) See Figure 11 and the accompanying discussion in Section 15.7 [ET 14.7].

(b) See Theorem 15.7.8 [ET 14.7.8].

(c) See the procedure outlined in (9) in Section 15.7 [ET 14.7].

19. See the discussion beginning on page 970 [ET 934]; see “Two Constraints” on page 974 [ET 938].

TRUE-FALSE QUIZ

1. True. $f_y(a,b) = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$ from Equation 15.3.3 [ET 14.3.3]. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a,b) = \lim\limits_{y\to b} \dfrac{f(a,y) - f(a,b)}{y - b}$.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y\,\partial x}$.

5. False. See Example 15.2.3 [ET 14.2.3].

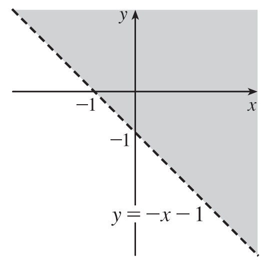
7. True. If f has a local minimum and f is differentiable at (a,b) then by Theorem 15.7.2 [ET 14.7.2], $f_x(a,b) = 0$ and $f_y(a,b) = 0$, so $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = \langle 0,0\rangle = \mathbf{0}$.

9. False. $\nabla f(x,y) = \langle 0, 1/y\rangle$.

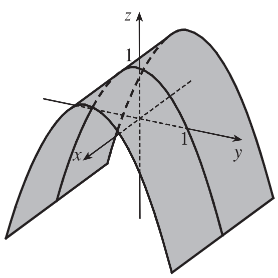
11. True. $\nabla f = \langle \cos x, \cos y\rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos\theta| \le 1$, so $|\nabla f| \le \sqrt{2}$. Now $D_{\mathbf{u}}\, f(x,y) = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}\, f(x,y)| \le \sqrt{2}\cdot 1\cdot 1 = \sqrt{2}$.

EXERCISES

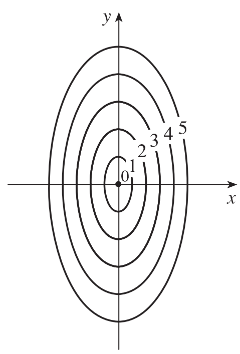
1. $\ln(x+y+1)$ is defined only when $x + y + 1 > 0 \quad\Rightarrow\quad y > -x - 1$, so the domain of f is $\{(x,y) \mid y > -x - 1\}$, all those points above the line $y = -x - 1$.



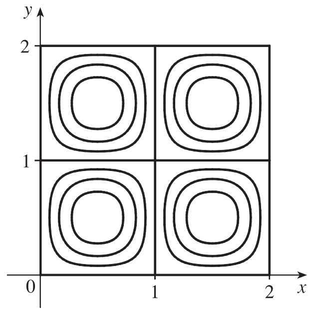
3. $z = f(x,y) = 1 - y^2$, a parabolic cylinder



5. The level curves are $\sqrt{4x^2 + y^2} = k$ or $4x^2 + y^2 = k^2$, $k \ge 0$, a family of ellipses.



7.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1,1)$, we use direct substitution to evaluate the limit: $\lim\limits_{(x,y)\to(1,1)} \dfrac{2xy}{x^2+2y^2} = \dfrac{2(1)(1)}{1^2+2(1)^2} = \dfrac{2}{3}$.

11. (a) $T_x(6,4) = \lim\limits_{h\to 0} \dfrac{T(6+h,4)-T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h=\pm 2$ and using the values given in the table: $T_x(6,4) \approx \dfrac{T(8,4)-T(6,4)}{2} = \dfrac{86-80}{2} = 3$,

$T_x(6,4) \approx \dfrac{T(4,4)-T(6,4)}{-2} = \dfrac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately 3.5°C/m. Similarly, $T_y\,(6,4) = \lim\limits_{h\to 0} \dfrac{T(6,4+h)-T(6,4)}{h}$, which we can approximate with $h=\pm 2$:

$T_y(6,4) \approx \dfrac{T(6,6)-T(6,4)}{2} = \dfrac{75-80}{2} = -2.5$, $T_y(6,4) \approx \dfrac{T(6,2)-T(6,4)}{-2} = \dfrac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to be approximately -3.0°C/m.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 15.6.9 [ET 14.6.9], $D_\mathbf{u}\,T(6,4) = \nabla T(6,4)\cdot\mathbf{u} = T_x(6,4)\,\frac{1}{\sqrt{2}} + T_y(6,4)\,\frac{1}{\sqrt{2}}$.

Using our estimates from part (a), we have $D_\mathbf{u}\,T(6,4) \approx (3.5)\,\frac{1}{\sqrt{2}} + (-3.0)\,\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately 0.35°C/m.

Alternatively, we can use Definition 15.6.2 [ET 14.6.2]: $D_\mathbf{u}\,T(6,4) = \lim\limits_{h\to 0} \dfrac{T\left(6+h\,\frac{1}{\sqrt{2}}, 4+h\,\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_\mathbf{u}\,T(6,4) \approx \dfrac{T(8,6)-T(6,4)}{2\sqrt{2}} = \dfrac{80-80}{2\sqrt{2}} = 0$,

$D_\mathbf{u}\,T(6,4) \approx \dfrac{T(4,2)-T(6,4)}{-2\sqrt{2}} = \dfrac{74-80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$. Averaging these values, we have $D_\mathbf{u}\,T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1^\circ\text{C/m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[T_x(x,y)] = \lim\limits_{h\to 0} \dfrac{T_x(x,y+h)-T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0} \dfrac{T_x(6,4+h)-T_x(6,4)}{h}$ which we can estimate with $h=\pm 2$. We have $T_x(6,4)\approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h=\pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \dfrac{T(8,6)-T(6,6)}{2} = \dfrac{80-75}{2} = 2.5$, $T_x(6,6) \approx \dfrac{T(4,6)-T(6,6)}{-2} = \dfrac{68-75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$$T_x(6,2) \approx \frac{T(8,2) - T_x(6,2)}{2} = \frac{90 - 87}{2} = 1.5, T_x(6,2) \approx \frac{T(4,2) - T(6,2)}{-2} = \frac{74 - 87}{-2} = 6.5.$$

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{2} = \frac{3.0 - 3.5}{2} = -0.25, T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25.$$

Averaging these values, we have $T_{xy}\,(6,4) \approx -0.25$.

13. $f(x,y) = \sqrt{2x + y^2} \quad \Rightarrow \quad f_x = \frac{1}{2}(2x + y^2)^{-1/2}(2) = \dfrac{1}{\sqrt{2x + y^2}},\; f_y = \frac{1}{2}(2x + y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{2x + y^2}}$

15. $g(u,v) = u\tan^{-1} v \quad \Rightarrow \quad g_u = \tan^{-1} v,\; g_v = \dfrac{u}{1 + v^2}$

17. $T(p,q,r) = p\ln(q + e^r) \quad \Rightarrow \quad T_p = \ln(q + e^r),\; T_q = \dfrac{p}{q + e^r},\; T_r = \dfrac{pe^r}{q + e^r}$

19. $f(x,y) = 4x^3 - xy^2 \quad \Rightarrow \quad f_x = 12x^2 - y^2,\; f_y = -2xy,\; f_{xx} = 24x,\; f_{yy} = -2x,\; f_{xy} = f_{yx} = -2y$

21. $f(x,y,z) = x^k y^l z^m \quad \Rightarrow \quad f_x = kx^{k-1}y^l z^m,\; f_y = lx^k y^{l-1}z^m,\; f_z = mx^k y^l z^{m-1},\; f_{xx} = k(k-1)x^{k-2}y^l z^m,$

$f_{yy} = l(l-1)x^k y^{l-2}z^m,\; f_{zz} = m(m-1)x^k y^l z^{m-2},\; f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m, f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1},$

$f_{yz} = f_{zy} = lmx^k y^{l-1}z^{m-1}$

23. $z = xy + xe^{y/x} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = y - \dfrac{y}{x}e^{y/x} + e^{y/x},\; \dfrac{\partial z}{\partial y} = x + e^{y/x}$ and

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = x\Big(y - \frac{y}{x}e^{y/x} + e^{y/x}\Big) + y\Big(x + e^{y/x}\Big) = xy - ye^{y/x} + xe^{y/x} + xy + ye^{y/x} = xy + xy + xe^{y/x} = xy + z.$$

25. (a) $z_x = 6x + 2 \quad \Rightarrow \quad z_x(1,-2) = 8$ and $z_y = -2y \quad \Rightarrow \quad z_y(1,-2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1,-2,1)$ is $\langle 8, 4, -1\rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are $\dfrac{x - 1}{8} = \dfrac{y + 2}{4} = \dfrac{z - 1}{-1}$.

27. (a) Let $F(x,y,z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2,-1,1) = 4$, $F_y(2,-1,1) = -4$, $F_z(2,-1,1) = -6$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\dfrac{x - 2}{4} = \dfrac{y + 1}{-4} = \dfrac{z - 1}{-6}$.

29. (a) $\mathbf{r}(u,v) = (u + v)\,\mathbf{i} + u^2\,\mathbf{j} + v^2\,\mathbf{k}$ and the point $(3,4,1)$ corresponds to $u = 2$, $v = 1$. Then $\mathbf{r}_u = \mathbf{i} + 2u\,\mathbf{j} \quad \Rightarrow$ $\mathbf{r}_u(2,1) = \mathbf{i} + 4\,\mathbf{j}$ and $\mathbf{r}_v = \mathbf{i} + 2v\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}_v(2,1) = \mathbf{i} + 2\,\mathbf{j}$. A normal vector to the surface at $(3,4,1)$ is $\mathbf{r}_u \times \mathbf{r}_v = 8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so an equation of the tangent plane there is $8(x - 3) - 2(y - 4) - 4(z - 1) = 0$ or equivalently $4x - y - 2z = 6$.

(b) A direction vector for the normal line through $(3,4,1)$ is $8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so a vector equation is $\mathbf{r}(t) = (3\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}) + t\,(8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k})$, and the corresponding parametric equations are $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$.

31. The hyperboloid is a level surface of the function $F(x,y,z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0 \rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1 \rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0 \rangle = k\,\langle 2, 2, 1 \rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \quad \Rightarrow \quad k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \quad \Rightarrow \quad k^2 = 4 \quad \Rightarrow \quad k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

33. $f(x,y,z) = x^3\sqrt{y^2+z^2} \quad \Rightarrow \quad f_x(x,y,z) = 3x^2\sqrt{y^2+z^2}$, $f_y(x,y,z) = \dfrac{yx^3}{\sqrt{y^2+z^2}}$, $f_z(x,y,z) = \dfrac{zx^3}{\sqrt{y^2+z^2}}$,

so $f(2,3,4) = 8(5) = 40$, $f_x(2,3,4) = 3(4)\sqrt{25} = 60$, $f_y(2,3,4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2,3,4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned} f(x,y,z) &\approx f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \tfrac{24}{5}(y-3) + \tfrac{32}{5}(z-4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

35. $\dfrac{du}{dp} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dp} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dp} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dp} = 2xy^3(1+6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

37. By the Chain Rule, $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\dfrac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so

$\dfrac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

39. $\dfrac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$, $\quad \dfrac{\partial z}{\partial y} = 1 - 2yf'(x^2-y^2) \quad \left[\text{where } f' = \dfrac{df}{d(x^2-y^2)}\right]$. Then

$y\,\dfrac{\partial z}{\partial x} + x\,\dfrac{\partial z}{\partial y} = 2xyf'(x^2-y^2) + x - 2xyf'(x^2-y^2) = x$.

41. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\,y + \dfrac{\partial z}{\partial v}\,\dfrac{-y}{x^2}$ and

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}\,y + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\,\partial v}\,y\right) \\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2} \end{aligned}$$

Also $\dfrac{\partial z}{\partial y} = x\,\dfrac{\partial z}{\partial u} + \dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\,\partial u}\,\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\,\frac{1}{x} + \frac{\partial^2 z}{\partial u\,\partial v}\,x\right) = x^2\,\frac{\partial^2 z}{\partial u^2} + 2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\,\frac{\partial^2 z}{\partial v^2}$$

Thus

$$x^2\,\frac{\partial^2 z}{\partial x^2} - y^2\,\frac{\partial^2 z}{\partial y^2} = \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\,\frac{\partial^2 z}{\partial u^2} - 2y^2\,\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}$$
$$= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\,\frac{\partial^2 z}{\partial u\,\partial v} = 2v\,\frac{\partial z}{\partial v} - 4uv\,\frac{\partial^2 z}{\partial u\,\partial v}$$

since $y = xv = \dfrac{uv}{y}$ or $y^2 = uv$.

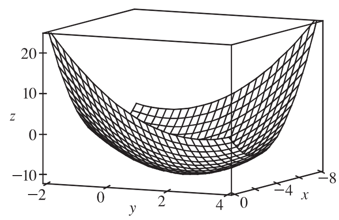
43. $\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}}, \dfrac{xz^2e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}}\right\rangle = ze^{x\sqrt{y}}\left\langle z\sqrt{y}, \dfrac{xz}{2\sqrt{y}}, 2\right\rangle$

45. $\nabla f = \langle 1/\sqrt{x}, -2y\rangle$, $\nabla f(1,5) = \langle 1, -10\rangle$, $\mathbf{u} = \frac{1}{5}\langle 3, -4\rangle$. Then $D_{\mathbf{u}}\,f(1,5) = \frac{43}{5}$.

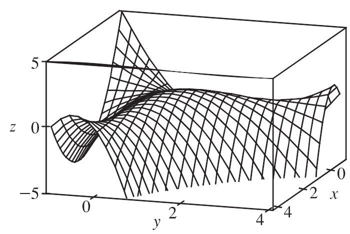
47. $\nabla f = \left\langle 2xy, x^2 + 1/(2\sqrt{y})\right\rangle$, $|\nabla f(2,1)| = \left|\left\langle 4, \frac{9}{2}\right\rangle\right|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2}\right\rangle$.

49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

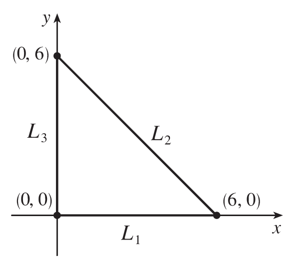
51. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \quad\Rightarrow\quad f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



53. $f(x,y) = 3xy - x^2y - xy^2 \quad\Rightarrow\quad f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$. $D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



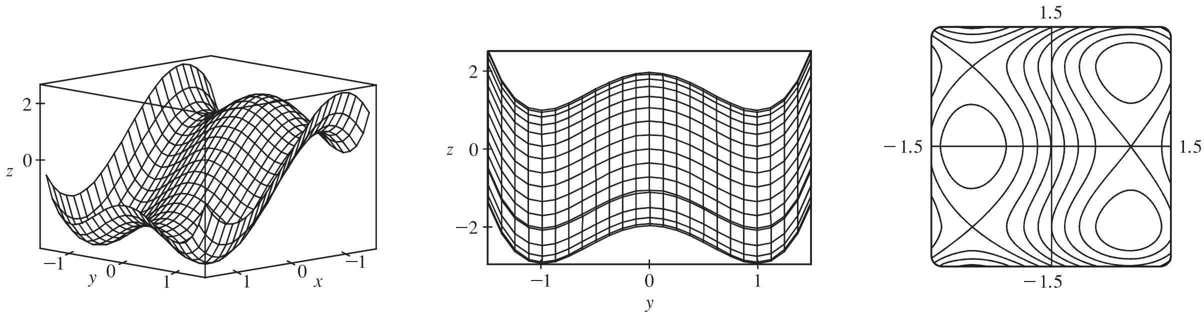
55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D.



On L_1: $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2: $x = -y + 6$ and

$f(-y+6,y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has critical pointsat $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3: $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$



From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minima $f(1,\pm1) \approx -3$, and saddle points at $(-1,\pm1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad \Leftrightarrow \quad x = \pm1$ and $f_y = 4y^3 - 4y = 0 \quad \Leftrightarrow$ $y = 0, \pm1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm1) = 48 > 0$ and $f_{xx}(1,\pm1) = 6 > 0$ indicating local minima $f(1,\pm1) = -3$; and $D(-1,\pm1) = -48$ and $D(1,0) = -24$, indicating saddle points.

59. $f(x,y) = x^2 y$, $g(x,y) = x^2 + y^2 = 1 \quad \Rightarrow \quad \nabla f = \left\langle 2xy, x^2 \right\rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y\rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f\,(0,0) = 0$.] Thus the possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

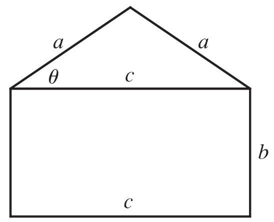
61. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle yz, xz, xy\rangle = \lambda\langle 2x, 2y, 2z\rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \quad \Rightarrow \quad 2y^2 z = 2x^2 z \quad \Rightarrow$ $y^2 = x^2$, and similarly $2yz^2 = 2x^2 y \quad \Rightarrow \quad z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \quad \Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1,1,\pm1)$, $(1,-1,\pm1)$, $(-1,1,\pm1)$, $(-1,-1,\pm1)$. The absolute maximum is $f(1,1,1) = f(1,-1,-1) = f(-1,1,-1) = f(-1,-1,1) = 1$ and the absolute minimum is $f(1,1,-1) = f(1,-1,1) = f(-1,1,1) = f(-1,-1,-1) = -1$.

63. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = xy^2z^3 = 2 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z\rangle = \lambda\nabla g = \left\langle \lambda y^2 z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \right\rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so $2x = \lambda y^2 z^3$ **(1)**, $1 = \lambda xz^3$ **(2)**, $2 = 3\lambda xy^2 z$ **(3)**. Then **(2)** and **(3)** imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly **(1)** and **(3)** imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But

$xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x, y, z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4})$, $(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2, 1, 1)$ also satisfies $g(x, y, z) = 2$ and $f(2, 1, 1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x, y, z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize $f(x, z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then $f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and $f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2 + 4)\left(2 + \frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

65.



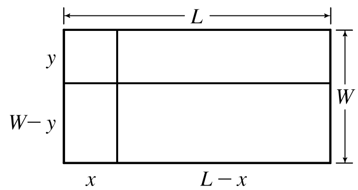
The area of the triangle is $\frac{1}{2}ca\sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a, b, c) = \frac{1}{2}ca\sin\theta + bc$. The perimeter of the object is $g(a, b, c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \quad\Rightarrow\quad \sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a, b, c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda\nabla g$ which gives the following equations: $ca(4a^2 - c^2)^{-1/2} = 2\lambda$ **(1)**, $c = 2\lambda$ **(2)**, $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$ **(3)**, and $2a + 2b + c = P$ **(4)**. From **(2)**, $\lambda = \frac{1}{2}c$ and so **(1)** produces $ca(4a^2 - c^2)^{-1/2} = c \quad\Rightarrow\quad (4a^2 - c^2)^{1/2} = a \quad\Rightarrow$ $4a^2 - c^2 = a^2 \quad\Rightarrow\quad c = \sqrt{3}\,a$ **(5)**. Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, **(3)** gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from **(5)**, $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}\,a}{2} \quad\Rightarrow\quad -\dfrac{a}{2} - \dfrac{\sqrt{3}\,a}{2} = -b \quad\Rightarrow\quad b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$ **(6)**. Substituting **(5)** and **(6)** into **(4)** we get:

$$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \quad\Rightarrow\quad 3a + 2\sqrt{3}\,a = P \quad\Rightarrow\quad a = \frac{P}{3 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{3}P \text{ and thus}$$

$$b = \frac{\left(2\sqrt{3} - 3\right)\left(1 + \sqrt{3}\right)}{6}P = \frac{3 - \sqrt{3}}{6}P \text{ and } c = \left(2 - \sqrt{3}\right)P.$$

PROBLEMS PLUS

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L-x)y$, $A_3 = (L-x)(W-y)$, $A_4 = x(W-y)$. For $0 \le x \le L$, $0 \le y \le W$, let



$$\begin{aligned} f(x,y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2y^2 + (L-x)^2y^2 + (L-x)^2(W-y)^2 + x^2(W-y)^2 \\ &= [x^2 + (L-x)^2][y^2 + (W-y)^2] \end{aligned}$$

Then we need to find the maximum and minimum values of $f(x,y)$. Here

$f_x(x,y) = [2x - 2(L-x)][y^2 + (W-y)^2] = 0 \quad\Rightarrow\quad 4x - 2L = 0$ or $x = \frac{1}{2}L$, and

$f_y(x,y) = [x^2 + (L-x)^2][2y - 2(W-y)] = 0 \quad\Rightarrow\quad 4y - 2W = 0$ or $y = W/2$. Also

$f_{xx} = 4[y^2 + (W-y)^2]$, $f_{yy} = 4[x^2 + (L-x)^2]$, and $f_{xy} = (4x - 2L)(4y - 2W)$. Then

$D = 16[y^2 + (W-y)^2][x^2 + (L-x)^2] - (4x-2L)^2(4y-2W)^2$. Thus when $x = \frac{1}{2}L$ and $y = \frac{1}{2}W$, $D > 0$ and $f_{xx} = 2W^2 > 0$. Thus a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2W^2$.

There are no other critical points, so the maximum must occur on the boundary. Now along the width of the rectangle let $g(y) = f(0,y) = f(L,y) = L^2[y^2 + (W-y)^2]$, $0 \le y \le W$. Then $g'(y) = L^2[2y - 2(W-y)] = 0 \quad\Leftrightarrow\quad y = \frac{1}{2}W$.

And $g\left(\frac{1}{2}\right) = \frac{1}{2}L^2W^2$. Checking the endpoints, we get $g(0) = g(W) = L^2W^2$. Along the length of the rectangle let $h(x) = f(x,0) = f(x,W) = W^2[x^2 + (L-x)^2]$, $0 \le x \le L$. By symmetry $h'(x) = 0 \quad\Leftrightarrow\quad x = \frac{1}{2}L$ and $h\left(\frac{1}{2}L\right) = \frac{1}{2}L^2W^2$. At the endpoints we have $h(0) = h(L) = L^2W^2$. Therefore L^2W^2 is the maximum value of f.

This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x\sin\theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x\cos\theta$. Therefore the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x,\theta) &= \tfrac{1}{2}x\sin\theta\,[(w-2x) + (w - 2x + 2x\cos\theta)] = (x\sin\theta)(w - 2x + x\cos\theta) \\ &= wx\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\ \ 0 < x \le \tfrac{1}{2}w, 0 < \theta \le \tfrac{\pi}{2} \end{aligned}$$

We look for the critical points of A: $\partial A/\partial x = w\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta$ and

$\partial A/\partial\theta = wx\cos\theta - 2x^2\cos\theta + x^2(\cos^2\theta - \sin^2\theta)$, so $\partial A/\partial x = 0 \quad\Leftrightarrow\quad \sin\theta\,(w - 4x + 2x\cos\theta) = 0 \quad\Leftrightarrow$

$\cos\theta = \dfrac{4x - w}{2x} = 2 - \dfrac{w}{2x} \quad \left(0 < \theta \le \frac{\pi}{2} \quad\Rightarrow\quad \sin\theta > 0\right)$. If, in addition, $\partial A/\partial\theta = 0$, then

$$\begin{aligned} 0 &= wx\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) \\ &= wx\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right] \\ &= 2wx - \tfrac{1}{2}w^2 - 4x^2 + wx + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$, $b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 15.7.6 [ET 14.7.6], we can argue from the physical nature of this problem that we have found a local maximum of A. Now checking the boundary of A, let $g(\theta) = A(w/2, \theta) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta$, $0 < \theta \le \frac{\pi}{2}$. Clearly g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also along the line $\theta = \frac{\pi}{2}$, let $h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2$, $0 < x < \frac{1}{2}w \quad\Rightarrow$ $h'(x) = w - 4x = 0 \quad\Leftrightarrow\quad x = \frac{1}{4}w$, and $h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}w^2$, we conclude that the local maximum found earlier was an absolute maximum.

(b) If the metal were bent into a semi-circular gutter of radius r, we would have $w = \pi r$ and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{w}{\pi}\right)^2 = \dfrac{w^2}{2\pi}$.

Since $\dfrac{w^2}{2\pi} > \dfrac{\sqrt{3}\,w^2}{12}$, it *would* be better to bend the metal into a gutter with a semicircular cross-section.

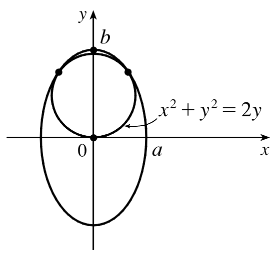
5. Let $g(x,y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x,y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$ and

$g_y(x,y) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$. Thus the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) \quad\Rightarrow$$

$\left[f\left(\frac{y_0}{x_0}\right) - y_0x_0^{-1}f'\left(\frac{y_0}{x_0}\right)\right]x + \left[f'\left(\frac{y_0}{x_0}\right)\right]y - z = 0$. But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Thus the origin is the common point of intersection.

7. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad\Rightarrow\quad x^2 = \dfrac{a^2}{b^2}\left(b^2 - y^2\right)$. Substituting into the equation of the circle gives $\dfrac{a^2}{b^2}\left(b^2 - y^2\right) + y^2 - 2y = 0 \quad\Rightarrow\quad \left(\dfrac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$.



In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2\dfrac{b^2 - a^2}{b^2} = 0 \quad\Rightarrow$ $b^2 - a^2b^2 + a^4 = 0$. The area of the ellipse is $A(a,b) = \pi ab$, and we minimize this function subject to the constraint $g(a,b) = b^2 - a^2b^2 + a^4 = 0$.

Now $\nabla A = \lambda\nabla g \quad\Leftrightarrow\quad \pi b = \lambda(4a^3 - 2ab^2)$, $\pi a = \lambda(2b - 2ba^2) \quad\Rightarrow\quad \lambda = \dfrac{\pi b}{2a(2a^2 - b^2)}$ **(1)**,

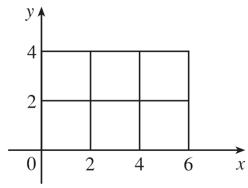
$\lambda = \dfrac{\pi a}{2b(1 - a^2)}$ **(2)**, $b^2 - a^2b^2 + a^4 = 0$ **(3)**. Comparing **(1)** and **(2)** gives $\dfrac{\pi b}{2a(2a^2 - b^2)} = \dfrac{\pi a}{2b(1 - a^2)} \quad\Rightarrow$

$2\pi b^2 = 4\pi a^4 \quad\Leftrightarrow\quad a^2 = \frac{1}{\sqrt{2}}b$. Substitute this into **(3)** to get $b = \frac{3}{\sqrt{2}} \quad\Rightarrow\quad a = \sqrt{\frac{3}{2}}$.

16 □ MULTIPLE INTEGRALS □ ET 15

16.1 Double Integrals over Rectangles ET 15.1

1. (a) The subrectangles are shown in the figure.

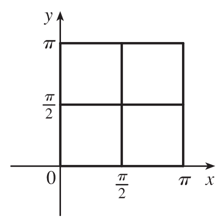


The surface is the graph of $f(x,y)=xy$ and $\Delta A=4$, so we estimate

$$\begin{aligned}V &\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f(x_i,y_j)\,\Delta A\\ &= f(2,2)\,\Delta A+f(2,4)\,\Delta A+f(4,2)\,\Delta A+f(4,4)\,\Delta A+f(6,2)\,\Delta A+f(6,4)\,\Delta A\\ &= 4(4)+8(4)+8(4)+16(4)+12(4)+24(4)=288\end{aligned}$$

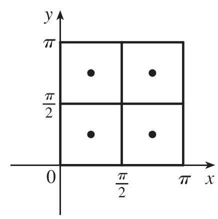
(b) $$\begin{aligned}V &\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A = f(1,1)\,\Delta A+f(1,3)\,\Delta A+f(3,1)\,\Delta A+f(3,3)\,\Delta A+f(5,1)\,\Delta A+f(5,3)\,\Delta A\\ &= 1(4)+3(4)+3(4)+9(4)+5(4)+15(4)=144\end{aligned}$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A=\pi^2/4$, we estimate

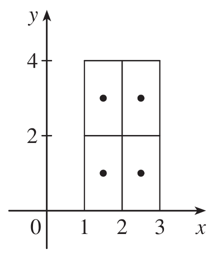


$$\begin{aligned}\iint_R \sin(x+y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*,y_{ij}^*\right)\Delta A\\ &= f(0,0)\,\Delta A+f\left(0,\tfrac{\pi}{2}\right)\Delta A+f\left(\tfrac{\pi}{2},0\right)\Delta A+f\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\Delta A\\ &= 0\left(\tfrac{\pi^2}{4}\right)+1\left(\tfrac{\pi^2}{4}\right)+1\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)=\tfrac{\pi^2}{2}\approx 4.935\end{aligned}$$

(b) $$\begin{aligned}\iint_R \sin(x+y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A\\ &= f\left(\tfrac{\pi}{4},\tfrac{\pi}{4}\right)\Delta A+f\left(\tfrac{\pi}{4},\tfrac{3\pi}{4}\right)\Delta A+f\left(\tfrac{3\pi}{4},\tfrac{\pi}{4}\right)\Delta A+f\left(\tfrac{3\pi}{4},\tfrac{3\pi}{4}\right)\Delta A\\ &= 1\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)+(-1)\left(\tfrac{\pi^2}{4}\right)=0\end{aligned}$$

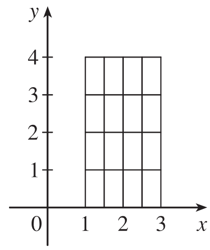


5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A=2$, so we evaluate f at each midpoint and estimate



$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i,\overline{y}_j\right)\Delta A\\ &= f(1.5,1)\,\Delta A+f(1.5,3)\,\Delta A\\ &\qquad + f(2.5,1)\,\Delta A+f(2.5,3)\,\Delta A\\ &= 1(2)+(-8)(2)+5(2)+(-1)(2)=-6\end{aligned}$$

(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(x_i,y_j)\,\Delta A \\ &= f(1.5,1)\,\Delta A + f(1.5,2)\,\Delta A + f(1.5,3)\,\Delta A + f(1.5,4)\,\Delta A \\ &\quad + f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A + f(2,4)\,\Delta A \\ &\quad + f(2.5,1)\,\Delta A + f(2.5,2)\,\Delta A + f(2.5,3)\,\Delta A + f(2.5,4)\,\Delta A \\ &\quad + f(3,1)\,\Delta A + f(3,2)\,\Delta A + f(3,3)\,\Delta A + f(3,4)\,\Delta A \\ &= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) \\ &\quad + 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) \\ &= -3.5\end{aligned}$$

7. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

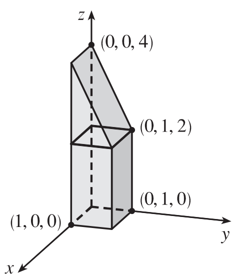
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \approx 4(27 + 4 + 14 + 17) = 248$$

(b) $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2,2]\times[1,6]$. S is a rectangular solid, thus $\iint_R 3\,dA = 4\cdot 5\cdot 3 = 60$.

13. $z = f(x,y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1]\times[0,1]\times[0,4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y)\,dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = \sqrt{1 + xe^{-y}}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                         y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	1.141606
4	1.143191
16	1.143535
64	1.143617
256	1.143637
1024	1.143642

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$.

But $f\left(x_{ij}^*, y_{ij}^*\right) = k$ always and $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta A =$ area of $R = (b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = k\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta A = k(b-a)(d-c)$ and so

$$\iint_R k\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A = \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c).$$

16.2 Iterated Integrals ET 15.2

1. $\int_0^5 12x^2y^3\,dx = \left[12\,\dfrac{x^3}{3}\,y^3\right]_{x=0}^{x=5} = 4x^3y^3\Big]_{x=0}^{x=5} = 4(5)^3\,y^3 - 4(0)^3\,y^3 = 500y^3,$

$\int_0^1 12x^2y^3\,dy = \left[12x^2\,\dfrac{y^4}{4}\right]_{y=0}^{y=1} = 3x^2y^4\Big]_{y=0}^{y=1} = 3x^2(1)^4 - 3x^2(0)^4 = 3x^2$

3. $\int_1^3\int_0^1(1+4xy)\,dx\,dy = \int_1^3\left[x+2x^2y\right]_{x=0}^{x=1}dy = \int_1^3(1+2y)\,dy = \left[y+y^2\right]_1^3 = (3+9)-(1+1) = 10$

5. $\int_0^2\int_0^{\pi/2} x\sin y\,dy\,dx = \int_0^2 x\,dx\int_0^{\pi/2}\sin y\,dy$ [as in Example 5] $= \left[\dfrac{x^2}{2}\right]_0^2\left[-\cos y\right]_0^{\pi/2} = (2-0)(0+1) = 2$

7. $\int_0^2\int_0^1(2x+y)^8dx\,dy = \displaystyle\int_0^2\left[\frac{1}{2}\,\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1}dy$ [substitute $u = 2x+y \;\Rightarrow\; dx = \frac{1}{2}\,du$]

$$= \frac{1}{18}\int_0^2\left[(2+y)^9-(0+y)^9\right]dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10}-\frac{y^{10}}{10}\right]_0^2$$

$$= \tfrac{1}{180}\left[(4^{10}-2^{10})-(2^{10}-0^{10})\right] = \tfrac{1{,}046{,}528}{180} = \tfrac{261{,}632}{45}$$

9. $\displaystyle\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^4\left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}\,y^2\right]_{y=1}^{y=2}dx = \int_1^4\left(x\ln 2+\frac{3}{2x}\right)dx = \left[\tfrac{1}{2}x^2\ln 2+\tfrac{3}{2}\ln|x|\right]_1^4$

$$= 8\ln 2+\tfrac{3}{2}\ln 4-\tfrac{1}{2}\ln 2 = \tfrac{15}{2}\ln 2+3\ln 4^{1/2} = \tfrac{21}{2}\ln 2$$

11. $\int_0^1\int_0^1(u-v)^5\,du\,dv = \int_0^1\left[\frac{1}{6}(u-v)^6\right]_{u=0}^{u=1}dv = \frac{1}{6}\int_0^1\left[(1-v)^6-(0-v)^6\right]dv$
$= \frac{1}{6}\int_0^1\left[(1-v)^6-v^6\right]dv = \frac{1}{6}\left[-\frac{1}{7}(1-v)^7-\frac{1}{7}v^7\right]_0^1$
$= -\frac{1}{42}\left[(0+1)-(1+0)\right]=0$

13. $\int_0^2\int_0^\pi r\sin^2\theta\,d\theta\,dr = \int_0^2 r\,dr\int_0^\pi \sin^2\theta\,d\theta$ [as in Example 5] $= \int_0^2 r\,dr\int_0^\pi \frac{1}{2}(1-\cos 2\theta)\,d\theta$
$= \left[\frac{1}{2}r^2\right]_0^2\cdot\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^\pi = (2-0)\cdot\frac{1}{2}\left[\left(\pi-\frac{1}{2}\sin 2\pi\right)-\left(0-\frac{1}{2}\sin 0\right)\right]$
$= 2\cdot\frac{1}{2}\left[(\pi-0)-(0-0)\right]=\pi$

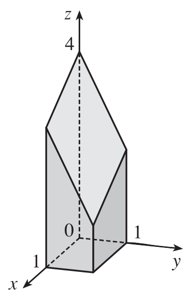
15. $\iint_R(6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1(6x^2y^3-5y^4)\,dy\,dx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx = \int_0^3\left(\frac{3}{2}x^2-1\right)dx$
$= \left[\frac{1}{2}x^3-x\right]_0^3 = \frac{27}{2}-3=\frac{21}{2}$

17. $\displaystyle\iint_R\frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3\frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1\frac{x}{x^2+1}\,dx\int_{-3}^3 y^2\,dy = \left[\frac{1}{2}\ln(x^2+1)\right]_0^1\left[\frac{1}{3}y^3\right]_{-3}^3$
$= \frac{1}{2}(\ln 2-\ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$

19. $\int_0^{\pi/6}\int_0^{\pi/3}x\sin(x+y)\,dy\,dx$
$= \int_0^{\pi/6}\left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}dx = \int_0^{\pi/6}\left[x\cos x - x\cos\left(x+\frac{\pi}{3}\right)\right]dx$
$= x\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\left[\sin x-\sin\left(x+\frac{\pi}{3}\right)\right]dx$ [by integrating by parts separately for each term]
$= \frac{\pi}{6}\left[\frac{1}{2}-1\right]-\left[-\cos x+\cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12}-\left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2}-\frac{\pi}{12}$

21. $\iint_R xye^{x^2y}\,dA = \int_0^2\int_0^1 xye^{x^2y}\,dx\,dy = \int_0^2\left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1}dy = \frac{1}{2}\int_0^2(e^y-1)\,dy = \frac{1}{2}\left[e^y-y\right]_0^2$
$= \frac{1}{2}\left[(e^2-2)-(1-0)\right] = \frac{1}{2}(e^2-3)$

23. $z=f(x,y)=4-x-2y\geq 0$ for $0\leq x\leq 1$ and $0\leq y\leq 1$. So the solid is the region in the first octant which lies below the plane $z=4-x-2y$ and above $[0,1]\times[0,1]$.



25. $V = \iint_R(12-3x-2y)\,dA = \int_{-2}^3\int_0^1(12-3x-2y)\,dx\,dy = \int_{-2}^3\left[12x-\frac{3}{2}x^2-2xy\right]_{x=0}^{x=1}dy$
$= \int_{-2}^3\left(\frac{21}{2}-2y\right)dy = \left[\frac{21}{2}y-y^2\right]_{-2}^3 = \frac{95}{2}$

27. $V = \int_{-2}^2\int_{-1}^1\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy = 4\int_0^2\int_0^1\left(1-\frac{1}{4}x^2-\frac{1}{9}y^2\right)dx\,dy$
$= 4\int_0^2\left[x-\frac{1}{12}x^3-\frac{1}{9}y^2x\right]_{x=0}^{x=1}dy = 4\int_0^2\left(\frac{11}{12}-\frac{1}{9}y^2\right)dy = 4\left[\frac{11}{12}y-\frac{1}{27}y^3\right]_0^2 = 4\cdot\frac{83}{54}=\frac{166}{27}$

29. Here we need the volume of the solid lying under the surface $z = x\sec^2 y$ and above the rectangle $R = [0,2] \times [0, \pi/4]$ in the xy-plane.

$$V = \int_0^2 \int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/4} \sec^2 y\,dy = \left[\tfrac{1}{2}x^2\right]_0^2 \left[\tan y\right]_0^{\pi/4}$$
$$= (2-0)(\tan\tfrac{\pi}{4} - \tan 0) = 2(1-0) = 2$$

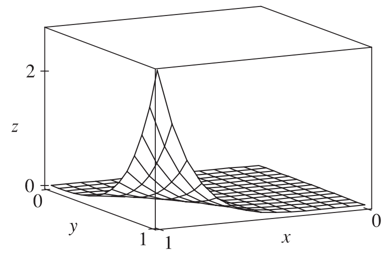
31. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1,1] \times [0,4]$ and the solid that lies under $z = 1$ over R.

$$V = \int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\,dx\,dy - \int_0^4 \int_{-1}^1 (1)\,dx\,dy = \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy$$
$$= \int_0^4 \left[\left(2 + \tfrac{1}{3} + (y-2)^2\right) - \left(-2 - \tfrac{1}{3} - (y-2)^2\right)\right] dy - [x]_{-1}^1\,[y]_0^4$$
$$= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right] dy - [1-(-1)][4-0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4)$$
$$= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3}$$

33. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command

```
Integrate[f,{x,0,1},{y,0,1}]
```

We find that $\iint_R x^5 y^3 e^{xy}\,dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3D` (in Mathematica) to graph the function.



35. R is the rectangle $[-1,1] \times [0,5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \tfrac{1}{10}\int_0^5 \int_{-1}^1 x^2 y\,dx\,dy = \tfrac{1}{10}\int_0^5 \left[\tfrac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10}\int_0^5 \tfrac{2}{3}y\,dy = \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}.$$

37. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

16.3 Double Integrals over General Regions ET 15.3

1. $\int_0^4 \int_0^{\sqrt{y}} xy^2\,dx\,dy = \int_0^4 \left[\frac{1}{2}x^2 y^2\right]_{x=0}^{x=\sqrt{y}} dy = \int_0^4 \frac{1}{2}y^2[(\sqrt{y})^2 - 0^2]dy = \frac{1}{2}\int_0^4 y^3\,dy = \frac{1}{2}\left[\frac{1}{4}y^4\right]_0^4 = \frac{1}{2}(64-0) = 32$

3. $\int_0^1 \int_{x^2}^x (1+2y)dy\,dx = \int_0^1 \left[y + y^2\right]_{y=x^2}^{y=x} dx = \int_0^1 \left[x + x^2 - x^2 - (x^2)^2\right] dx$

$$= \int_0^1 (x - x^4)dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{5} - 0 + 0 = \tfrac{3}{10}$$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta}d\theta = e^{\sin\theta}\Big]_0^{\pi/2} = e^{\sin(\pi/2)} - e^0 = e - 1$

7. $\iint_D y^2\,dA = \int_{-1}^{1}\int_{-y-2}^{y} y^2\,dx\,dy = \int_{-1}^{1}\left[xy^2\right]_{x=-y-2}^{x=y}dy = \int_{-1}^{1} y^2\,[y-(-y-2)]\,dy$

$= \int_{-1}^{1}(2y^3+2y^2)dy = \left[\frac{1}{2}y^4+\frac{2}{3}y^3\right]_{-1}^{1} = \frac{1}{2}+\frac{2}{3}-\frac{1}{2}+\frac{2}{3} = \frac{4}{3}$

9. $\iint_D x\,dA = \int_0^{\pi}\int_0^{\sin x} x\,dy\,dx = \int_0^{\pi}\left[xy\right]_{y=0}^{y=\sin x}dx = \int_0^{\pi} x\sin x\,dx$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{with } u=x,\, dv=\sin x\,dx\end{array}\right]$

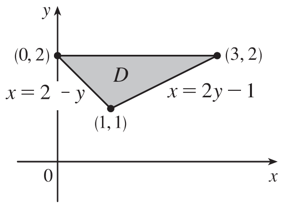
$= \left[-x\cos x+\sin x\right]_0^{\pi} = -\pi\cos\pi+\sin\pi+0-\sin 0 = \pi$

11. $\iint_D y^2e^{xy}\,dA = \int_0^4\int_0^y y^2e^{xy}\,dx\,dy = \int_0^4\left[ye^{xy}\right]_{x=0}^{x=y}dy = \int_0^4\left(ye^{y^2}-y\right)dy$

$= \left[\frac{1}{2}e^{y^2}-\frac{1}{2}y^2\right]_0^4 = \frac{1}{2}e^{16}-8-\frac{1}{2}+0 = \frac{1}{2}e^{16}-\frac{17}{2}$

13. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1\left[x\sin y\right]_{y=0}^{y=x^2}dx = \int_0^1 x\sin x^2\,dx = -\frac{1}{2}\cos x^2\Big]_0^1 = \frac{1}{2}(1-\cos 1)$

15.

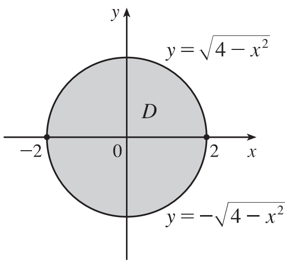


$$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2\left[xy^3\right]_{x=2-y}^{x=2y-1}dy = \int_1^2\left[(2y-1)-(2-y)\right]y^3\,dy$$

$$= \int_1^2(3y^4-3y^3)\,dy = \left[\frac{3}{5}y^5-\frac{3}{4}y^4\right]_1^2$$

$$= \frac{96}{5}-12-\frac{3}{5}+\frac{3}{4} = \frac{147}{20}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

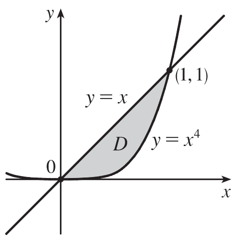
$$= \int_{-2}^{2}\left[2xy-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$

$$= \int_{-2}^{2}\left[2x\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\frac{1}{2}\left(4-x^2\right)\right]dx$$

$$= \int_{-2}^{2}4x\sqrt{4-x^2}\,dx = -\frac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2} = 0$$

[Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx = 0$.]

19.

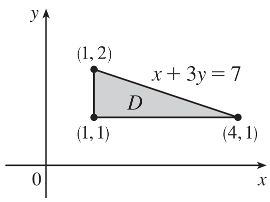


$$V = \int_0^1\int_{x^4}^{x}(x+2y)\,dy\,dx$$

$$= \int_0^1\left[xy+y^2\right]_{y=x^4}^{y=x}dx = \int_0^1(2x^2-x^5-x^8)\,dx$$

$$= \left[\frac{2}{3}x^3-\frac{1}{6}x^6-\frac{1}{9}x^9\right]_0^1 = \frac{2}{3}-\frac{1}{6}-\frac{1}{9} = \frac{7}{18}$$

21.

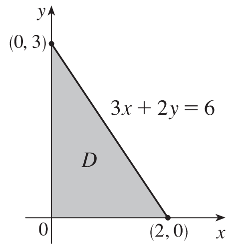


$$V = \int_1^2\int_1^{7-3y} xy\,dx\,dy = \int_1^2\left[\frac{1}{2}x^2y\right]_{x=1}^{x=7-3y}dy$$

$$= \frac{1}{2}\int_1^2(48y-42y^2+9y^3)\,dy$$

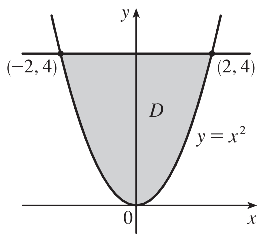
$$= \frac{1}{2}\left[24y^2-14y^3+\frac{9}{4}y^4\right]_1^2 = \frac{31}{8}$$

23.



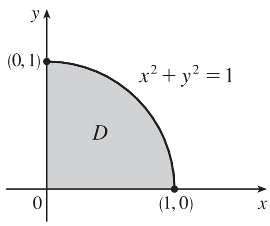
$$\begin{aligned} V &= \int_0^2 \int_0^{3-\frac{3}{2}x} (6-3x-2y)\,dy\,dx \\ &= \int_0^2 \left[6y-3xy-y^2\right]_{y=0}^{y=3-\frac{3}{2}x} dx \\ &= \int_0^2 \left[6(3-\tfrac{3}{2}x)-3x(3-\tfrac{3}{2}x)-(3-\tfrac{3}{2}x)^2\right] dx \\ &= \int_0^2 \left(\tfrac{9}{4}x^2-9x+9\right) dx = \left[\tfrac{3}{4}x^3-\tfrac{9}{2}x^2+9x\right]_0^2 = 6-0=6 \end{aligned}$$

25.



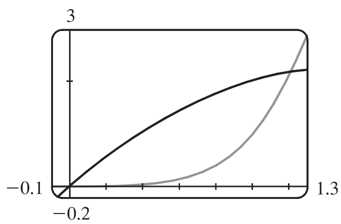
$$\begin{aligned} V &= \int_{-2}^2 \int_{x^2}^4 x^2\,dy\,dx \\ &= \int_{-2}^2 x^2 \left[\,y\,\right]_{y=x^2}^{y=4} dx = \int_{-2}^2 (4x^2-x^4)\,dx \\ &= \left[\tfrac{4}{3}x^3-\tfrac{1}{5}x^5\right]_{-2}^2 = \tfrac{32}{3}-\tfrac{32}{5}+\tfrac{32}{3}-\tfrac{32}{5}=\tfrac{128}{15} \end{aligned}$$

27.



$$\begin{aligned} V &= \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{1-x^2}{2}\,dx = \tfrac{1}{2}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} \end{aligned}$$

29.



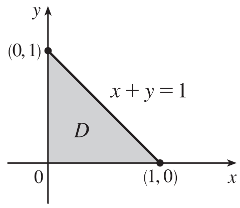
From the graph, it appears that the two curves intersect at $x=0$ and at $x\approx 1.213$. Thus the desired integral is

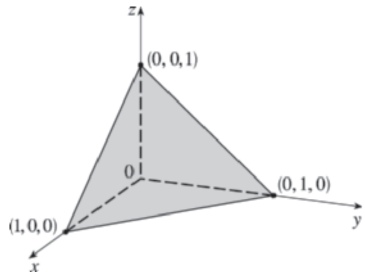
$$\begin{aligned} \iint_D x\,dA &\approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\Big[\,xy\,\Big]_{y=x^4}^{y=3x-x^2} dx \\ &= \int_0^{1.213}(3x^2-x^3-x^5)\,dx = \left[x^3-\tfrac{1}{4}x^4-\tfrac{1}{6}x^6\right]_0^{1.213} \\ &\approx 0.713 \end{aligned}$$

31. The two bounding curves $y=1-x^2$ and $y=x^2-1$ intersect at $(\pm1,0)$ with $1-x^2\ge x^2-1$ on $[-1,1]$. Within this region, the plane $z=2x+2y+10$ is above the plane $z=2-x-y$, so

$$\begin{aligned} V &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10)\,dy\,dx - \int_{-1}^1\int_{x^2-1}^{1-x^2}(2-x-y)\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(2x+2y+10-(2-x-y))\,dy\,dx \\ &= \int_{-1}^1\int_{x^2-1}^{1-x^2}(3x+3y+8)\,dy\,dx = \int_{-1}^1\left[3xy+\tfrac{3}{2}y^2+8y\right]_{y=x^2-1}^{y=1-x^2} dx \\ &= \int_{-1}^1\left[3x(1-x^2)+\tfrac{3}{2}(1-x^2)^2+8(1-x^2)-3x(x^2-1)-\tfrac{3}{2}(x^2-1)^2-8(x^2-1)\right] dx \\ &= \int_{-1}^1(-6x^3-16x^2+6x+16)\,dx = \left[-\tfrac{3}{2}x^4-\tfrac{16}{3}x^3+3x^2+16x\right]_{-1}^1 \\ &= -\tfrac{3}{2}-\tfrac{16}{3}+3+16+\tfrac{3}{2}-\tfrac{16}{3}-3+16=\tfrac{64}{3} \end{aligned}$$

33. The solid lies below the plane $z = 1 - x - y$ or $x + y + z = 1$ and above the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$ in the xy-plane. The solid is a tetrahedron.





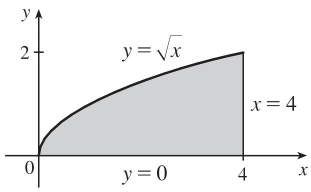
35. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} z\, dy\, dx = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + x y^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

37. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

39.

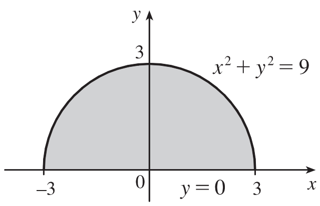


Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 4\} = \{(x, y) \mid y^2 \le x \le 4, 0 \le y \le 2\}$

we have $\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^2 \int_{y^2}^4 f(x, y)\, dx\, dy$.

41.



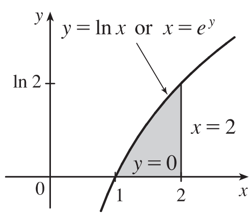
Because the region of integration is

$$\begin{aligned} D &= \left\{(x, y) \mid -\sqrt{9 - y^2} \le x \le \sqrt{9 - y^2}, 0 \le y \le 3\right\} \\ &= \left\{(x, y) \mid 0 \le y \le \sqrt{9 - x^2}, -3 \le x \le 3\right\} \end{aligned}$$

we have

$$\begin{aligned} \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)\, dx\, dy &= \iint_D f(x, y)\, dA \\ &= \int_{-3}^{3} \int_0^{\sqrt{9-x^2}} f(x, y)\, dy\, dx \end{aligned}$$

43.



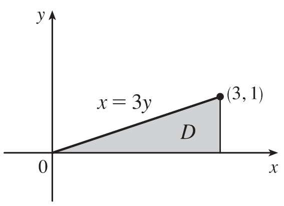
Because the region of integration is

$D = \{(x, y) \mid 0 \le y \le \ln x, 1 \le x \le 2\} = \{(x, y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$

we have

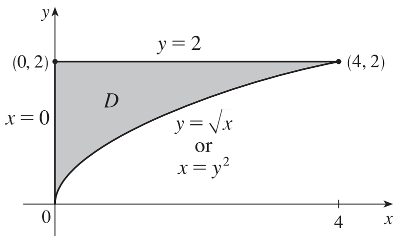
$$\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^{\ln 2} \int_{e^y}^{2} f(x, y)\, dx\, dy$$

45.



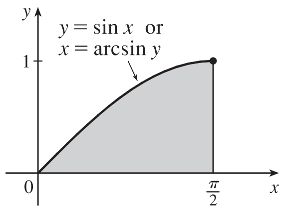
$$\begin{aligned} \int_0^1 \int_{3y}^{3} e^{x^2}\, dx\, dy &= \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx \\ &= \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\, dx = \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6} \end{aligned}$$

47.



$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$
$$= \int_0^2 \frac{1}{y^3+1}\left[x\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\,dy$$
$$= \tfrac{1}{3}\ln\left|y^3+1\right|\Big]_0^2 = \tfrac{1}{3}(\ln 9 - \ln 1) = \tfrac{1}{3}\ln 9$$

49.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\left[y\right]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \left[\begin{array}{c}\text{Let } u = \cos x,\ du = -\sin x\,dx,\\ dx = du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\,\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

51. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0,\ x+1 \le y \le 1\}$

$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

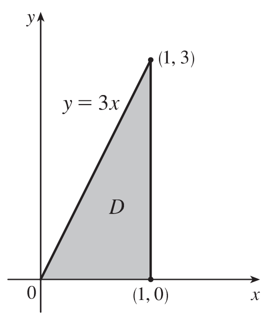
$$\iint_D x^2\,dA = \int_0^1\int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0\int_{x+1}^1 x^2\,dy\,dx + \int_0^1\int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0\int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1\int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

53. Here $Q = \left\{(x,y) \mid x^2+y^2 \le \frac{1}{4}, x \ge 0, y \ge 0\right\}$, and $0 \le (x^2+y^2)^2 \le \left(\frac{1}{4}\right)^2 \Rightarrow -\frac{1}{16} \le -(x^2+y^2)^2 \le 0$ so $e^{-1/16} \le e^{-(x^2+y^2)^2} \le e^0 = 1$ since e^t is an increasing function. We have $A(Q) = \frac{1}{4}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$, so by Property 11, $e^{-1/16}\,A(Q) \le \iint_Q e^{-(x^2+y^2)^2} dA \le 1 \cdot A(Q) \Rightarrow \frac{\pi}{16}e^{-1/16} \le \iint_Q e^{-(x^2+y^2)^2} dA \le \frac{\pi}{16}$ or we can say $0.1844 < \iint_Q e^{-(x^2+y^2)^2} dA < 0.1964$. (We have rounded the lower bound down and the upper bound up to preserve the inequalities.)

55. The average value of a function f of two variables defined on a rectangle R was defined in Section 16.1 [ET 15.1] as $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)dA$. Extending this definition to general regions D, we have $f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA$.

Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$ and

$$f_{\text{ave}} = \tfrac{1}{A(D)}\iint_D f(x,y)dA = \tfrac{1}{3/2}\int_0^1\int_0^{3x} xy\,dy\,dx$$
$$= \tfrac{2}{3}\int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \tfrac{1}{3}\int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \tfrac{3}{4}$$



57. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$ $m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$ $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

59. $\iint_D (x^2 \tan x + y^3 + 4)\, dA = \iint_D x^2 \tan x\, dA + \iint_D y^3\, dA + \iint_D 4\, dA$. But $x^2 \tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2 \tan x\, dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\, dA = 0$. Thus

$$\iint_D (x^2 \tan x + y^3 + 4)\, dA = 4 \iint_D dA = 4(\text{area of } D) = 4 \cdot \pi\left(\sqrt{2}\right)^2 = 8\pi$$

61. Since $\sqrt{1 - x^2 - y^2} \geq 0$, we can interpret $\iint_D \sqrt{1 - x^2 - y^2}\, dA$ as the volume of the solid that lies below the graph of $z = \sqrt{1 - x^2 - y^2}$ and above the region D in the xy-plane. $z = \sqrt{1 - x^2 - y^2}$ is equivalent to $x^2 + y^2 + z^2 = 1$, $z \geq 0$ which meets the xy-plane in the circle $x^2 + y^2 = 1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

16.4 Double Integrals in Polar Coordinates — ET 15.4

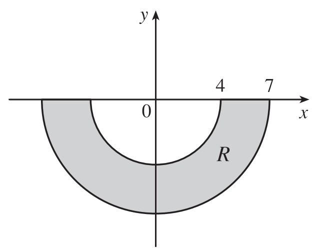
1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x, y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}x + \frac{1}{2}\right\}$.

Thus $\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\, dy\, dx$.

5. The integral $\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$, the lower half of a ring.



$$\int_\pi^{2\pi} \int_4^7 r\, dr\, d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\, dr\right)$$
$$= \left[\,\theta\,\right]_\pi^{2\pi} \left[\tfrac{1}{2}r^2\right]_4^7 = \pi \cdot \tfrac{1}{2}(49 - 16) = \tfrac{33\pi}{2}$$

7. The disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. Then

$\iint_D xy\, dA = \int_0^{2\pi} \int_0^3 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^{2\pi} \sin\theta\cos\theta\, d\theta\right)\left(\int_0^3 r^3\, dr\right) = \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^3 = 0.$

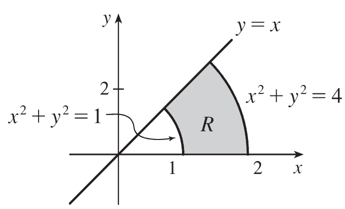
9. $\iint_R \cos(x^2 + y^2)\, dA = \int_0^\pi \int_0^3 \cos(r^2)\, r\, dr\, d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\, dr\right)$

$$= \left[\,\theta\,\right]_0^\pi \left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi \cdot \tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

11. $\iint_D e^{-x^2 - y^2}\, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2}\, r\, dr\, d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\, dr\right)$

$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus



$\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$.

Also, $\arctan(\tan\theta) = \theta$ for $0 \leq \theta \leq \pi/4$, so the integral becomes

$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4} \left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64}\pi^2.$

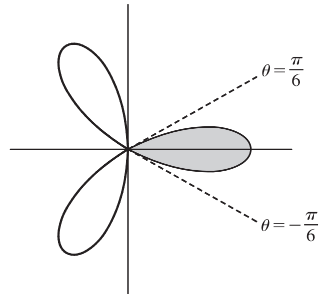
15. One loop is given by the region

$D = \{(r,\theta)\,|\,-\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



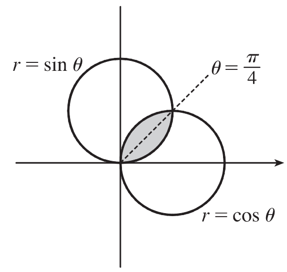
17. By symmetry,

$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$



19. $V = \iint_{x^2+y^2\le 4}\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r^2\,dr = \left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^2 = 2\pi\left(\tfrac{8}{3}\right) = \tfrac{16}{3}\pi$

21. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ intersects the plane $z = 2$ when $-x^2 - y^2 + 4 = 1$ or $x^2 + y^2 = 3$. So the solid region lies above the surface $z = \sqrt{1 + x^2 + y^2}$ and below the plane $z = 2$ for $x^2 + y^2 \le 3$, and its volume is

$$V = \iint\limits_{x^2+y^2\le 3}\left(2 - \sqrt{1+x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{\sqrt{3}}(2 - \sqrt{1+r^2})\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{3}}(2r - r\sqrt{1+r^2})dr = \left[\theta\right]_0^{2\pi}\left[r^2 - \tfrac{1}{3}(1+r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

23. By symmetry,

$$V = 2\iint\limits_{x^2+y^2\le a^2}\sqrt{a^2 - x^2 - y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2 - r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2 - r^2}\,dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2 - r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

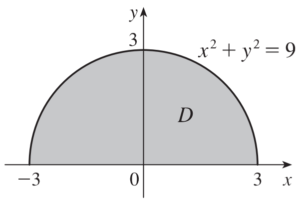
$$V = \iint\limits_{x^2+y^2\le 1/2}\left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2+y^2}\right)dA = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\left(\sqrt{1-r^2} - r\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{1/\sqrt{2}}\left(r\sqrt{1-r^2} - r^2\right)dr = \left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(1-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

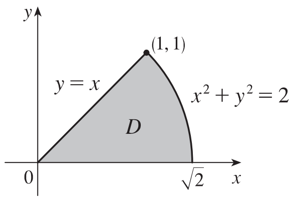
$$\begin{aligned} V &= \iint\limits_{x^2+y^2 \le 4} \left[\sqrt{64 - 4x^2 - 4y^2} - \left(-\sqrt{64 - 4x^2 - 4y^2}\right)\right] dA = \iint\limits_{x^2+y^2 \le 4} 2\sqrt{64 - 4x^2 - 4y^2}\, dA \\ &= 4 \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2}\, r\, dr\, d\theta = 4 \int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2}\, dr = 4\left[\,\theta\,\right]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^2 \\ &= 8\pi\left(-\tfrac{1}{3}\right)\left(12^{3/2} - 16^{2/3}\right) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right) \end{aligned}$$

29.



$$\begin{aligned} \int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2)\,dy\,dx &= \int_0^{\pi}\int_0^3 \sin\left(r^2\right) r\,dr\,d\theta \\ &= \int_0^{\pi} d\theta \int_0^3 r\sin\left(r^2\right)\,dr = [\theta]_0^{\pi}\left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3 \\ &= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1 - \cos 9) \end{aligned}$$

31.



$$\begin{aligned} \int_0^{\pi/4}\int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta &= \int_0^{\pi/4} (\cos\theta + \sin\theta)\,d\theta \int_0^{\sqrt{2}} r^2\,dr \\ &= [\sin\theta - \cos\theta]_0^{\pi/4}\left[\tfrac{1}{3}r^3\right]_0^{\sqrt{2}} \\ &= \left[\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} - 0 + 1\right]\cdot\tfrac{1}{3}\left(2\sqrt{2} - 0\right) = \tfrac{2\sqrt{2}}{3} \end{aligned}$$

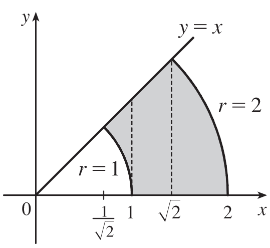
33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x, y)$ to be the depth of the water at (x, y), then the volume of water in the pool is the volume of the solid that lies above $D = \left\{(x, y) \mid x^2 + y^2 \le 400\right\}$ and below the graph of $f(x, y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0, -20) = 2$ to $f(0, 20) = 7$. The trace in the yz-plane is a line segment from $(0, -20, 2)$ to $(0, 20, 7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \quad\Rightarrow\quad z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x, y)$ is independent of x, $f(x, y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x, y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r, \theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\begin{aligned} \int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta &= \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta \\ &= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi \end{aligned}$$

Thus the pool contains $1800\pi \approx 5655\ \text{ft}^3$ of water.

35.

$$\begin{aligned} &\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx \\ &\quad = \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta \\ &\quad = \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16} \end{aligned}$$



37. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\begin{aligned}\int_0^\infty x^2e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}\\ &= \tfrac{1}{4}\int_{-\infty}^\infty e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}\\ &= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}\end{aligned}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \;\Rightarrow\; dx = 2u\,du \;\Rightarrow$

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du = 2\left(\tfrac{1}{4}\sqrt{\pi}\right) \quad \text{[by part(a)]} = \tfrac{1}{2}\sqrt{\pi}.$$

16.5 Applications of Double Integrals ET 15.5

1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3\int_0^2 (2xy + y^2)\,dy\,dx = \int_1^3 \left[xy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=2} dx$

$= \int_1^3 \left(4x + \frac{8}{3}\right) dx = \left[2x^2 + \frac{8}{3}x\right]_1^3 = 16 + \frac{16}{3} = \frac{64}{3}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2\int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx\,\int_{-1}^1 y^2\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = 2\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx = \frac{3}{4}\int_0^2 x^2\,dx\,\int_{-1}^1 y^2\,dy = \frac{3}{4}\left[\frac{1}{3}x^3\right]_0^2\left[\frac{1}{3}y^3\right]_{-1}^1 = \frac{3}{4}\cdot\frac{8}{3}\cdot\frac{2}{3} = \frac{4}{3}$,

$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx = \frac{3}{4}\int_0^2 x\,dx\,\int_{-1}^1 y^3\,dy = \frac{3}{4}\left[\frac{1}{2}x^2\right]_0^2\left[\frac{1}{4}y^4\right]_{-1}^1 = \frac{3}{4}\cdot 2\cdot 0 = 0.$

Hence, $(\overline{x},\overline{y}) = \left(\frac{4}{3},0\right)$.

5. $m = \int_0^2\int_{x/2}^{3-x}(x+y)\,dy\,dx = \int_0^2\left[xy + \frac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left[x\left(3 - \frac{3}{2}x\right) + \frac{1}{2}(3-x)^2 - \frac{1}{8}x^2\right] dx$

$= \int_0^2\left(-\frac{9}{8}x^2 + \frac{9}{2}\right) dx = \left[-\frac{9}{8}\left(\frac{1}{3}x^3\right) + \frac{9}{2}x\right]_0^2 = 6,$

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2+xy)\,dy\,dx = \int_0^2\left[x^2y + \frac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(\frac{9}{2}x - \frac{9}{8}x^3\right) dx = \frac{9}{2}$,

$M_x = \int_0^2\int_{x/2}^{3-y}(xy+y^2)\,dy\,dx = \int_0^2\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(9 - \frac{9}{2}x\right) dx = 9.$

Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{3}{4},\dfrac{3}{2}\right)$.

7. $m = \int_0^1\int_0^{e^x} y\,dy\,dx = \int_0^1\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \left.\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{4}(e^2-1)$,

$M_y = \int_0^1\int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2+1)$,

$M_x = \int_0^1\int_0^{e^x} y^2\,dy\,dx = \int_0^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x} dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3-1).$

Hence $m = \frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)},\dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)},\dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

9. Note that $\sin(\pi x/L) \geq 0$ for $0 \leq x \leq L$.

$m = \int_0^L \int_0^{\sin(\pi x/L)} y\,dy\,dx = \int_0^L \frac{1}{2}\sin^2(\pi x/L)\,dx = \frac{1}{2}\left[\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right]_0^L = \frac{1}{4}L,$

$$M_y = \int_0^L \int_0^{\sin(\pi x/L)} x \cdot y\,dy\,dx = \frac{1}{2}\int_0^L x\sin^2(\pi x/L)\,dx \quad \left[\begin{array}{c}\text{integrate by parts with}\\ u = x,\, dv = \sin^2(\pi x/L)\,dx\end{array}\right]$$

$$= \frac{1}{2}\cdot x\left(\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right)\Big]_0^L - \frac{1}{2}\int_0^L \left[\frac{1}{2}x - \frac{L}{4\pi}\sin(2\pi x/L)\right]dx$$

$$= \frac{1}{4}L^2 - \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{L^2}{4\pi^2}\cos(2\pi x/L)\right]_0^L = \frac{1}{4}L^2 - \frac{1}{2}\left(\frac{1}{4}L^2 + \frac{L^2}{4\pi^2} - \frac{L^2}{4\pi^2}\right) = \frac{1}{8}L^2,$$

$$M_x = \int_0^L \int_0^{\sin(\pi x/L)} y \cdot y\,dy\,dx = \int_0^L \frac{1}{3}\sin^3(\pi x/L)\,dx = \frac{1}{3}\int_0^L \left[1 - \cos^2(\pi x/L)\right]\sin(\pi x/L)\,dx$$

$$[\text{substitute } u = \cos(\pi x/L)] \quad \Rightarrow \quad du = -\tfrac{\pi}{L}\sin(\pi x/L)]$$

$$= \frac{1}{3}\left(-\frac{L}{\pi}\right)\left[\cos(\pi x/L) - \frac{1}{3}\cos^3(\pi x/L)\right]_0^L = -\frac{L}{3\pi}\left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right) = \frac{4}{9\pi}L.$$

Hence $m = \dfrac{L}{4}$, $(\overline{x}, \overline{y}) = \left(\dfrac{L^2/8}{L/4}, \dfrac{4L/(9\pi)}{L/4}\right) = \left(\dfrac{L}{2}, \dfrac{16}{9\pi}\right)$.

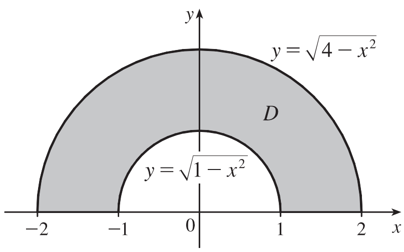
11. $\rho(x,y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,dr\,d\theta = \frac{1}{3}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{3}k\big[-\cos\theta\big]_0^{\pi/2} = \frac{1}{3}k,$

$M_y = \int_0^{\pi/2}\int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{1}{8}k\big[-\cos 2\theta\big]_0^{\pi/2} = \frac{1}{8}k,$

$M_x = \int_0^{\pi/2}\int_0^1 kr^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}k\big[\theta + \sin 2\theta\big]_0^{\pi/2} = \frac{\pi}{16}k.$

Hence $(\overline{x}, \overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

13.



$\rho(x,y) = k\sqrt{x^2 + y^2} = kr,$

$m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_1^2 kr \cdot r\,dr\,d\theta$

$\quad = k\int_0^\pi d\theta \int_1^2 r^2\,dr = k(\pi)\left[\frac{1}{3}r^3\right]_1^2 = \frac{7}{3}\pi k,$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\cos\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \cos\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[\sin\theta\right]_0^\pi \left[\frac{1}{4}r^4\right]_1^2 = k(0)\left(\frac{15}{4}\right) = 0$$

[this is to be expected as the region and density function are symmetric about the y-axis]

$$M_x = \iint_D y\rho(x,y)\,dA = \int_0^\pi \int_1^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = k\int_0^\pi \sin\theta\,d\theta \int_1^2 r^3\,dr$$

$$= k\left[-\cos\theta\right]_0^\pi \left[\frac{1}{4}r^4\right]_1^2 = k(1+1)\left(\frac{15}{4}\right) = \frac{15}{2}k.$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.

15. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2 + y^2)$. Then

$m = \int_0^a \int_0^{a-x} k(x^2 + y^2)\,dy\,dx = k\int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right]dx = k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 - \frac{1}{12}(a-x)^4\right]_0^a = \frac{1}{6}ka^4.$

By symmetry,

$$M_y = M_x = \int_0^a \int_0^{a-x} ky(x^2 + y^2)\,dy\,dx = k\int_0^a \left[\frac{1}{2}(a-x)^2x^2 + \frac{1}{4}(a-x)^4\right]dx$$

$$= k\left[\frac{1}{6}a^2x^3 - \frac{1}{4}ax^4 + \frac{1}{10}x^5 - \frac{1}{20}(a-x)^5\right]_0^a = \frac{1}{15}ka^5$$

Hence $(\overline{x}, \overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

17. $I_x = \iint_D y^2 \rho(x,y)dA = \int_0^1 \int_0^{e^x} y^2 \cdot y\, dy\, dx = \int_0^1 \left[\frac{1}{4}y^4\right]_{y=0}^{y=e^x} dx = \frac{1}{4}\int_0^1 e^{4x}\, dx = \frac{1}{4}\left[\frac{1}{4}e^{4x}\right]_0^1 = \frac{1}{16}(e^4 - 1),$

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^1 \int_0^{e^x} x^2 y\, dy\, dx = \int_0^1 x^2 \left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 x^2 e^{2x}\, dx$

$= \frac{1}{2}\left[\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x}\right]_0^1$ [integrate by parts twice] $= \frac{1}{8}(e^2 - 1),$

and $I_0 = I_x + I_y = \frac{1}{16}(e^4 - 1) + \frac{1}{8}(e^2 - 1) = \frac{1}{16}(e^4 + 2e^2 - 3).$

19. As in Exercise 15, we place the vertex opposite the hypotenuse at $(0,0)$ and the equal sides along the positive axes.

$I_x = \int_0^a \int_0^{a-x} y^2 k(x^2 + y^2)\, dy\, dx = k\int_0^a \int_0^{a-x} (x^2y^2 + y^4)\, dy\, dx = k\int_0^a \left[\frac{1}{3}x^2y^3 + \frac{1}{5}y^5\right]_{y=0}^{y=a-x} dx$

$= k\int_0^a \left[\frac{1}{3}x^2(a-x)^3 + \frac{1}{5}(a-x)^5\right] dx = k\left[\frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right) - \frac{1}{30}(a-x)^6\right]_0^a = \frac{7}{180}ka^6,$

$I_y = \int_0^a \int_0^{a-x} x^2 k(x^2 + y^2)\, dy\, dx = k\int_0^a \int_0^{a-x} (x^4 + x^2y^2)\, dy\, dx = k\int_0^a \left[x^4 y + \frac{1}{3}x^2y^3\right]_{y=0}^{y=a-x} dx$

$= k\int_0^a \left[x^4(a-x) + \frac{1}{3}x^2(a-x)^3\right] dx = k\left[\frac{1}{5}ax^5 - \frac{1}{6}x^6 + \frac{1}{3}\left(\frac{1}{3}a^3x^3 - \frac{3}{4}a^2x^4 + \frac{3}{5}ax^5 - \frac{1}{6}x^6\right)\right]_0^a = \frac{7}{180}ka^6,$

and $I_0 = I_x + I_y = \frac{7}{90}ka^6.$

21. Using a CAS, we find $m = \iint_D \rho(x,y)\, dA = \int_0^\pi \int_0^{\sin x} xy\, dy\, dx = \dfrac{\pi^2}{8}$. Then

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\, dA = \frac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} x^2 y\, dy\, dx = \frac{2\pi}{3} - \frac{1}{\pi} \text{ and}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\, dA = \frac{8}{\pi^2}\int_0^\pi \int_0^{\sin x} xy^2\, dy\, dx = \frac{16}{9\pi}, \text{ so } (\overline{x}, \overline{y}) = \left(\frac{2\pi}{3} - \frac{1}{\pi}, \frac{16}{9\pi}\right).$$

The moments of inertia are $I_x = \iint_D y^2 \rho(x,y)\, dA = \int_0^\pi \int_0^{\sin x} xy^3\, dy\, dx = \dfrac{3\pi^2}{64}$,

$I_y = \iint_D x^2 \rho(x,y)\, dA = \int_0^\pi \int_0^{\sin x} x^3 y\, dy\, dx = \dfrac{\pi^2}{16}(\pi^2 - 3)$, and $I_0 = I_x + I_y = \dfrac{\pi^2}{64}(4\pi^2 - 9).$

23. $I_x = \iint_D y^2 \rho(x,y)dA = \int_0^h \int_0^b \rho y^2\, dx\, dy = \rho\int_0^b dx \int_0^h y^2\, dy = \rho\left[x\right]_0^b \left[\frac{1}{3}y^3\right]_0^h = \rho b\left(\frac{1}{3}h^3\right) = \frac{1}{3}\rho bh^3,$

$I_y = \iint_D x^2 \rho(x,y)dA = \int_0^h \int_0^b \rho x^2\, dx\, dy = \rho\int_0^b x^2\, dx \int_0^h dy = \rho\left[\frac{1}{3}x^3\right]_0^b [y]_0^h = \frac{1}{3}\rho b^3 h,$

and $m = \rho\,(\text{area of rectangle}) = \rho bh$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{3}\rho b^3 h}{\rho bh} = \dfrac{b^2}{3} \quad\Rightarrow\quad \overline{\overline{x}} = \dfrac{b}{\sqrt{3}}$

and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{3}\rho bh^3}{\rho bh} = \dfrac{h^2}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \dfrac{h}{\sqrt{3}}.$

25. In polar coordinates, the region is $D = \left\{(r,\theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}\right\}$, so

$I_x = \iint_D y^2 \rho\, dA = \int_0^{\pi/2} \int_0^a \rho(r\sin\theta)^2\, r\, dr\, d\theta = \rho\int_0^{\pi/2} \sin^2 d\theta \int_0^a r^3\, dr$

$= \rho\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

$I_y = \iint_D x^2 \rho\, dA = \int_0^{\pi/2} \int_0^a \rho(r\cos\theta)^2\, r\, dr\, d\theta = \rho\int_0^{\pi/2} \cos^2 d\theta \int_0^a r^3\, dr$

$= \rho\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^a = \rho\left(\frac{\pi}{4}\right)\left(\frac{1}{4}a^4\right) = \frac{1}{16}\rho a^4 \pi,$

and $m = \rho \cdot A(D) = \rho \cdot \frac{1}{4}\pi a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \overline{\overline{y}}^2 = \dfrac{\frac{1}{16}\rho a^4 \pi}{\frac{1}{4}\rho a^2 \pi} = \dfrac{a^2}{4} \quad\Rightarrow\quad \overline{\overline{x}} = \overline{\overline{y}} = \dfrac{a}{2}.$

27. (a) $f(x,y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,1]\times[0,2]$, we can say

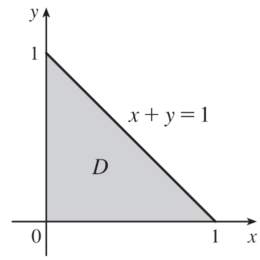
$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^2 Cx(1+y)\,dy\,dx \\ &= C\int_0^1 x\left[y+\tfrac{1}{2}y^2\right]_{y=0}^{y=2}dx = C\int_0^1 4x\,dx = C\left[2x^2\right]_0^1 = 2C\end{aligned}$$

Then $2C = 1 \quad\Rightarrow\quad C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y)\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}x(1+y)\,dy\,dx$

$$= \int_0^1 \tfrac{1}{2}x\left[y+\tfrac{1}{2}y^2\right]_{y=0}^{y=1}dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{3}{2}\right)dx = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X+Y \le 1) = P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus



$$\begin{aligned}P(X+Y\le 1) &= \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{2}x(1+y)\,dy\,dx \\ &= \int_0^1 \tfrac{1}{2}x\left[y+\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x}dx = \int_0^1 \tfrac{1}{2}x\left(\tfrac{1}{2}x^2-2x+\tfrac{3}{2}\right)dx \\ &= \tfrac{1}{4}\int_0^1\left(x^3-4x^2+3x\right)dx = \tfrac{1}{4}\left[\tfrac{x^4}{4}-4\tfrac{x^3}{3}+3\tfrac{x^2}{2}\right]_0^1 \\ &= \tfrac{5}{48}\approx 0.1042\end{aligned}$$

29. (a) $f(x,y)\ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\,dA &= \int_0^\infty\int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx = 0.1\int_0^\infty\int_0^\infty e^{-0.5x}e^{-0.2y}\,dy\,dx = 0.1\int_0^\infty e^{-0.5x}\,dx\int_0^\infty e^{-0.2y}\,dy \\ &= 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \\ &= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] = (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1\end{aligned}$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$\begin{aligned}P(Y\ge 1) &= \int_{-\infty}^{\infty}\int_1^\infty f(x,y)\,dy\,dx = \int_0^\infty\int_1^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx \\ &= 0.1\int_0^\infty e^{-0.5x}\,dx\int_1^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_1^t e^{-0.2y}\,dy \\ &= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t = 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right] \\ &(0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2}\approx 0.8187\end{aligned}$$

(ii) $P(X\le 2, Y\le 4) = \int_{-\infty}^{2}\int_{-\infty}^{4} f(x,y)\,dy\,dx = \int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$

$$\begin{aligned}&= 0.1\int_0^2 e^{-0.5x}\,dx\int_0^4 e^{-0.2y}\,dy = 0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4 \\ &= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1) \\ &= (e^{-1}-1)(e^{-0.8}-1) = 1+e^{-1.8}-e^{-0.8}-e^{-1}\approx 0.3481\end{aligned}$$

(c) The expected value of X is given by

$$\begin{aligned}\mu_1 &= \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^\infty\int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right]dy\,dx \\ &= 0.1\int_0^\infty xe^{-0.5x}\,dx\int_0^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}\,dx\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy\end{aligned}$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\,dx$ (or we can use Formula 96

in the Table of Integrals): $\int xe^{-0.5x}\,dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\,dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$.

Thus

$$\begin{aligned}\mu_1 &= 0.1 \lim_{t\to\infty} \left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_0^t \\ &= 0.1 \lim_{t\to\infty} (-2)\left[(t+2)e^{-0.5t} - 2\right] \lim_{t\to\infty} (-5)\left[e^{-0.2t} - 1\right] \\ &= 0.1(-2)\left(\lim_{t\to\infty} \frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}\end{aligned}$$

The expected value of Y is given by

$$\begin{aligned}\mu_2 &= \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^\infty \int_0^\infty y\left[0.1e^{-(0.5x+0.2y)}\right] dy\,dx \\ &= 0.1 \int_0^\infty e^{-0.5x}\,dx \int_0^\infty ye^{-0.2y}\,dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\,dy\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\,dy$ (or again we can use Formula 96 in the Table of Integrals) which gives $\int ye^{-0.2y}\,dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\,dy = -5(y+5)e^{-0.2y}$. Then

$$\begin{aligned}\mu_2 &= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t \\ &= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t} - 5\right]\right) \\ &= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}\end{aligned}$$

31. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$.

The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}$ and

$f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$f(x,y) = f_1(x)f_2(y) = \dfrac{1}{0.5\sqrt{2\pi}}\,e^{-(x-45)^2/0.5}\,\dfrac{1}{0.1\sqrt{2\pi}}\,e^{-(y-20)^2/0.02} = \frac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}$.

Then $P(40 \le X \le 50,\, 20 \le Y \le 25) = \int_{40}^{50}\int_{20}^{25} f(x,y)\,dy\,dx = \frac{10}{\pi}\int_{40}^{50}\int_{20}^{25} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx$.

Using a CAS or calculator to evaluate the integral, we get $P(40 \le X \le 50,\, 20 \le Y \le 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \le 2) = \iint_D \frac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

$$\iint_D \tfrac{10}{\pi}e^{-2(x-45)^2-50(y-20)^2}\,dA = \frac{10}{\pi}\int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20-\frac{1}{10}\sqrt{2-4(x-45)^2}}^{20+\frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2-50(y-20)^2}\,dy\,dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \le 2) \approx 0.632$.

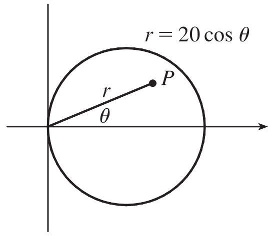
33. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's

center), the exposure of a person at A is

$$E = \iint_D kf(P,A)\,dA = k\iint_D \frac{20-d(P,A)}{20}\,dA = k\iint_D \left[1-\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{20}\right]dx\,dy$$

(b) If $A=(0,0)$, then

$$\begin{aligned} E &= k\iint_D \left[1-\frac{1}{20}\sqrt{x^2+y^2}\right]dx\,dy \\ &= k\int_0^{2\pi}\int_0^{10}\left(1-\frac{r}{20}\right)r\,dr\,d\theta = 2\pi k\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_0^{10} \\ &= 2\pi k\left(50-\tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k \end{aligned}$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$\begin{aligned} E &= k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1-\frac{r}{20}\right)r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}-\frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta}d\theta \\ &= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta-\tfrac{400}{3}\cos^3\theta\right)d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac12+\tfrac12\cos2\theta-\tfrac23\left(1-\sin^2\theta\right)\cos\theta\right]d\theta \\ &= 200k\left[\tfrac12\theta+\tfrac14\sin2\theta-\tfrac23\sin\theta+\tfrac23\cdot\tfrac13\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4}+0-\tfrac23+\tfrac29+\tfrac{\pi}{4}+0-\tfrac23+\tfrac29\right] \\ &= 200k\left(\tfrac{\pi}{2}-\tfrac89\right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

16.6 Triple Integrals ET 15.6

1. $$\begin{aligned}\iiint_B xyz^2\,dV &= \int_0^1\int_0^3\int_{-1}^2 xyz^2\,dy\,dz\,dx = \int_0^1\int_0^3\left[\tfrac12xy^2z^2\right]_{y=-1}^{y=2}dz\,dx = \int_0^1\int_0^3\tfrac32xz^2\,dz\,dx \\ &= \int_0^1\left[\tfrac12xz^3\right]_{z=0}^{z=3}dx = \int_0^1\tfrac{27}{2}x\,dx = \tfrac{27}{4}x^2\Big]_0^1 = \tfrac{27}{4}\end{aligned}$$

3. $$\begin{aligned}\int_0^1\int_0^z\int_0^{x+z}6xz\,dy\,dx\,dz &= \int_0^1\int_0^z\left[6xyz\right]_{y=0}^{y=x+z}dx\,dz = \int_0^1\int_0^z 6xz(x+z)\,dx\,dz \\ &= \int_0^1\left[2x^3z+3x^2z^2\right]_{x=0}^{x=z}dz = \int_0^1(2z^4+3z^4)\,dz = \int_0^1 5z^4\,dz = z^5\Big]_0^1 = 1\end{aligned}$$

5. $$\begin{aligned}\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}}ze^y\,dx\,dz\,dy &= \int_0^3\int_0^1\left[xze^y\right]_{x=0}^{x=\sqrt{1-z^2}}dz\,dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2}\,dz\,dy \\ &= \int_0^3\left[-\tfrac13(1-z^2)^{3/2}e^y\right]_{z=0}^{z=1}dy = \int_0^3\tfrac13e^y\,dy = \tfrac13e^y\Big]_0^3 = \tfrac13(e^3-1)\end{aligned}$$

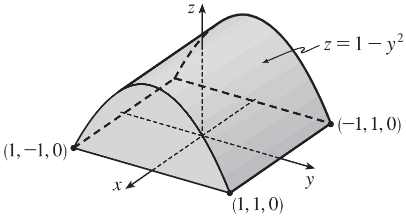
7. $$\begin{aligned}\int_0^{\pi/2}\int_0^y\int_0^x\cos(x+y+z)dz\,dx\,dy &= \int_0^{\pi/2}\int_0^y\left[\sin(x+y+z)\right]_{z=0}^{z=x}dx\,dy \\ &= \int_0^{\pi/2}\int_0^y\left[\sin(2x+y)-\sin(x+y)\right]dx\,dy \\ &= \int_0^{\pi/2}\left[-\tfrac12\cos(2x+y)+\cos(x+y)\right]_{x=0}^{x=y}dy \\ &= \int_0^{\pi/2}\left[-\tfrac12\cos3y+\cos2y+\tfrac12\cos y-\cos y\right]dy \\ &= \left[-\tfrac16\sin3y+\tfrac12\sin2y-\tfrac12\sin y\right]_0^{\pi/2} = \tfrac16-\tfrac12 = -\tfrac13\end{aligned}$$

9. $$\begin{aligned}\iiint_E 2x\,dV &= \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}\left[2xz\right]_{z=0}^{z=y}dx\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}2xy\,dx\,dy \\ &= \int_0^2\left[x^2y\right]_{x=0}^{x=\sqrt{4-y^2}}dy = \int_0^2(4-y^2)y\,dy = \left[2y^2-\tfrac14y^4\right]_0^2 = 4\end{aligned}$$

11. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx$$
$$= \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \tfrac{3}{4}x^4 + \tfrac{4}{7}x^{7/2}\right]_0^1 = \tfrac{65}{28}$$

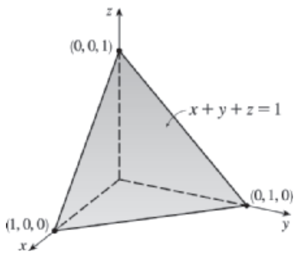
13.



E is the region below the parabolic cylinder $z = 1 - y^2$ and above the square $[-1,1] \times [-1,1]$ in the xy-plane.

$$\begin{aligned}
\iiint_E x^2 e^y\,dV &= \int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} x^2 e^y\,dz\,dy\,dx \\
&= \int_{-1}^1 \int_{-1}^1 x^2 e^y (1-y^2)\,dy\,dx \\
&= \int_{-1}^1 x^2\,dx \int_{-1}^1 (e^y - y^2 e^y)\,dy \\
&= \left[\tfrac{1}{3}x^3\right]_{-1}^1 \left[e^y - (y^2 - 2y + 2)e^y\right]_{-1}^1 \qquad \left[\begin{array}{l}\text{integrate by}\\ \text{parts twice}\end{array}\right] \\
&= \tfrac{1}{3}(2)[e - e - e^{-1} + 5e^{-1}] = \tfrac{8}{3e}
\end{aligned}$$

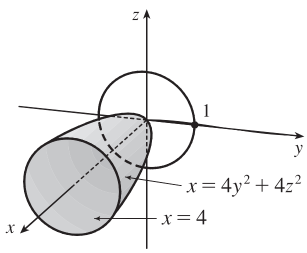
15.



Here $T = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$, so

$$\begin{aligned}
\iiint_T x^2\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} x^2(1-x-y)dy\,dx \\
&= \int_0^1 \int_0^{1-x} (x^2 - x^3 - x^2y)dy\,dx = \int_0^1 \left[x^2y - x^3y - \tfrac{1}{2}x^2y^2\right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left[x^2(1-x) - x^3(1-x) - \tfrac{1}{2}x^2(1-x)^2\right] dx \\
&= \int_0^1 \left(\tfrac{1}{2}x^4 - x^3 + \tfrac{1}{2}x^2\right) dx = \left[\tfrac{1}{10}x^5 - \tfrac{1}{4}x^4 + \tfrac{1}{6}x^3\right]_0^1 \\
&= \tfrac{1}{10} - \tfrac{1}{4} + \tfrac{1}{6} = \tfrac{1}{60}
\end{aligned}$$

17.



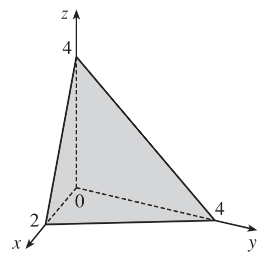
The projection E on the yz-plane is the disk $y^2 + z^2 \le 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\begin{aligned}
\iiint_E x\,dV &= \iint_D \left[\int_{4y^2+4z^2}^4 x\,dx\right] dA = \tfrac{1}{2}\iint_D \left[4^2 - (4y^2 + 4z^2)^2\right] dA \\
&= 8\int_0^{2\pi}\int_0^1 (1 - r^4)\,r\,dr\,d\theta = 8\int_0^{2\pi} d\theta \int_0^1 (r - r^5)\,dr \\
&= 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}
\end{aligned}$$

19. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad\Rightarrow\quad y = 4 - 2x$, so

$E = \{(x,y,z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and



$$\begin{aligned}
V &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx \\
&= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx \\
&= \int_0^2 \left[4(4-2x) - 2x(4-2x) - \tfrac{1}{2}(4-2x)^2\right] dx \\
&= \int_0^2 (2x^2 - 8x + 8)\,dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}
\end{aligned}$$

21. $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} dz\,dy\,dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-y-1)\,dy\,dx = \int_{-3}^{3} \left[4y - \tfrac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$

$= \int_{-3}^{3} 8\sqrt{9-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$= 8\left[\frac{9}{2}\sin^{-1}(1) - \frac{9}{2}\sin^{-1}(-1)\right] = 36\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 36\pi$

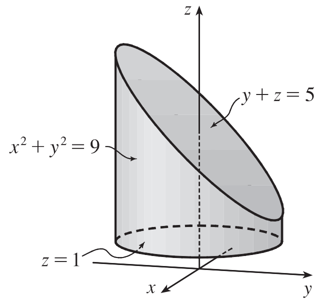
Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-y)\,dy\,dx = \int_{0}^{2\pi} \int_{0}^{3} (4 - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} (18 - 9\sin\theta)\,d\theta$$

$$= 18\theta + 9\cos\theta\Big]_0^{2\pi} = 36\pi$$



23. (a) The wedge can be described as the region

$D = \left\{(x,y,z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\right\}$

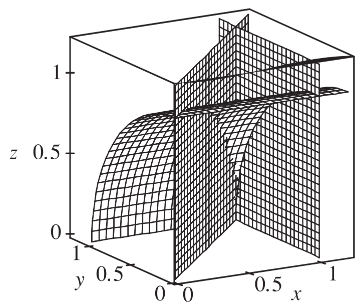
$= \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1-y^2}\right\}$

So the integral expressing the volume of the wedge is

$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx.$

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)



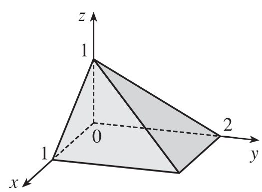
25. Here $f(x,y,z) = \dfrac{1}{\ln(1+x+y+z)}$ and $\Delta V = 2 \cdot 4 \cdot 2 = 16$, so the Midpoint Rule gives

$$\iiint_B f(x,y,z)\,dV \approx \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\overline{x}_i, \overline{y}_j, \overline{z}_k\right)\Delta V$$

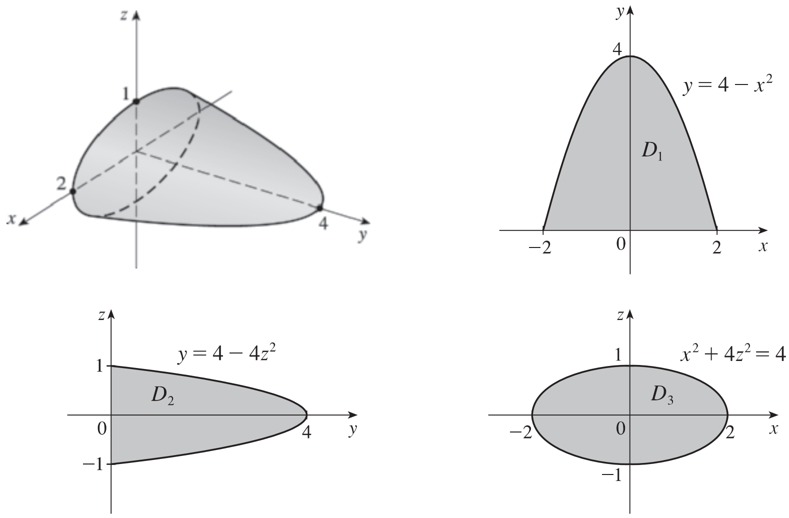
$$= 16[f(1,2,1) + f(1,2,3) + f(1,6,1) + f(1,6,3) + f(3,2,1) + f(3,2,3) + f(3,6,1) + f(3,6,3)]$$

$$= 16\left[\tfrac{1}{\ln 5} + \tfrac{1}{\ln 7} + \tfrac{1}{\ln 9} + \tfrac{1}{\ln 11} + \tfrac{1}{\ln 7} + \tfrac{1}{\ln 9} + \tfrac{1}{\ln 11} + \tfrac{1}{\ln 13}\right] \approx 60.533$$

27. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1-x, 0 \le y \le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1 - x$, $y = 2 - 2z$.



29.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\right\} = \left\{(x,y) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y}\,\right\}$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\right\} = \left\{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\right\}$$

$$D_3 = \left\{(x,z) \mid x^2 + 4z^2 \le 4\right\}$$

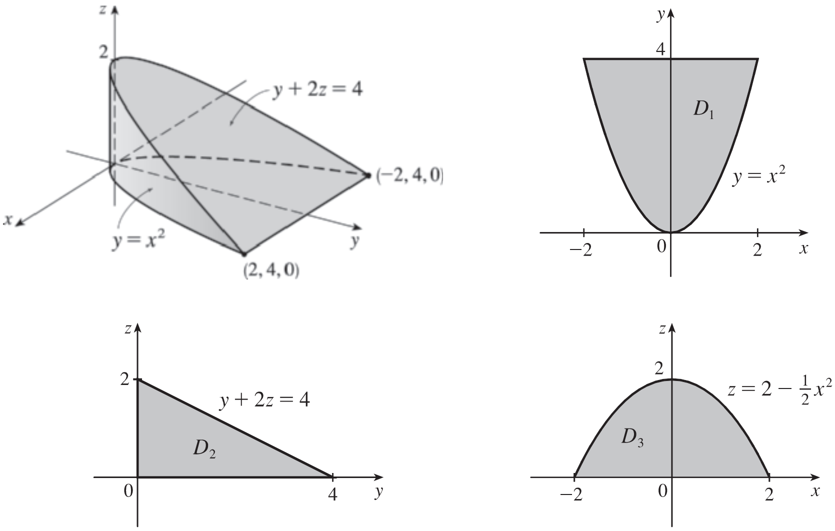
Therefore

$$\begin{aligned}
E &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2,\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y},\ -\tfrac{1}{2}\sqrt{4-x^2-y} \le z \le \tfrac{1}{2}\sqrt{4-x^2-y}\,\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2,\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y},\ -\sqrt{4-y-4z^2} \le x \le \sqrt{4-y-4z^2}\,\right\} \\
&= \left\{(x,y,z) \mid -2 \le x \le 2,\ -\tfrac{1}{2}\sqrt{4-x^2} \le z \le \tfrac{1}{2}\sqrt{4-x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1,\ -\sqrt{4-4z^2} \le x \le \sqrt{4-4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dx\,dy \\
&= \int_{-1}^{1}\int_0^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz = \int_0^4\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx = \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

31.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, x^2 \le y \le 4\right\} = \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}\right\},$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} = \left\{(y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z\right\}, \text{ and}$$

$$D_3 = \left\{(x,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2\right\} = \left\{(x,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}\right\}$$

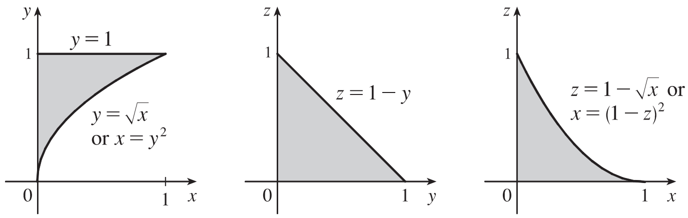
Therefore

$$\begin{aligned} E &= \left\{(x,y,z) \mid -2 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, -\sqrt{y} \le x \le \sqrt{y}, 0 \le z \le 2 - \tfrac{1}{2}y\right\} \\ &= \left\{(x,y,z) \mid 0 \le y \le 4, 0 \le z \le 2 - \tfrac{1}{2}y, -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, 0 \le y \le 4 - 2z, -\sqrt{y} \le x \le \sqrt{y}\right\} \\ &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le z \le 2 - \tfrac{1}{2}x^2, x^2 \le y \le 4 - 2z\right\} \\ &= \left\{(x,y,z) \mid 0 \le z \le 2, -\sqrt{4-2z} \le x \le \sqrt{4-2z}, x^2 \le y \le 4 - 2z\right\} \end{aligned}$$

Then

$$\begin{aligned} \iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{2-y/2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{2-y/2} f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^4\int_0^{2-y/2}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy = \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz \\ &= \int_{-2}^{2}\int_0^{2-x^2/2}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx = \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

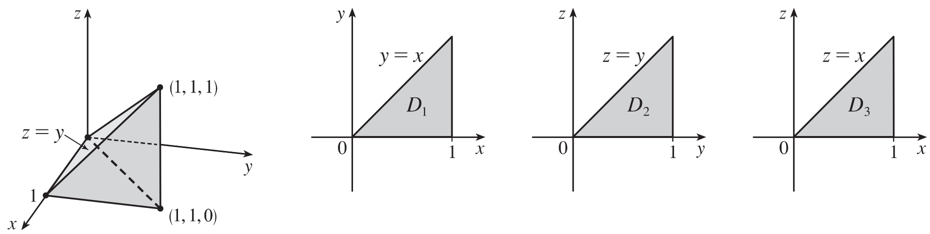
33.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\begin{aligned}\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz\\ &= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

35.



$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

Thus we also have

$$\begin{aligned}E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\}\\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\}\\ &= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^y\int_y^1 f(x,y,z)\,dx\,dz\,dy\\ &= \int_0^1\int_z^1\int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1\int_0^x\int_z^x f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^1\int_z^1\int_z^x f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

37. $m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx$

$= \int_0^1 \left[2y + 2xy + y^2\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(2\sqrt{x} + 2x^{3/2} + x\right) dx = \left[\frac{4}{3}x^{3/2} + \frac{4}{5}x^{5/2} + \frac{1}{2}x^2\right]_0^1 = \frac{79}{30}$

$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx$

$= \int_0^1 \left[2xy + 2x^2y + xy^2\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 (2x^{3/2} + 2x^{5/2} + x^2)\,dx = \left[\frac{4}{5}x^{5/2} + \frac{4}{7}x^{7/2} + \frac{1}{3}x^3\right]_0^1 = \frac{179}{105}$

$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx$

$= \int_0^1 \left[y^2 + xy^2 + \frac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(x + x^2 + \frac{2}{3}x^{3/2}\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{4}{15}x^{5/2}\right]_0^1 = \frac{11}{10}$

$M_{xy} = \iiint_E z\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2z\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} \left[z^2\right]_{z=0}^{z=1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} (1+x+y)^2\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} (1 + 2x + 2y + 2xy + x^2 + y^2)\,dy\,dx = \int_0^1 \left[y + 2xy + y^2 + xy^2 + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \left(\sqrt{x} + \frac{7}{3}x^{3/2} + x + x^2 + x^{5/2}\right) dx = \left[\frac{2}{3}x^{3/2} + \frac{14}{15}x^{5/2} + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{2}{7}x^{7/2}\right]_0^1 = \frac{571}{210}$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(\dfrac{358}{553}, \dfrac{33}{79}, \dfrac{571}{553}\right)$.

39. $m = \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^a \int_0^a \left[\frac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a} dy\,dz = \int_0^a \int_0^a \left(\frac{1}{3}a^3 + ay^2 + az^2\right) dy\,dz$

$= \int_0^a \left[\frac{1}{3}a^3y + \frac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a} dz = \int_0^a \left(\frac{2}{3}a^4 + a^2z^2\right) dz = \left[\frac{2}{3}a^4z + \frac{1}{3}a^2z^3\right]_0^a = \frac{2}{3}a^5 + \frac{1}{3}a^5 = a^5$

$M_{yz} = \int_0^a \int_0^a \int_0^a \left[x^3 + x(y^2 + z^2)\right] dx\,dy\,dz = \int_0^a \int_0^a \left[\frac{1}{4}a^4 + \frac{1}{2}a^2(y^2 + z^2)\right] dy\,dz$

$= \int_0^a \left(\frac{1}{4}a^5 + \frac{1}{6}a^5 + \frac{1}{2}a^3z^2\right) dz = \frac{1}{4}a^6 + \frac{1}{3}a^6 = \frac{7}{12}a^6 = M_{xz} = M_{xy}$ by symmetry of E and $\rho(x,y,z)$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

41. $I_x = \int_0^L \int_0^L \int_0^L k(y^2 + z^2)\,dz\,dy\,dx = k\int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3\right) dy\,dx = k\int_0^L \frac{2}{3}L^4\,dx = \frac{2}{3}kL^5$.

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

43. $I_z = \iiint_E (x^2 + y^2)\,\rho(x,y,z)\,dV = \iint\limits_{x^2+y^2 \le a^2} \left[\int_0^h k(x^2 + y^2)\,dz\right] dA = \iint\limits_{x^2+y^2 \le a^2} k(x^2 + y^2)h\,dA$

$= kh\int_0^{2\pi} \int_0^a (r^2)\,r\,dr\,d\theta = kh\int_0^{2\pi} d\theta \int_0^a r^3\,dr = kh(2\pi)\left[\frac{1}{4}r^4\right]_0^a = 2\pi kh \cdot \frac{1}{4}a^4 = \frac{1}{2}\pi kha^4$

45. (a) $m = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} \sqrt{x^2 + y^2}\,dz\,dy\,dx$

(b) $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right)$ where

$M_{yz} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} x\sqrt{x^2 + y^2}\,dz\,dy\,dx$, $M_{xz} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} y\sqrt{x^2 + y^2}\,dz\,dy\,dx$, and

$M_{xy} = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} z\sqrt{x^2 + y^2}\,dz\,dy\,dx$.

(c) $I_z = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2 + y^2)\sqrt{x^2 + y^2}\,dz\,dy\,dx = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} (x^2 + y^2)^{3/2}\,dz\,dy\,dx$

47. (a) $m = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (1 + x + y + z)\, dz\, dy\, dx = \frac{3\pi}{32} + \frac{11}{24}$

(b) $$(\overline{x}, \overline{y}, \overline{z}) = \left(m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y x(1 + x + y + z)\, dz\, dy\, dx, \right.$$
$$m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y y(1 + x + y + z)\, dz\, dy\, dx,$$
$$\left. m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y z(1 + x + y + z)\, dz\, dy\, dx \right)$$
$$= \left(\frac{28}{9\pi + 44}, \frac{30\pi + 128}{45\pi + 220}, \frac{45\pi + 208}{135\pi + 660} \right)$$

(c) $$I_z = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2 + y^2)(1 + x + y + z)\, dz\, dy\, dx = \frac{68 + 15\pi}{240}$$

49. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z)\, dV = 1$. Here we have

$$\begin{aligned} \iiint_{\mathbb{R}^3} f(x, y, z)\, dV &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = \int_0^2 \int_0^2 \int_0^2 Cxyz\, dz\, dy\, dx \\ &= C \int_0^2 x\, dx \int_0^2 y\, dy \int_0^2 z\, dz = C\left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^2 \left[\tfrac{1}{2}z^2\right]_0^2 = 8C \end{aligned}$$

Then we must have $8C = 1 \quad \Rightarrow \quad C = \frac{1}{8}$.

(b) $$\begin{aligned} P(X \leq 1, Y \leq 1, Z \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 \int_{-\infty}^1 f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^1 \int_0^1 \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8} \int_0^1 x\, dx \int_0^1 y\, dy \int_0^1 z\, dz = \tfrac{1}{8}\left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_0^1 \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64} \end{aligned}$$

(c) $P(X + Y + Z \leq 1) = P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the xy-plane in the line $x + y = 1$, so we have

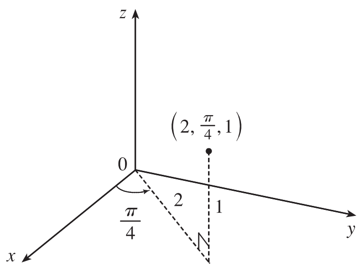
$$\begin{aligned} P(X + Y + Z \leq 1) &= \iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \tfrac{1}{8}xyz\, dz\, dy\, dx \\ &= \tfrac{1}{8} \int_0^1 \int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\, dx = \tfrac{1}{16} \int_0^1 \int_0^{1-x} xy(1 - x - y)^2\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \int_0^{1-x} [(x^3 - 2x^2 + x)y + (2x^2 - 2x)y^2 + xy^3]\, dy\, dx \\ &= \tfrac{1}{16} \int_0^1 \left[(x^3 - 2x^2 + x)\tfrac{1}{2}y^2 + (2x^2 - 2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx \\ &= \tfrac{1}{192} \int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\, dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760} \end{aligned}$$

51. $$V(E) = L^3 \quad \Rightarrow \quad f_{\text{ave}} = \frac{1}{L^3} \int_0^L \int_0^L \int_0^L xyz\, dx\, dy\, dz = \frac{1}{L^3} \int_0^L x\, dx \int_0^L y\, dy \int_0^L z\, dz$$
$$= \frac{1}{L^3} \left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3} \frac{L^2}{2} \frac{L^2}{2} \frac{L^2}{2} = \frac{L^3}{8}$$

53. The triple integral will attain its maximum when the integrand $1 - x^2 - 2y^2 - 3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2 + 2y^2 + 3z^2 \leq 1$. This describes the region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

16.7 Triple Integrals in Cylindrical Coordinates ET 15.7

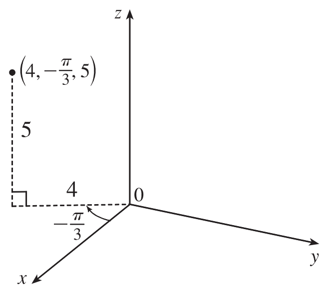
1. (a)



$x = 2\cos\frac{\pi}{4} = \sqrt{2}$, $y = 2\sin\frac{\pi}{4} = \sqrt{2}$, $z = 1$,

so the point is $\left(\sqrt{2}, \sqrt{2}, 1\right)$ in rectangular coordinates.

(b)



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$, $y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$,

and $z = 5$, so the point is $\left(2, -2\sqrt{3}, 5\right)$ in rectangular coordinates.

3. (a) $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \dfrac{y}{x} = \dfrac{-1}{1} = -1$ and the point $(1, -1)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}, 4\right)$.

(b) $r^2 = (-1)^2 + \left(-\sqrt{3}\right)^2 = 4$ so $r = 2$; $\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $\left(-1, -\sqrt{3}\right)$ is in the third quadrant of the xy-plane, so $\theta = \frac{4\pi}{3} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(2, \frac{4\pi}{3}, 2\right)$.

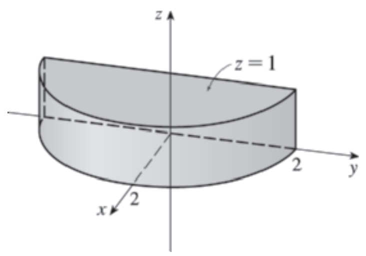
5. Since $\theta = \frac{\pi}{4}$ but r and z may vary, the surface is a vertical half-plane including the z-axis and intersecting the xy-plane in the half-line $y = x$, $x \geq 0$.

7. $z = 4 - r^2 = 4 - (x^2 + y^2)$ or $4 - x^2 - y^2$, so the surface is a circular paraboloid with vertex $(0, 0, 4)$, axis the z-axis, and opening downward.

9. (a) $x^2 + y^2 = r^2$, so the equation becomes $z = r^2$.

(b) Substituting $x^2 + y^2 = r^2$ and $y = r\sin\theta$, the equation $x^2 + y^2 = 2y$ becomes $r^2 = 2r\sin\theta$ or $r = 2\sin\theta$.

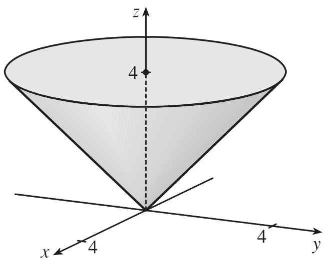
11.



$0 \leq r \leq 2$ and $0 \leq z \leq 1$ describe a solid circular cylinder with radius 2, axis the z-axis, and height 1, but $-\pi/2 \leq \theta \leq \pi/2$ restricts the solid to the first and fourth quadrants of the xy-plane, so we have a half-cylinder.

13. We can position the cylindrical shell vertically so that its axis coincides with the z-axis and its base lies in the xy-plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \leq r \leq 7$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 20$.

15.



The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, r \le z \le 4\}$. This represents the solid region bounded below by the cone $z = r$ and above by the horizontal plane $z = 4$.

$$\int_0^4 \int_0^{2\pi} \int_r^4 r\,dz\,d\theta\,dr = \int_0^4 \int_0^{2\pi} \left[rz\right]_{z=r}^{z=4} d\theta\,dr = \int_0^4 \int_0^{2\pi} r(4-r)\,d\theta\,dr$$
$$= \int_0^4 (4r - r^2)\,dr \int_0^{2\pi} d\theta = \left[2r^2 - \tfrac{1}{3}r^3\right]_0^4 \left[\theta\right]_0^{2\pi}$$
$$= \left(32 - \tfrac{64}{3}\right)(2\pi) = \tfrac{64\pi}{3}$$

17. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2 + y^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^4 \left[z\right]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

19. In cylindrical coordinates E is bounded by the paraboloid $z = 1 + r^2$, the cylinder $r^2 = 5$ or $r = \sqrt{5}$, and the xy-plane, so E is given by $\left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{5}, 0 \le z \le 1 + r^2\right\}$. Thus

$$\iiint_E e^z\,dV = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} r\left[e^z\right]_{z=0}^{z=1+r^2} dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{5}} r(e^{1+r^2} - 1)\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} \left(re^{1+r^2} - r\right) dr = 2\pi\left[\tfrac{1}{2}e^{1+r^2} - \tfrac{1}{2}r^2\right]_0^{\sqrt{5}} = \pi(e^6 - e - 5)$$

21. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\iiint_E x^2\,dV = \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \left[r^3 \cos^2\theta\,z\right]_{z=0}^{z=2r} dr\,d\theta = \int_0^{2\pi} \int_0^1 2r^4 \cos^2\theta\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{2}{5}r^5 \cos^2\theta\right]_{r=0}^{r=1} d\theta = \tfrac{2}{5} \int_0^{2\pi} \cos^2\theta\,d\theta = \frac{2}{5} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2}\,d\theta = \frac{1}{5}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{2\pi}{5}$$

23. (a) The paraboloids intersect when $x^2 + y^2 = 36 - 3x^2 - 3y^2 \;\Rightarrow\; x^2 + y^2 = 9$, so the region of integration is $D = \left\{(x, y) \mid x^2 + y^2 \le 9\right\}$. Then, in cylindrical coordinates, $E = \left\{(r, \theta, z) \mid r^2 \le z \le 36 - 3r^2, 0 \le r \le 3, 0 \le \theta \le 2\pi\right\}$ and

$$V = \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \left(36r - 4r^3\right) dr\,d\theta = \int_0^{2\pi} \left[18r^2 - r^4\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} 81\,d\theta = 162\pi.$$

(b) For constant density K, $m = KV = 162\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} (zK)\,r\,dz\,dr\,d\theta = K \int_0^{2\pi} \int_0^3 r\left[\tfrac{1}{2}z^2\right]_{z=r^2}^{z=36-3r^2} dr\,d\theta$$
$$= \tfrac{K}{2} \int_0^{2\pi} \int_0^3 r((36 - 3r^2)^2 - r^4)\,dr\,d\theta = \tfrac{K}{2} \int_0^{2\pi} d\theta \int_0^3 (8r^5 - 216r^3 + 1296r)\,dr$$
$$= \tfrac{K}{2}(2\pi)\left[\tfrac{8}{6}r^6 - \tfrac{216}{4}r^4 + \tfrac{1296}{2}r^2\right]_0^3 = \pi K(2430) = 2430\pi K$$

Thus $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \frac{2430\pi K}{162\pi K}\right) = (0, 0, 15)$.

25. The paraboloid $z = 4x^2 + 4y^2$ intersects the plane $z = a$ when $a = 4x^2 + 4y^2$ or $x^2 + y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \left\{(r, \theta, z) \mid 0 \le r \le \frac{1}{2}\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\right\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(ar - 4r^3\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\tfrac{1}{2}ar^2 - r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\tfrac{1}{2}a^2 r - 8r^5\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\tfrac{1}{4}a^2r^2 - \tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{24}a^3\,d\theta = \tfrac{1}{12}a^3\pi K$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{2}{3}a\right)$.

27. The region of integration is the region above the cone $z = \sqrt{x^2 + y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \le y \le 2$ with $-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0, 0)$. Thus,

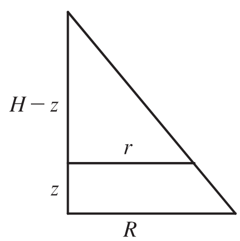
$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy = \int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)\,z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_r^2 r^2(\cos\theta)\,z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left[\tfrac{1}{2}z^2\right]_{z=r}^{z=2}dr\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\int_0^2 r^2(\cos\theta)\left(4 - r^2\right)dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\cos\theta\,d\theta\int_0^2\left(4r^2 - r^4\right)dr = \tfrac{1}{2}\left[\sin\theta\right]_0^{2\pi}\left[\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right]_0^2 = 0$$

29. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\,dV$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200$ lb/ft^3 at all points P in C. We use cylindrical coordinates:



$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

$$W = \int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta = 2\pi\int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz$$

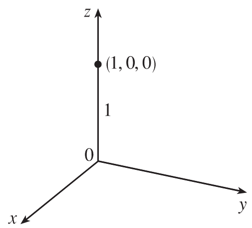
$$= 400\pi\int_0^H z\,\frac{R^2}{2}\left(1 - \frac{z}{H}\right)^2 dz = 200\pi R^2\int_0^H\left(z - \frac{2z^2}{H} + \frac{z^3}{H^2}\right)dz$$

$$= 200\pi R^2\left[\frac{z^2}{2} - \frac{2z^3}{3H} + \frac{z^4}{4H^2}\right]_0^H = 200\pi R^2\left(\frac{H^2}{2} - \frac{2H^2}{3} + \frac{H^2}{4}\right)$$

$$= \tfrac{50}{3}\pi R^2H^2 = \tfrac{50}{3}\pi(62{,}000)^2(12{,}400)^2 \approx 3.1\times 10^{19}\text{ ft-lb}$$

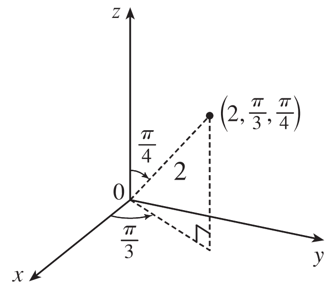
16.8 Triple Integrals in Spherical Coordinates ET 15.8

1. (a)



$x = \rho \sin\phi \cos\theta = (1)\sin 0 \cos 0 = 0$,
$y = \rho \sin\phi \sin\theta = (1)\sin 0 \sin 0 = 0$, and
$z = \rho \cos\phi = (1)\cos 0 = 1$ so the point is
$(0, 0, 1)$ in rectangular coordinates.

(b)



$x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$,
$z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.

3. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \dfrac{\pi}{6}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{4\sin(\pi/6)} = \dfrac{1}{2} \quad\Rightarrow\quad \theta = \dfrac{\pi}{3}$ [since $y > 0$]. Thus spherical coordinates are $\left(4, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{0+1+1} = \sqrt{2}$, $\cos\phi = \dfrac{-1}{\sqrt{2}} \quad\Rightarrow\quad \phi = \dfrac{3\pi}{4}$, and $\cos\theta = \dfrac{0}{\sqrt{2}\sin(3\pi/4)} = 0 \quad\Rightarrow\quad \theta = \dfrac{3\pi}{2}$ [since $y < 0$].

Thus spherical coordinates are $\left(\sqrt{2}, \dfrac{3\pi}{2}, \dfrac{3\pi}{4}\right)$.

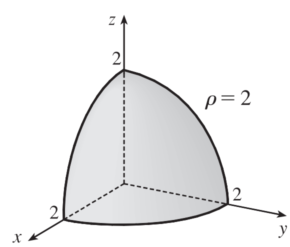
5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

7. $\rho = \sin\theta\sin\phi \quad\Rightarrow\quad \rho^2 = \rho\sin\theta\sin\phi \quad\Leftrightarrow\quad x^2+y^2+z^2 = y \quad\Leftrightarrow\quad x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4} \quad\Leftrightarrow$
$x^2 + (y-\frac{1}{2})^2 + z^2 = \frac{1}{4}$. Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}, 0\right)$.

9. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z^2 = x^2+y^2$ becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho^2\cos^2\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos^2\phi = \sin^2\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.) There are many equivalent equations in spherical coordinates, such as $\tan^2\phi = 1$, $2\cos^2\phi = 1$, $\cos 2\phi = 0$, or even $\phi = \frac{\pi}{4}$, $\phi = \frac{3\pi}{4}$.

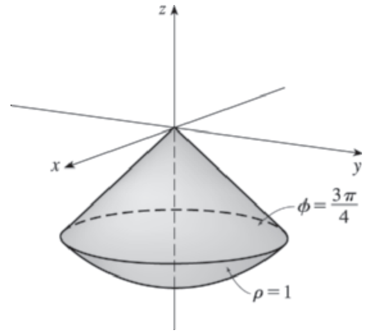
(b) $x^2+z^2 = 9 \quad\Leftrightarrow\quad (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9 \quad\Leftrightarrow\quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 9$ or
$\rho^2\left(\sin^2\phi\cos^2\theta + \cos^2\phi\right) = 9$.

11. $\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy-plane, and $0 \leq \theta \leq \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.



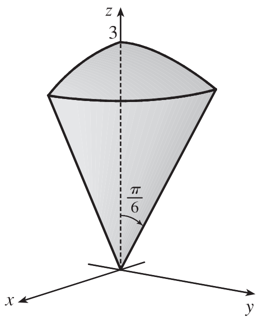
13. $\rho \le 1$ represents the solid sphere of radius 1 centered at the origin.

$\frac{3\pi}{4} \le \phi \le \pi$ restricts the solid to that portion on or below the cone $\phi = \frac{3\pi}{4}$.



15. $z \ge \sqrt{x^2+y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \ge x^2+y^2 \Rightarrow 2z^2 \ge x^2+y^2+z^2 = \rho^2 \Rightarrow z^2 = \rho^2\cos^2\phi \ge \frac{1}{2}\rho^2 \Rightarrow \cos^2\phi \ge \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \ge \frac{1}{\sqrt{2}}$, or equivalently $0 \le \phi \le \frac{\pi}{4}$. In spherical coordinates the sphere $z = x^2+y^2+z^2$ is $\rho\cos\phi = \rho^2 \Rightarrow \rho = \cos\phi$. $0 \le \rho \le \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \le \rho \le \cos\phi$, $0 \le \phi \le \frac{\pi}{4}$.

17.



The region of integration is given in spherical coordinates by $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned}\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/6}\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_0^3\rho^2\,d\rho \\ &= \big[-\cos\phi\big]_0^{\pi/6}\big[\theta\big]_0^{\pi/2}\big[\tfrac{1}{3}\rho^3\big]_0^3 \\ &= \left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}\left(2-\sqrt{3}\right)\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then
$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$.

21. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2+y^2+z^2)^2\,dV &= \int_0^\pi\int_0^{2\pi}\int_0^5 (\rho^2)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^5\rho^6\,d\rho \\ &= \big[-\cos\phi\big]_0^\pi\big[\theta\big]_0^{2\pi}\big[\tfrac{1}{7}\rho^7\big]_0^5 = (2)(2\pi)\left(\tfrac{78{,}125}{7}\right) \\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7\end{aligned}$$

23. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi) \,\middle|\, 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}\iiint_E z\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho \\ &= \big[\tfrac{1}{2}\sin^2\phi\big]_0^{\pi/2}\big[\theta\big]_0^{\pi/2}\big[\tfrac{1}{4}\rho^4\big]_1^2 = \left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{15}{4}\right) = \tfrac{15\pi}{16}\end{aligned}$$

25. $\iiint_E x^2\,dV = \int_0^\pi \int_0^\pi \int_3^4 (\rho\sin\phi\cos\theta)^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^\pi \cos^2\theta\,d\theta\,\int_0^\pi \sin^3\phi\,d\phi\,\int_3^4 \rho^4\,d\rho$

$$= \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^\pi \left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^\pi \left[\tfrac{1}{5}\rho^5\right]_3^4 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\tfrac{1}{5}(4^5-3^5) = \tfrac{1562}{15}\pi$$

27. The solid region is given by $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, \frac{\pi}{6} \le \phi \le \frac{\pi}{3}\right\}$ and its volume is

$$V = \iiint_E dV = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^a \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/3} \sin\phi\,d\phi\,\int_0^{2\pi} d\theta\,\int_0^a \rho^2\,d\rho$$

$$= [-\cos\phi]_{\pi/6}^{\pi/3}\,[\theta]_0^{2\pi}\,\left[\tfrac{1}{3}\rho^3\right]_0^a = \left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)(2\pi)\left(\tfrac{1}{3}a^3\right) = \tfrac{\sqrt{3}-1}{3}\pi a^3$$

29. (a) Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{\rho=4\cos\phi} \sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left(\tfrac{64}{3}\cos^3\phi\right)\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{16}{3}\cos^4\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} -\tfrac{16}{3}\left(\tfrac{1}{16}-1\right)d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} \cos\phi\sin\phi\left(64\cos^4\phi\right)d\phi\,d\theta$$

$$= \int_0^{2\pi} 64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \tfrac{21}{2}\,d\theta = 21\pi$$

Hence $(\overline{x},\overline{y},\overline{z}) = (0,0,2.1)$.

31. By the symmetry of the region, $M_{xy} = 0$ and $M_{yz} = 0$. Assuming constant density K,

$$m = \iiint_E KV = K\int_0^\pi\int_0^\pi\int_3^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^\pi d\theta\,\int_0^\pi \sin\phi\,d\phi\,\int_3^4 \rho^2\,d\rho$$

$$= K\pi\left[-\cos\phi\right]_0^\pi \left[\tfrac{1}{3}\rho^3\right]_3^4 = 2K\pi\cdot\tfrac{37}{3} = \tfrac{74}{3}\pi K$$

and $M_{xz} = \iiint_E y\,K\,dV = K\int_0^\pi\int_0^\pi\int_3^4 (\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = K\int_0^\pi \sin\theta\,d\theta\int_0^\pi \sin^2\phi\,d\phi\int_3^4 \rho^3\,d\rho$

$$= K\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^\pi \left[\tfrac{1}{4}\rho^4\right]_3^4 = K(2)\left(\tfrac{\pi}{2}\right)\tfrac{1}{4}(256-81) = \tfrac{175}{4}\pi K$$

Thus the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, \dfrac{175\pi K/4}{74\pi K/3}, 0\right) = \left(0, \frac{525}{296}, 0\right)$.

33. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then $M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3\sin\phi\,\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{2}\pi Ka^4\int_0^{\pi/2}\sin\phi\,\cos\phi\,d\phi = \frac{1}{8}\pi Ka^4$. But the mass is K(volume of the hemisphere) $= \frac{2}{3}\pi Ka^3$, so the centroid is $\left(0,0,\frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$I_x = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho^2\sin\phi)\,\rho^2\left(\sin^2\phi\,\sin^2\theta + \cos^2\phi\right)d\rho\,d\phi\,d\theta$$

$$= K\int_0^{2\pi}\int_0^{\pi/2} (\sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi)\left(\tfrac{1}{5}a^5\right)d\phi\,d\theta$$

$$= \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\sin^2\theta\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) + \left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2} d\theta = \tfrac{1}{5}Ka^5\int_0^{2\pi}\left[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\right]d\theta$$

$$= \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{3}\theta\right]_0^{2\pi} = \tfrac{1}{5}Ka^5\left[\tfrac{2}{3}(\pi-0) + \tfrac{1}{3}(2\pi-0)\right] = \tfrac{4}{15}Ka^5\pi$$

35. In spherical coordinates $z = \sqrt{x^2 + y^2}$ becomes $\cos\phi = \sin\phi$ or $\phi = \frac{\pi}{4}$. Then

$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^1 \rho^2 \, d\rho = \frac{1}{3}\pi\left(2 - \sqrt{2}\right)$,

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \dfrac{3}{8\left(2 - \sqrt{2}\right)}\right)$.

37. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r\sin\theta$ and they intersect in the circle $r = 2\sin\theta$. Then $\iiint_E z \, dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz \, dz \, dr \, d\theta = \frac{5\pi}{6}$ [using a CAS].

39. The region E of integration is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. So the integral becomes

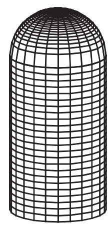
$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} \left(\rho\sin\phi\cos\theta\right)\left(\rho\sin\phi\sin\theta\right)\rho^2\sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/4}\sin^3\phi \, d\phi \int_0^{\pi/2}\sin\theta\cos\theta \, d\theta \int_0^{\sqrt{2}}\rho^4 \, d\rho = \left(\int_0^{\pi/4}\left(1 - \cos^2\phi\right)\sin\phi \, d\phi\right)\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \left(\tfrac{1}{3} - 1\right)\right] \cdot \tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}$$

41. In cylindrical coordinates, the equation of the cylinder is $r = 3$, $0 \le z \le 10$.

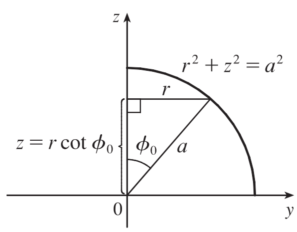
The hemisphere is the upper part of the sphere radius 3, center $(0, 0, 10)$, equation $r^2 + (z - 10)^2 = 3^2$, $z \ge 10$. In Maple, we can use the `coords=cylindrical` option in a regular `plot3d` command. In Mathematica, we can use `ParametricPlot3D`.



43. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5}\sin 6\theta \, \sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le 1 + \frac{1}{5}\sin 6\theta\sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by

$V(E) = \iiint_E dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1 + (\sin 6\theta\sin 5\phi)/5} \rho^2\sin\phi \, d\rho \, d\theta \, d\phi = \frac{136\pi}{99}$ [using a CAS].

45. (a) From the diagram, $z = r\cot\phi_0$ to $z = \sqrt{a^2 - r^2}$, $r = 0$ to $r = a\sin\phi_0$ (or use $a^2 - r^2 = r^2\cot^2\phi_0$). Thus



$$V = \int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$$

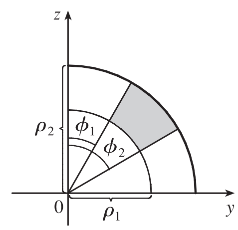
$$= 2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2 - r^2} - r^2\cot\phi_0\right)dr$$

$$= \tfrac{2\pi}{3}\left[-(a^2 - r^2)^{3/2} - r^3\cot\phi_0\right]_0^{a\sin\phi_0}$$

$$= \tfrac{2\pi}{3}\left[-\left(a^2 - a^2\sin^2\phi_0\right)^{3/2} - a^3\sin^3\phi_0\cot\phi_0 + a^3\right]$$

$$= \tfrac{2}{3}\pi a^3\left[1 - \left(\cos^3\phi_0 + \sin^2\phi_0\cos\phi_0\right)\right] = \tfrac{2}{3}\pi a^3(1 - \cos\phi_0)$$

(b) The wedge in question is the shaded area rotated from $\theta = \theta_1$ to $\theta = \theta_2$. Letting



V_{ij} = volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta = \theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$\begin{aligned}
V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\rho_2^3(1 - \cos\phi_2) - \rho_2^3(1 - \cos\phi_1) - \rho_1^3(1 - \cos\phi_2) + \rho_1^3(1 - \cos\phi_1)\right] \\
&= \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(1 - \cos\phi_2) - \left(\rho_2^3 - \rho_1^3\right)(1 - \cos\phi_1)\right] = \tfrac{1}{3}(\theta_2 - \theta_1)\left[\left(\rho_2^3 - \rho_1^3\right)(\cos\phi_1 - \cos\phi_2)\right]
\end{aligned}$$

Or: Show that $V = \displaystyle\int_{\theta_1}^{\theta_2} \int_{\rho_1 \sin\phi_1}^{\rho_2 \sin\phi_2} \int_{r\cot\phi_2}^{r\cot\phi_1} r\,dz\,dr\,d\theta$.

(c) By the Mean Value Theorem with $f(\rho) = \rho^3$ there exists some $\tilde{\rho}$ with $\rho_1 \le \tilde{\rho} \le \rho_2$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1)$ or $\rho_1^3 - \rho_2^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly there exists ϕ with $\phi_1 \le \tilde{\phi} \le \phi_2$ such that $\cos\phi_2 - \cos\phi_1 = \left(-\sin\tilde{\phi}\right)\Delta\phi$. Substituting into the result from (b) gives $\Delta V = (\tilde{\rho}^2\,\Delta\rho)(\theta_2 - \theta_1)(\sin\tilde{\phi})\;\Delta\phi = \tilde{\rho}^2 \sin\tilde{\phi}\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

16.9 Change of Variables in Multiple Integrals ET 15.9

1. $x = 5u - v,\ y = u + 3v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 5(3) - (-1)(1) = 16.$

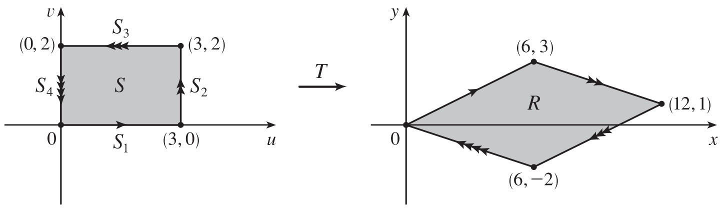
3. $x = e^{-r}\sin\theta,\ y = e^r\cos\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^r\cos\theta & -e^r\sin\theta \end{vmatrix} = e^{-r}e^r\sin^2\theta - e^{-r}e^r\cos^2\theta = \sin^2\theta - \cos^2\theta \text{ or } -\cos 2\theta$$

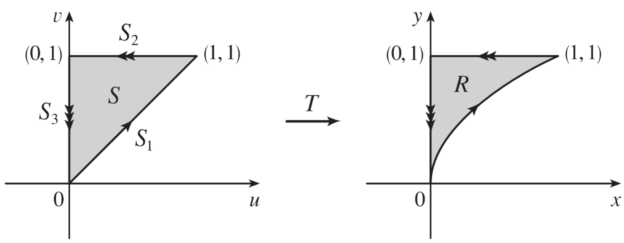
5. $x = u/v,\ y = v/w,\ z = w/u$.

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{vmatrix} \\
&= \frac{1}{v}\begin{vmatrix} 1/w & -v/w^2 \\ 0 & 1/u \end{vmatrix} - \left(-\frac{u}{v^2}\right)\begin{vmatrix} 0 & -v/w^2 \\ -w/u^2 & 1/u \end{vmatrix} + 0\begin{vmatrix} 0 & 1/w \\ -w/u^2 & 0 \end{vmatrix} \\
&= \frac{1}{v}\left(\frac{1}{uw} - 0\right) + \frac{u}{v^2}\left(0 - \frac{v}{u^2w}\right) + 0 = \frac{1}{uvw} - \frac{1}{uvw} = 0
\end{aligned}$$

7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0 \quad [0 \le u \le 3]$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad \Rightarrow \quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad \Rightarrow \quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad \Rightarrow \quad x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \quad \Rightarrow \quad 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



11. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0,0)$ and $(2,1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \quad \Rightarrow \quad v = 0$; the line through $(2,1)$ and $(1,2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \quad \Rightarrow \quad u + v = 1$; the line through $(0,0)$ and $(1,2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \quad \Rightarrow \quad u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\, dA = \int_0^1 \int_0^{1-u} (-u - 5v)\, |3|\, dv\, du = -3 \int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$

$$= -3 \int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3$$

13. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\iint_R x^2\, dA = \iint\limits_{u^2+v^2 \le 1} (4u^2)(6)\, du\, dv = \int_0^{2\pi} \int_0^1 (24r^2 \cos^2\theta)\, r\, dr\, d\theta = 24 \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^1 r^3\, dr$$

$$= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi$$

15. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \leq 1$. So

$$\iiint_E dV = \iiint\limits_{u^2+v^2+w^2\leq 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

19. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4 \int_1^8 \frac{u}{v}\left|\frac{1}{5}\right| dv\ du = \frac{1}{5}\int_0^4 u\,du \int_1^8 \frac{1}{v}\,dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \left[\ln|v|\right]_1^8 = \tfrac{8}{5}\ln 8.$$

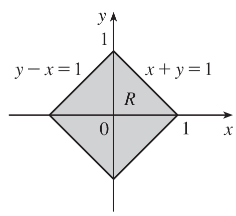
21. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\iint_R \cos\frac{y-x}{y+x}\,dA = \int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 \left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 2v\sin(1)\,dv = \tfrac{3}{2}\sin 1$$

23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \quad\Rightarrow\quad y = \frac{1}{2}(u + v)$ and $u - v = 2x \quad\Rightarrow\quad x = \frac{1}{2}(u - v)$.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \leq |x| + |y| \leq 1 \quad\Rightarrow\quad -1 \leq u \leq 1$, and $|v| = |-x + y| \leq |x| + |y| \leq 1 \quad\Rightarrow\quad -1 \leq v \leq 1$. R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^1 \int_{-1}^1 e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^1 \left[v\right]_{-1}^1 = e - e^{-1}$.



16 Review ET 15

CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$, where ΔA is the area of each subrectangle and $\left(x_{ij}^*, y_{ij}^*\right)$ is a sample point in each subrectangle. If $f(x,y) \geq 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

(c) If $f(x,y) \geq 0$, $\iint_R f(x,y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$. If f takes on both positive and negative values, $\iint_R f(x,y)\,dA$ is the difference of the volume above R but below the surface $z = f(x,y)$ and the volume below R but above the surface $z = f(x,y)$.

(d) We usually evaluate $\iint_R f(x,y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x,y)\,dA$ by the double Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$ where the sample points $\left(\overline{x}_i, \overline{y}_j\right)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \dfrac{1}{A\,(R)}\displaystyle\iint_R f(x,y)\,dA$ where $A\,(R)$ is the area of R.

2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

(b) See (3) and the accompanying discussion in Section 16.3 [ET 15.3].

(c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].

(d) See (6)–(11) in Section 16.3 [ET 15.3].

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where R is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$.

4. (a) $m = \iint_D \rho(x,y)\,dA$

(b) $M_x = \iint_D y\rho(x,y)\,dA$, $M_y = \iint_D x\rho(x,y)\,dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2\rho(x,y)\,dA$, $I_y = \iint_D x^2\rho(x,y)\,dA$, $I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA$

5. (a) $P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(b) $f(x,y) \geq 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)\,dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} yf(x,y)\,dA$.

6. (a) $\iiint_B f(x,y,z)\,dV = \lim\limits_{l,m,n\to\infty} \sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

(b) We usually evaluate $\iiint_B f(x,y,z)\,dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.6.4 [ET 15.6.4]).

(c) See the paragraph following Example 16.6.1 [ET 15.6.1].

(d) See (5) and (6) and the accompanying discussion in Section 16.6 [ET 15.6].

(e) See (10) and the accompanying discussion in Section 16.6 [ET 15.6].

(f) See (11) and the preceding discussion in Section 16.6 [ET 15.6].

7. (a) $m = \iiint_E \rho(x,y,z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x,y,z)\,dV,\; M_{xz} = \iiint_E y\rho(x,y,z)\,dV,\; M_{xy} = \iiint_E z\rho(x,y,z)\,dV.$

(c) The center of mass is $(\overline{x},\overline{y},\overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV,\; I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV,\; I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV.$

8. (a) See Formula 16.7.4 [ET 15.7.4] and the accompanying discussion.

(b) See Formula 16.8.3 [ET 15.8.3] and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

9. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

3. True by Equation 16.2.5 [ET 15.2.5].

5. True: $\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane $= \frac{1}{2}$ (the volume of the sphere $x^2+y^2+z^2=4$) $= \frac{1}{2}\cdot\frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

7. The volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates, $V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta \neq \int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$, so the assertion is false.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m = n = 3$ and the sample points the upper right corners of each square, so

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A$$
$$= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)]$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

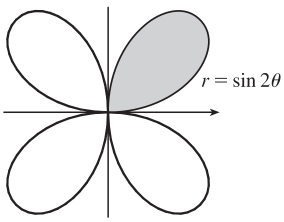
3. $\int_1^2\int_0^2 (y+2xe^y)\,dx\,dy = \int_1^2 \left[xy+x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y+4e^y)\,dy = \left[y^2+4e^y\right]_1^2$
$= 4+4e^2-1-4e = 4e^2-4e+3$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\big]_0^1 = \frac{1}{2}\sin 1$

7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi\int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$
$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\big]_0^\pi = \frac{2}{3}$

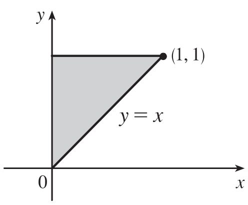
9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4,\ 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

11.



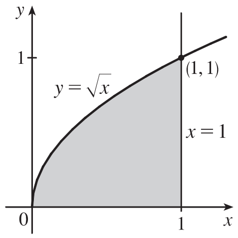
The region whose area is given by $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sin 2\theta\right\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

13.

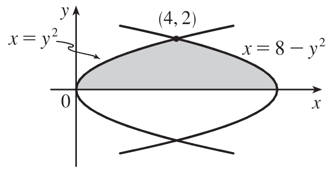


$\int_0^1\int_x^1 \cos(y^2)\,dy\,dx = \int_0^1\int_0^y \cos(y^2)\,dx\,dy$
$= \int_0^1 \cos(y^2)\left[x\right]_{x=0}^{x=y} dy = \int_0^1 y\cos(y^2)\,dy$
$= \left[\frac{1}{2}\sin(y^2)\right]_0^1 = \frac{1}{2}\sin 1$

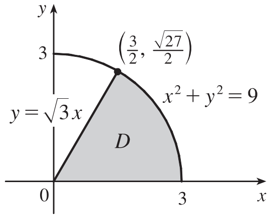
15. $\iint_R ye^{xy}\,dA = \int_0^3\int_0^2 ye^{xy}\,dx\,dy = \int_0^3 \left[e^{xy}\right]_{x=0}^{x=2} dy = \int_0^3 (e^{2y}-1)\,dy = \left[\frac{1}{2}e^{2y}-y\right]_0^3 = \frac{1}{2}e^6-3-\frac{1}{2} = \frac{1}{2}e^6-\frac{7}{2}$

17. 

$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$$
$$= \tfrac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

19. 

$$\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$
$$= \int_0^2 y\left[x\right]_{x=y^2}^{x=8-y^2}\,dy = \int_0^2 y(8-y^2-y^2)\,dy$$
$$= \int_0^2 (8y-2y^3)\,dy = \left[4y^2-\tfrac{1}{2}y^4\right]_0^2 = 8$$

21. 

$$\iint_D \left(x^2+y^2\right)^{3/2}dA = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2} r\,dr\,d\theta$$
$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \left[\,\theta\,\right]_0^{\pi/3}\left[\frac{1}{5}r^5\right]_0^3$$
$$= \frac{\pi}{3}\,\frac{3^5}{5} = \frac{81\pi}{5}$$

23. $\iiint_E xy\,dV = \int_0^3\int_0^x\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3\int_0^x xy\left[\,z\,\right]_{z=0}^{z=x+y}\,dy\,dx = \int_0^3\int_0^x xy(x+y)\,dy\,dx$

$= \int_0^3\int_0^x (x^2y+xy^2)\,dy\,dx = \int_0^3\left[\tfrac{1}{2}x^2y^2+\tfrac{1}{3}xy^3\right]_{y=0}^{y=x}\,dx = \int_0^3\left(\tfrac{1}{2}x^4+\tfrac{1}{3}x^4\right)dx$

$= \tfrac{5}{6}\int_0^3 x^4\,dx = \left[\tfrac{1}{6}x^5\right]_0^3 = \tfrac{81}{2} = 40.5$

25. $\iiint_E y^2z^2\,dV = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

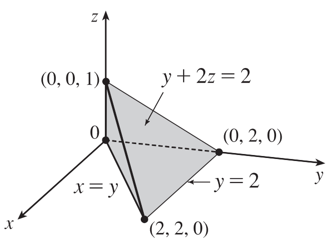
$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \tfrac{1}{4}\sin^2 2\theta(r^5-r^7)\,dr\,d\theta$

$= \int_0^{2\pi}\tfrac{1}{8}(1-\cos 4\theta)\left[\tfrac{1}{6}r^6-\tfrac{1}{8}r^8\right]_{r=0}^{r=1}\,d\theta = \tfrac{1}{192}\left[\theta-\tfrac{1}{4}\sin 4\theta\right]_0^{2\pi} = \tfrac{2\pi}{192} = \tfrac{\pi}{96}$

27. $\iiint_E yz\,dV = \int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \int_{-2}^2\int_0^{\sqrt{4-x^2}}\tfrac{1}{2}y^3dy\,dx = \int_0^\pi\int_0^2\tfrac{1}{2}r^3(\sin^3\theta)\,r\,dr\,d\theta$

$= \tfrac{16}{5}\int_0^\pi \sin^3\theta\,d\theta = \tfrac{16}{5}\left[-\cos\theta+\tfrac{1}{3}\cos^3\theta\right]_0^\pi = \tfrac{64}{15}$

29. $V = \int_0^2\int_1^4(x^2+4y^2)\,dy\,dx = \int_0^2\left[x^2y+\tfrac{4}{3}y^3\right]_{y=1}^{y=4}dx = \int_0^2(3x^2+84)\,dx = 176$

31. 

$$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2\int_0^y\left(1-\tfrac{1}{2}y\right)dx\,dy$$
$$= \int_0^2\left(y-\tfrac{1}{2}y^2\right)dy = \tfrac{2}{3}$$

33. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3} \frac{1}{2}m(a^2 - 9y^2)\,dy = m\big[a^2y - 3y^3\big]_0^{a/3} = m\big(\frac{1}{3}a^3 - \frac{1}{9}a^3\big) = \frac{2}{9}ma^3.$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\big]_0^1 = \frac{1}{24}$,

$I_0 = I_x + I_y = \frac{1}{8}$, $\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \quad\Rightarrow\quad \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

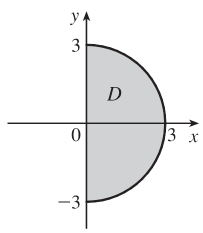
37. The equation of the cone with the suggested orientation is $(h - z) = \frac{h}{a}\sqrt{x^2 + y^2}$, $0 \le z \le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint\limits_{x^2+y^2\le a^2} \int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta = \pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr$$

$$= \frac{\pi h^2}{a^2}\int_0^a (a^2 r - 2ar^2 + r^3)\,dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

39.



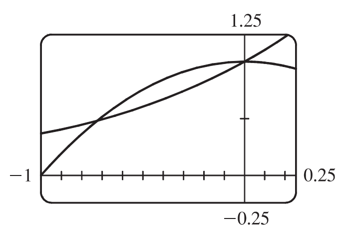
$$\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx = \int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2 + y^2)\,dy\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^3 (r\cos\theta)(r^2)\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\int_0^3 r^4\,dr$$

$$= \big[\sin\theta\big]_{-\pi/2}^{\pi/2}\left[\frac{1}{5}r^5\right]_0^3 = 2\cdot\frac{1}{5}(243) = \frac{486}{5} = 97.2$$

41. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is



$$\iint_D y^2\,dA \approx \int_{-0.71}^0\int_{e^x}^{1-x^2} y^2\,dy\,dx$$

$$= \frac{1}{3}\int_{-0.71}^0 [(1-x^2)^3 - e^{3x}]\,dx$$

$$= \frac{1}{3}\left[x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7 - \frac{1}{3}e^{3x}\right]_{-0.71}^0 \approx 0.0512$$

43. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3]\times[0,2]$, we can say

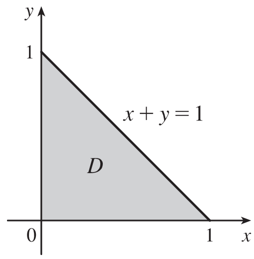
$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3\int_0^2 C(x+y)\,dy\,dx$$
$$= C\int_0^3 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C\int_0^3 (2x+2)\,dx = C\left[x^2+2x\right]_0^3 = 15C$$

Then $15C = 1 \quad\Rightarrow\quad C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2}\int_1^{\infty} f(x,y)\,dy\,dx = \int_0^2\int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15}\int_0^2 \left[xy+\frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

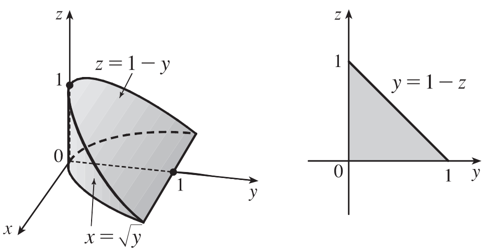
$$= \tfrac{1}{15}\int_0^2 \left(x+\tfrac{3}{2}\right) dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2+\tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X+Y\le 1) = P((X,Y)\in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y\le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx$$
$$= \tfrac{1}{15}\int_0^1 \left[xy+\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \tfrac{1}{15}\int_0^1 \left[x(1-x)+\tfrac{1}{2}(1-x)^2\right] dx$$
$$= \tfrac{1}{30}\int_0^1 (1-x^2)\,dx = \tfrac{1}{30}\left[x-\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$

45.



$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

47. Since $u = x-y$ and $v = x+y$, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$.

Thus $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$ and $\displaystyle\iint_R \frac{x-y}{x+y}\,dA = \int_2^4\int_{-2}^{0} \frac{u}{v}\left(\frac{1}{2}\right) du\,dv = -\int_2^4 \frac{dv}{v} = -\ln 2$.

49. Let $u = y-x$ and $v = y+x$ so $x = y-u = (v-x)-u \quad\Rightarrow\quad x = \frac{1}{2}(v-u)$ and $y = v-\frac{1}{2}(v-u) = \frac{1}{2}(v+u)$.

$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. So

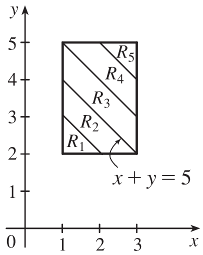
$$\iint_R xy\,dA = \int_0^2\int_{-2}^{0} \frac{v^2-u^2}{4}\left(\frac{1}{2}\right) du\,dv = \tfrac{1}{8}\int_0^2 \left[v^2u-\tfrac{1}{3}u^3\right]_{u=-2}^{u=0} dv = \tfrac{1}{8}\int_0^2 \left(2v^2-\tfrac{8}{3}\right) dv = \tfrac{1}{8}\left[\tfrac{2}{3}v^3-\tfrac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

51. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double Integrals there exists (x_r, y_r) in D_r such that $f(x_r,y_r) = \dfrac{1}{\pi r^2}\displaystyle\iint_{D_r} f(x,y)\,dA$. But $\lim\limits_{r\to 0^+}(x_r,y_r) = (a,b)$,

so $\displaystyle\lim_{r\to 0^+}\frac{1}{\pi r^2}\iint_{D_r} f(x,y)\,dA = \lim_{r\to 0^+} f(x_r,y_r) = f(a,b)$ by the continuity of f.

PROBLEMS PLUS

1. 

Let $R = \bigcup_{i=1}^{5} R_i$, where

$R_i = \{(x,y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x+y]\!] \, dA = \sum\limits_{i=1}^{5} \iint_{R_i} [\![x+y]\!] \, dA = \sum\limits_{i=1}^{5} [\![x+y]\!] \iint_{R_i} dA$, since

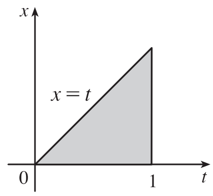
$[\![x+y]\!] = \text{constant} = i+2$ for $(x,y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x+y]\!] \, dA &= \textstyle\sum_{i=1}^{5} (i+2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

3. $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{1-0}\int_0^1 \left[\int_x^1 \cos(t^2)\,dt\right] dx$

$= \int_0^1 \int_x^1 \cos(t^2)\,dt\,dx = \int_0^1 \int_0^t \cos(t^2)\,dx\,dt$ [changing the order of integration]

$= \int_0^1 t\cos(t^2)\,dt = \frac{1}{2}\sin\!\left(t^2\right)\big]_0^1 = \frac{1}{2}\sin 1$



5. Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \sum\limits_{n=0}^{\infty} (xy)^n$, so

$$\int_0^1\!\int_0^1 \tfrac{1}{1-xy}\,dx\,dy = \int_0^1\!\int_0^1 \sum_{n=0}^{\infty} (xy)^n\,dx\,dy = \sum_{n=0}^{\infty} \int_0^1\!\int_0^1 (xy)^n\,dx\,dy = \sum_{n=0}^{\infty} \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]$$

$$= \sum_{n=0}^{\infty} \tfrac{1}{n+1} \cdot \tfrac{1}{n+1} = \sum_{n=0}^{\infty} \tfrac{1}{(n+1)^2} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \cdots = \textstyle\sum_{n=1}^{\infty} \tfrac{1}{n^2}$$

7. (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty} (xyz)^n$, so

$$\begin{aligned}\int_0^1\!\int_0^1\!\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz &= \int_0^1\!\int_0^1\!\int_0^1 \sum_{n=0}^{\infty} (xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty} \int_0^1\!\int_0^1\!\int_0^1 (xyz)^n\,dx\,dy\,dz \\ &= \sum_{n=0}^{\infty} \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot \frac{1}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}\end{aligned}$$

(b) Since $|-xyz| < 1$, except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \sum\limits_{n=0}^{\infty} (-xyz)^n$, so

$$\begin{aligned}\int_0^1\!\int_0^1\!\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz &= \int_0^1\!\int_0^1\!\int_0^1 \frac{1}{1+xyz} \sum_{n=0}^{\infty} (-xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty} \int_0^1\!\int_0^1\!\int_0^1 (-xyz)^n\,dx\,dy\,dz \\ &= \sum_{n=0}^{\infty} (-1)^n \left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \cdot \frac{1}{n+1} \cdot \frac{1}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^3}\end{aligned}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$. Notice that

$a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

9. (a) $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right]\\ &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}r\sin\theta + \dfrac{\partial u}{\partial y}r\cos\theta$ and

$\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \dfrac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\,\dfrac{\partial^2 u}{\partial y\,\partial x}r^2\sin\theta\,\cos\theta - \dfrac{\partial u}{\partial x}r\cos\theta - \dfrac{\partial u}{\partial y}r\sin\theta$. So

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r}\\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\,\cos\theta\\ &\quad - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2}\\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\end{aligned}$$

(b) $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Then

$\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \rho} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial \rho} = \dfrac{\partial u}{\partial x}\sin\phi\,\cos\theta + \dfrac{\partial u}{\partial y}\sin\phi\,\sin\theta + \dfrac{\partial u}{\partial z}\cos\phi$, and

$$\begin{aligned}\frac{\partial^2 u}{\partial \rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial \rho}\right]\\ &\quad + \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial \rho}\right]\\ &\quad + \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial \rho}\right]\\ &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\,\sin\theta\,\cos\theta + 2\,\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\,\cos\phi\,\cos\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\,\cos\phi\,\sin\theta\\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\rho\sin\phi$, and

$$\begin{aligned}\frac{\partial^2 u}{\partial \phi^2} &= 2\,\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\cos^2\phi\,\sin\theta\,\cos\theta - 2\,\frac{\partial^2 u}{\partial x\,\partial z}\rho^2\sin\phi\,\cos\phi\,\cos\theta\\ &\quad - 2\,\frac{\partial^2 u}{\partial y\,\partial z}\rho^2\sin\phi\,\cos\phi\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\cos^2\phi\,\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\phi\,\sin^2\theta\\ &\quad + \frac{\partial^2 u}{\partial z^2}\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\,\sin\theta - \frac{\partial u}{\partial z}\rho\cos\phi\end{aligned}$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\,\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\,\rho\sin\phi\cos\theta$, while

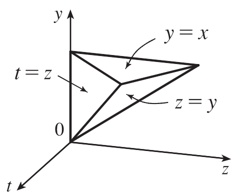
$$\frac{\partial^2 u}{\partial \theta^2} = -2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\sin^2\phi\,\cos\theta\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\,\rho^2\sin^2\phi\,\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\,\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta$$

Therefore

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\
&\quad = \frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\,\cos^2\theta) + (\cos^2\phi\,\cos^2\theta) + \sin^2\theta\right] \\
&\qquad + \frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\,\sin^2\theta) + (\cos^2\phi\,\sin^2\theta) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right] \\
&\qquad + \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\,\cos\theta + \cos^2\phi\,\cos\theta - \sin^2\phi\,\cos\theta - \cos\theta}{\rho\sin\phi}\right] \\
&\qquad + \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\,\sin\theta + \cos^2\phi\,\sin\theta - \sin^2\phi\,\sin\theta - \sin\theta}{\rho\sin\phi}\right]
\end{aligned}$$

But $2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = (\sin^2\phi + \cos^2\phi - 1)\,\cos\theta = 0$ and similarly the coefficient of $\partial u/\partial y$ is 0. Also $\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta\,(\sin^2\phi + \cos^2\phi) + \sin^2\theta = 1$, and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

11. $\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where



$E = \{(t,z,y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$.

If we let D be the projection of E on the yt-plane then

$D = \{(y,t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram

that $E = \{(t,z,y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So

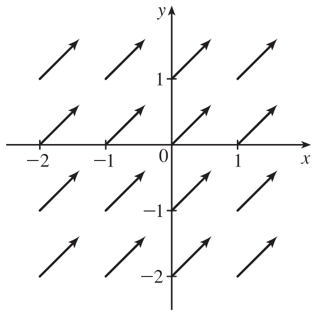
$$\begin{aligned}
\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy &= \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt = \int_0^x \left[\int_t^x (y-t)\,f(t)\,dy\right] dt \\
&= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\,dt \\
&= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right) f(t)\,dt \\
&= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt
\end{aligned}$$

17 □ VECTOR CALCULUS □ ET 16

17.1 Vector Fields ET 16.1

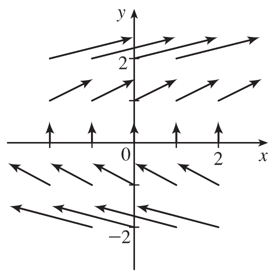
1. $\mathbf{F}(x,y) = \frac{1}{2}(\mathbf{i} + \mathbf{j})$

All vectors in this field are identical, with length $\frac{1}{\sqrt{2}}$ and direction parallel to the line $y = x$.



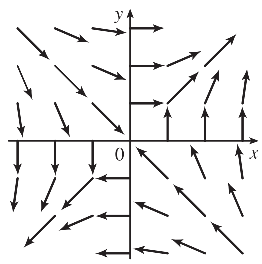
3. $\mathbf{F}(x,y) = y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$

The length of the vector $y\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$ is $\sqrt{y^2 + \frac{1}{4}}$. Vectors are tangent to parabolas opening about the x-axis.



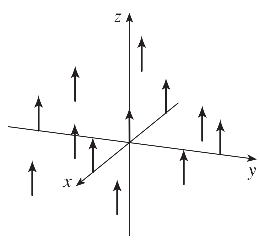
5. $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$ is 1.



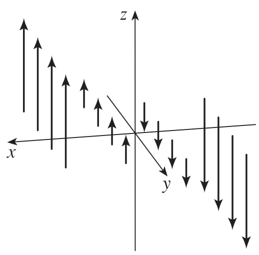
7. $\mathbf{F}(x,y,z) = \mathbf{k}$

All vectors in this field are parallel to the z-axis and have length 1.



9. $\mathbf{F}(x,y,z) = x\,\mathbf{k}$

At each point (x,y,z), $\mathbf{F}(x,y,z)$ is a vector of length $|x|$. For $x > 0$, all point in the direction of the positive z-axis, while for $x < 0$, all are in the direction of the negative z-axis. In each plane $x = k$, all the vectors are identical.



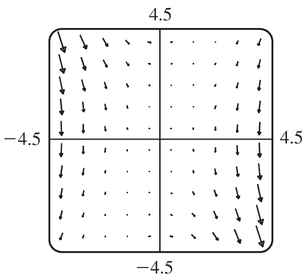
11. $\mathbf{F}(x, y) = \langle y, x \rangle$ corresponds to graph II. In the first quadrant all the vectors have positive x- and y-components, in the second quadrant all vectors have positive x-components and negative y-components, in the third quadrant all vectors have negative x- and y-components, and in the fourth quadrant all vectors have negative x-components and positive y-components. In addition, the vectors get shorter as we approach the origin.

13. $\mathbf{F}(x, y) = \langle x - 2, x + 1 \rangle$ corresponds to graph I since the vectors are independent of y (vectors along vertical lines are identical) and, as we move to the right, both the x- and the y-components get larger.

15. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

17. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the xy-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their z-components are all 3.

19.



The vector field seems to have very short vectors near the line $y = 2x$. For $\mathbf{F}(x, y) = \langle 0, 0 \rangle$ we must have $y^2 - 2xy = 0$ and $3xy - 6x^2 = 0$. The first equation holds if $y = 0$ or $y = 2x$, and the second holds if $x = 0$ or $y = 2x$. So both equations hold [and thus $\mathbf{F}(x, y) = \mathbf{0}$] along the line $y = 2x$.

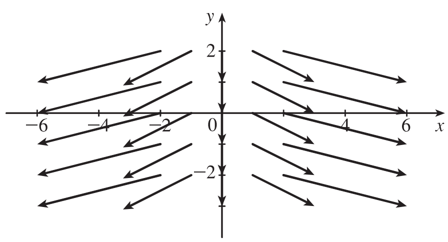
21. $f(x, y) = xe^{xy} \quad \Rightarrow$

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y\,(x, y)\,\mathbf{j} = (xe^{xy} \cdot y + e^{xy})\,\mathbf{i} + (xe^{xy} \cdot x)\,\mathbf{j} = (xy + 1)e^{xy}\,\mathbf{i} + x^2 e^{xy}\,\mathbf{j}$$

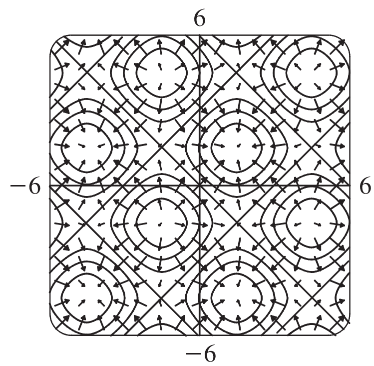
23. $\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\,\mathbf{k}$

25. $f(x, y) = x^2 - y \quad \Rightarrow \quad \nabla f(x, y) = 2x\,\mathbf{i} - \mathbf{j}$.

The length of $\nabla f(x, y)$ is $\sqrt{4x^2 + 1}$. When $x \neq 0$, the vectors point away from the y-axis in a slightly downward direction with length that increases as the distance from the y-axis increases.



27. We graph ∇f along with a contour map of f.



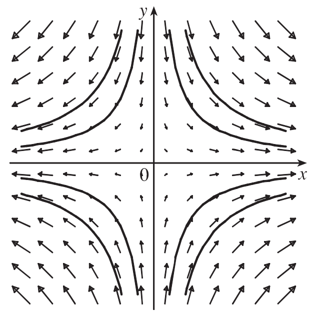
The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

29. $f(x, y) = x^2 + y^2 \quad \Rightarrow \quad \nabla f(x, y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x, y)$ has the same direction and twice the length of the position vector of the point (x, y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph II.

31. $f(x, y) = (x + y)^2 \quad \Rightarrow \quad \nabla f(x, y) = 2(x + y)\,\mathbf{i} + 2(x + y)\,\mathbf{j}$. The x- and y-components of each vector are equal, so all vectors are parallel to the line $y = x$. The vectors are $\mathbf{0}$ along the line $y = -x$ and their length increases as the distance from this line increases. Thus, ∇f is graph II.

33. At $t = 3$ the particle is at $(2, 1)$ so its velocity is $\mathbf{V}(2, 1) = \langle 4, 3\rangle$. After 0.01 units of time, the particle's change in location should be approximately $0.01\,\mathbf{V}(2, 1) = 0.01\,\langle 4, 3\rangle = \langle 0.04, 0.03\rangle$, so the particle should be approximately at the point $(2.04, 1.03)$.

35. (a) We sketch the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines.The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'\,(t)\;\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \quad \Rightarrow \quad dx/dt = x,\, dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \quad \Rightarrow \quad dx/x = dt \quad \Rightarrow \quad \ln|x| = t + C \quad \Rightarrow \quad x = \pm e^{t+C} = Ae^t$ for some constant A, and $dy/dt = -y \quad \Rightarrow \quad dy/y = -dt \quad \Rightarrow \quad \ln|y| = -t + K \quad \Rightarrow \quad y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^tBe^{-t} = AB =$ constant. If the flow line passes through $(1, 1)$ then $(1)\,(1) =$ constant $= 1 \quad \Rightarrow \quad xy = 1 \quad \Rightarrow$ $y = 1/x,\, x > 0$.

17.2 Line Integrals ET 16.2

1. $x = t^3$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y^3\,ds = \int_0^2 t^3\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \int_0^2 t^3\,\sqrt{(3t^2)^2 + (1)^2}\,dt = \int_0^2 t^3\,\sqrt{9t^4 + 1}\,dt$$

$$= \tfrac{1}{36}\cdot\tfrac{2}{3}\left(9t^4 + 1\right)^{3/2}\Big]_0^2 = \tfrac{1}{54}(145^{3/2} - 1) \text{ or } \tfrac{1}{54}\left(145\sqrt{145} - 1\right)$$

3. Parametric equations for C are $x = 4\cos t,\;\; y = 4\sin t,\;\; -\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^5\cos t\,\sin^4 t\,\sqrt{16(\sin^2 t + \cos^2 t)}\,dt$$

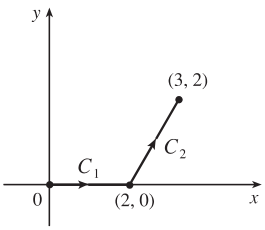
$$= 4^5\int_{-\pi/2}^{\pi/2}(\sin^4 t\cos t)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$

5. If we choose x as the parameter, parametric equations for C are $x = x,\ \ y = \sqrt{x}$ for $1 \le x \le 4$ and

$$\int_C \left(x^2y^3 - \sqrt{x}\right) dy = \int_1^4 \left[x^2 \cdot (\sqrt{x})^3 - \sqrt{x}\right] \frac{1}{2\sqrt{x}}\, dx = \tfrac{1}{2}\int_1^4 \left(x^3 - 1\right) dx$$

$$= \tfrac{1}{2}\left[\tfrac{1}{4}x^4 - x\right]_1^4 = \tfrac{1}{2}\left(64 - 4 - \tfrac{1}{4} + 1\right) = \tfrac{243}{8}$$

7.



$C = C_1 + C_2$

On C_1: $x = x, y = 0 \quad\Rightarrow\quad dy = 0\,dx,\ \ 0 \le x \le 2.$

On C_2: $x = x, y = 2x - 4 \quad\Rightarrow\quad dy = 2\,dx,\ \ 2 \le x \le 3.$

Then

$$\int_C xy\,dx + (x-y)\,dy = \int_{C_1} xy\,dx + (x-y)\,dy + \int_{C_2} xy\,dx + (x-y)\,dy$$

$$= \int_0^2 (0+0)\,dx + \int_2^3 \left[(2x^2 - 4x) + (-x+4)(2)\right] dx$$

$$= \int_2^3 (2x^2 - 6x + 8)\,dx = \tfrac{17}{3}$$

9. $x = 2\sin t,\ \ y = t,\ \ z = -2\cos t,\ \ 0 \le t \le \pi$. Then by Formula 9,

$$\int_C xyz\,ds = \int_0^\pi (2\sin t)(t)(-2\cos t)\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^\pi -4t\sin t\,\cos t\,\sqrt{(2\cos t)^2 + (1)^2 + (2\sin t)^2}\,dt = \int_0^\pi -2t\sin 2t\,\sqrt{4(\cos^2 t + \sin^2 t) + 1}\,dt$$

$$= -2\sqrt{5}\int_0^\pi t\sin 2t\,dt = -2\sqrt{5}\left[-\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t\right]_0^\pi \qquad \left[\begin{array}{c}\text{integrate by parts with}\\ u = t,\ dv = \sin 2t\,dt\end{array}\right]$$

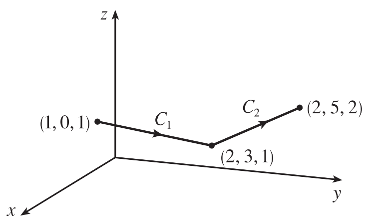
$$= -2\sqrt{5}\left(-\tfrac{\pi}{2} - 0\right) = \sqrt{5}\,\pi$$

11. Parametric equations for C are $x = t,\ \ y = 2t,\ \ z = 3t,\ 0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2 + 2^2 + 3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt = \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}(e^6 - 1).$$

13. $\int_C x^2y\sqrt{z}\,dz = \int_0^1 (t^3)^2(t)\sqrt{t^2}\cdot 2t\,dt = \int_0^1 2t^9\,dt = \left.\tfrac{1}{5}t^{10}\right]_0^1 = \tfrac{1}{5}$

15.



On C_1: $x = 1 + t \quad\Rightarrow\quad dx = dt,\ y = 3t \quad\Rightarrow$

$dy = 3\,dt,\ \ z = 1 \quad\Rightarrow\quad dz = 0\,dt,\ 0 \le t \le 1.$

On C_2: $x = 2 \quad\Rightarrow\quad dx = 0\,dt, y = 3 + 2t \quad\Rightarrow$

$dy = 2\,dt,\ \ z = 1 + t \quad\Rightarrow\quad dz = dt,\ 0 \le t \le 1.$

Then

$$\int_C (x + yz)\,dx + 2x\,dy + xyz\,dz$$

$$= \int_{C_1} (x+yz)\,dx + 2x\,dy + xyz\,dz + \int_{C_2} (x+yz)\,dx + 2x\,dy + xyz\,dz$$

$$= \int_0^1 (1 + t + (3t)(1))\,dt + 2(1+t)\cdot 3\,dt + (1+t)(3t)(1)\cdot 0\,dt$$

$$+ \int_0^1 (2 + (3+2t)(1+t))\cdot 0\,dt + 2(2)\cdot 2\,dt + (2)(3+2t)(1+t)\,dt$$

$$= \int_0^1 (10t + 7)\,dt + \int_0^1 (4t^2 + 10t + 14)\,dt = \left[5t^2 + 7t\right]_0^1 + \left[\tfrac{4}{3}t^3 + 5t^2 + 14t\right]_0^1 = 12 + \tfrac{61}{3} = \tfrac{97}{3}$$

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F} \cdot \mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F} \cdot \mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

19. $\mathbf{r}(t) = 11t^4\,\mathbf{i} + t^3\,\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (11t^4)(t^3)\,\mathbf{i} + 3(t^3)^2\,\mathbf{j} = 11t^7\,\mathbf{i} + 3t^6\,\mathbf{j}$ and $\mathbf{r}'(t) = 44t^3\,\mathbf{i} + 3t^2\,\mathbf{j}$. Then

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 (11t^7 \cdot 44t^3 + 3t^6 \cdot 3t^2)\, dt = \int_0^1 (484t^{10} + 9t^8)\, dt = \left[44t^{11} + t^9\right]_0^1 = 45.$

21. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle \sin t^3, \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle\, dt$

$$= \int_0^1 (3t^2 \sin t^3 - 2t \cos t^2 + t^4)\, dt = \left[-\cos t^3 - \sin t^2 + \tfrac{1}{5}t^5\right]_0^1 = \tfrac{6}{5} - \cos 1 - \sin 1$$

23. $\mathbf{F}(\mathbf{r}(t)) = (e^t)\left(e^{-t^2}\right)\mathbf{i} + \sin\left(e^{-t^2}\right)\mathbf{j} = e^{t-t^2}\,\mathbf{i} + \sin\left(e^{-t^2}\right)\mathbf{j}$, $\mathbf{r}'(t) = e^t\,\mathbf{i} - 2te^{-t^2}\,\mathbf{j}$. Then

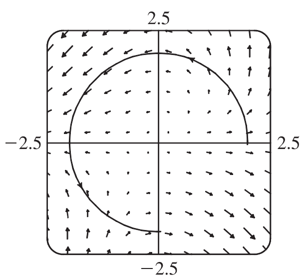
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_1^2 \left[e^{t-t^2} e^t + \sin\left(e^{-t^2}\right) \cdot \left(-2te^{-t^2}\right)\right] dt$$
$$= \int_1^2 \left[e^{2t-t^2} - 2te^{-t^2} \sin\left(e^{-t^2}\right)\right] dt \approx 1.9633$$

25. $x = t^2,\quad y = t^3,\quad z = t^4$ so by Formula 9,

$$\int_C x \sin(y + z)\, ds = \int_0^5 (t^2) \sin(t^3 + t^4) \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2}\, dt$$
$$= \int_0^5 t^2 \sin(t^3 + t^4)\, \sqrt{4t^2 + 9t^4 + 16t^6}\, dt \approx 15.0074$$

27. We graph $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F} \cdot \mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $\;0 \le t \le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t)\,\mathbf{i} + 4\cos t \sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Then



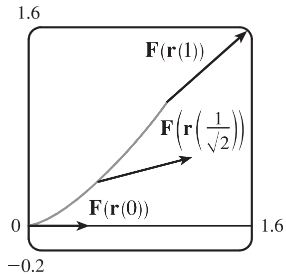
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$
$$= \int_0^{3\pi/2} [-2\sin t(2\cos t - 2\sin t) + 2\cos t(4\cos t \sin t)]\, dt$$
$$= 4\int_0^{3\pi/2} (\sin^2 t - \sin t \cos t + 2\sin t \cos^2 t)\, dt$$
$$= 3\pi + \tfrac{2}{3} \qquad \text{[using a CAS]}$$

29. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5\right\rangle \cdot \langle 2t, 3t^2\rangle\, dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right) dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0\rangle$;

$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle$;

$\mathbf{r}(1) = \langle 1, 1\rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1, 1\rangle$.



In order to generate the graph with Maple, we use the `PLOT` command (not to be confused with the `plot` command) to define each of the vectors. For example,

```
v1:=PLOT(CURVES([[0,0],[evalf(1/exp(1)),0]]));
```

generates the vector from the vector field at the point $(0, 0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

31. $x = e^{-t}\cos 4t$, $y = e^{-t}\sin 4t$, $z = e^{-t}$, $0 \le t \le 2\pi$.

Then $\dfrac{dx}{dt} = e^{-t}(-\sin 4t)(4) - e^{-t}\cos 4t = -e^{-t}(4\sin 4t + \cos 4t)$,

$\dfrac{dy}{dt} = e^{-t}(\cos 4t)(4) - e^{-t}\sin 4t = -e^{-t}(-4\cos 4t + \sin 4t)$, and $\dfrac{dz}{dt} = -e^{-t}$, so

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} &= \sqrt{(-e^{-t})^2[(4\sin 4t + \cos 4t)^2 + (-4\cos 4t + \sin 4t)^2 + 1]} \\ &= e^{-t}\sqrt{16(\sin^2 4t + \cos^2 4t) + \sin^2 4t + \cos^2 4t + 1} = 3\sqrt{2}\,e^{-t}\end{aligned}$$

Therefore

$$\begin{aligned}\int_C x^3y^2z\,ds &= \int_0^{2\pi} (e^{-t}\cos 4t)^3(e^{-t}\sin 4t)^2(e^{-t})\,(3\sqrt{2}\,e^{-t})\,dt \\ &= \int_0^{2\pi} 3\sqrt{2}\,e^{-7t}\cos^3 4t\sin^2 4t\,dt = \tfrac{172{,}704}{5{,}632{,}705}\sqrt{2}\,(1 - e^{-14\pi})\end{aligned}$$

33. We use the parametrization $x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = 2\,dt$, so $m = \int_C k\,ds = 2k\int_{-\pi/2}^{\pi/2} dt = 2k(\pi)$,

$\overline{x} = \frac{1}{2\pi k}\int_C xk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt = \frac{1}{2\pi}\left[4\sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$, $\overline{y} = \frac{1}{2\pi k}\int_C yk\,ds = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt = 0$.

Hence $(\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, 0\right)$.

35. (a) $\overline{x} = \dfrac{1}{m}\displaystyle\int_C x\rho(x,y,z)\,ds$, $\overline{y} = \dfrac{1}{m}\displaystyle\int_C y\rho(x,y,z)\,ds$, $\overline{z} = \dfrac{1}{m}\displaystyle\int_C z\rho(x,y,z)\,ds$ where $m = \int_C \rho(x,y,z)\,ds$.

(b) $m = \int_C k\,ds = k\int_0^{2\pi}\sqrt{4\sin^2 t + 4\cos^2 t + 9}\,dt = k\sqrt{13}\int_0^{2\pi} dt = 2\pi k\sqrt{13}$,

$\overline{x} = \dfrac{1}{2\pi k\sqrt{13}}\displaystyle\int_0^{2\pi} 2k\sqrt{13}\,\sin t\,dt = 0$, $\overline{y} = \dfrac{1}{2\pi k\sqrt{13}}\displaystyle\int_0^{2\pi} 2k\sqrt{13}\,\cos t\,dt = 0$,

$\overline{z} = \dfrac{1}{2\pi k\sqrt{13}}\displaystyle\int_0^{2\pi}\left(k\sqrt{13}\right)(3t)\,dt = \dfrac{3}{2\pi}\left(2\pi^2\right) = 3\pi$. Hence $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 3\pi)$.

37. From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \Rightarrow$

$$I_x = \int_C y^2 \rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi (\sin^2 t - \sin^3 t)\,dt$$

$$= \tfrac{1}{2}k\int_0^\pi (1-\cos 2t)\,dt - k\int_0^\pi (1-\cos^2 t)\sin t\,dt \qquad \left[\begin{array}{c}\text{Let } u = \cos t,\ du = -\sin t\,dt\\ \text{in the second integral}\end{array}\right]$$

$$= k\left[\frac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\frac{\pi}{2} - \frac{4}{3}\right)$$

$$I_y = \int_C x^2 \rho(x,y)\,ds = k\int_0^\pi \cos^2 t\,(1-\sin t)\,dt = \frac{k}{2}\int_0^\pi (1+\cos 2t)\,dt - k\int_0^\pi \cos^2 t\sin t\,dt$$

$= k\left(\frac{\pi}{2} - \frac{2}{3}\right)$, using the same substitution as above.

39. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle t-\sin t, 3-\cos t\rangle \cdot \langle 1-\cos t, \sin t\rangle\,dt$

$$= \int_0^{2\pi} (t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (t - t\cos t + 2\sin t)\,dt = \left[\tfrac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi} \qquad \left[\begin{array}{c}\text{integrate by parts}\\ \text{in the second term}\end{array}\right]$$

$$= 2\pi^2$$

41. $\mathbf{r}(t) = \langle 1+2t, 4t, 2t\rangle$, $0 \le t \le 1$,

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle 6t, 1+4t, 1+6t\rangle \cdot \langle 2,4,2\rangle\,dt = \int_0^1 (12t + 4(1+4t) + 2(1+6t))\,dt$$

$$= \int_0^1 (40t+6)\,dt = \left[20t^2 + 6t\right]_0^1 = 26$$

43. Let $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t$, $y = 20\sin t$, $z = \frac{90}{6\pi}t = \frac{15}{\pi}t$, $0 \le t \le 6\pi \Rightarrow$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{6\pi} \langle 0,0,185\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi}\right\rangle dt = (185)\tfrac{15}{\pi}\int_0^{6\pi} dt = (185)(90) \approx 1.67\times 10^4 \text{ ft-lb}$$

45. (a) $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, and let $\mathbf{F} = \langle a, b\rangle$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle a,b\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} (-a\sin t + b\cos t)\,dt = \left[a\cos t + b\sin t\right]_0^{2\pi}$$

$$= a + 0 - a + 0 = 0$$

(b) Yes. $\mathbf{F}(x,y) = k\,\mathbf{x} = \langle kx, ky\rangle$ and

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle k\cos t, k\sin t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} (-k\sin t\,\cos t + k\sin t\,\cos t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

47. The work done in moving the object is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F}\cdot\mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F}\cdot\mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is

$\int_C \mathbf{F}\cdot\mathbf{T}\,ds \approx \sum_{i=1}^{7} [\mathbf{F}(x_i^*, y_i^*)\cdot\mathbf{T}(x_i^*, y_i^*)]\,\Delta s = [2+2+2+2+1+1+1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

17.3 The Fundamental Theorem for Line Integrals ET 16.3

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f \cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

3. $\partial(2x - 3y)/\partial y = -3 = \partial(-3x + 4y - 8)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 2x - 3y$ and $f_y(x,y) = -3x + 4y - 8$. But $f_x(x,y) = 2x - 3y$ implies $f(x,y) = x^2 - 3xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = -3x + g'(y)$. Thus $-3x + 4y - 8 = -3x + g'(y)$ so $g'(y) = 4y - 8$ and $g(y) = 2y^2 - 8y + K$ where K is a constant. Hence $f(x,y) = x^2 - 3xy + 2y^2 - 8y + K$ is a potential function for $\mathbf{F}$.

5. $\partial(e^x \sin y)/\partial y = e^x \cos y = \partial(e^x \cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = e^x \sin y$ implies $f(x,y) = e^x \sin y + g(y)$ and $f_y(x,y) = e^x \cos y + g'(y)$. But $f_y(x,y) = e^x \cos y$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x,y) = e^x \sin y + K$ is a potential function for $\mathbf{F}$.

7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

9. $\partial(\ln y + 2xy^3)/\partial y = 1/y + 6xy^2 = \partial(3x^2y^2 + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid y > 0\}$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = \ln y + 2xy^3$ implies $f(x,y) = x\ln y + x^2y^3 + g(y)$ and $f_y(x,y) = x/y + 3x^2y^2 + g'(y)$. But $f_y(x,y) = 3x^2y^2 + x/y$ so $g'(y) = 0 \Rightarrow g(y) = K$ and $f(x,y) = x\ln y + x^2y^3 + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}\,2xy = 2x = \dfrac{\partial}{\partial x}\,(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant. Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

13. (a) $f_x(x,y) = xy^2$ implies $f(x,y) = \frac{1}{2}x^2y^2 + g(y)$ and $f_y(x,y) = x^2y + g'(y)$. But $f_y(x,y) = x^2y$ so $g'(y) = 0 \Rightarrow g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{2}x^2y^2$.

(b) The initial point of C is $\mathbf{r}(0) = (0, 1)$ and the terminal point is $\mathbf{r}(1) = (2, 1)$, so

$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(0, 1) = 2 - 0 = 2$.

15. (a) $f_x(x, y, z) = yz$ implies $f(x, y, z) = xyz + g(y, z)$ and so $f_y(x, y, z) = xz + g_y(y, z)$. But $f_y(x, y, z) = xz$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xyz + h(z)$ and $f_z(x, y, z) = xy + h'(z)$. But $f_z(x, y, z) = xy + 2z$, so $h'(z) = 2z \Rightarrow h(z) = z^2 + K$. Hence $f(x, y, z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77$.

17. (a) $f_x(x, y, z) = y^2 \cos z$ implies $f(x, y, z) = xy^2 \cos z + g(y, z)$ and so $f_y(x, y, z) = 2xy \cos z + g_y(y, z)$. But $f_y(x, y, z) = 2xy \cos z$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xy^2 \cos z + h(z)$ and $f_z(x, y, z) = -xy^2 \sin z + h'(z)$. But $f_z(x, y, z) = -xy^2 \sin z$, so $h'(z) = 0 \Rightarrow h(z) = K$. Hence $f(x, y, z) = xy^2 \cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(\pi) = \langle \pi^2, 0, \pi \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^2, 0, \pi) - f(0, 0, 0) = 0 - 0 = 0$.

19. Here $\mathbf{F}(x, y) = \tan y\, \mathbf{i} + x \sec^2 y\, \mathbf{j}$. Then $f(x, y) = x \tan y$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence

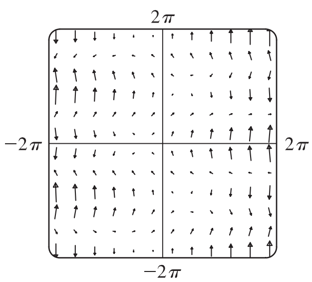
$\int_C \tan y\, dx + x \sec^2 y\, dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f\left(2, \frac{\pi}{4}\right) - f(1, 0) = 2 \tan \frac{\pi}{4} - \tan 0 = 2$.

21. $\mathbf{F}(x, y) = 2y^{3/2}\, \mathbf{i} + 3x\sqrt{y}\, \mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\partial(2y^{3/2})/\partial y = 3\sqrt{y} = \partial(3x\sqrt{y})/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x, y) = 2y^{3/2} \Rightarrow f(x, y) = 2xy^{3/2} + g(y) \Rightarrow f_y(x, y) = 3xy^{1/2} + g'(y)$. But $f_y(x, y) = 3x\sqrt{y}$ so $g'(y) = 0$ or $g(y) = K$. We can take $K = 0 \Rightarrow f(x, y) = 2xy^{3/2}$. Thus

$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 4) - f(1, 1) = 2(2)(8) - 2(1) = 30$.

23. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on C are roughly in the direction of motion along C, so the integral around C will be positive. Therefore the field is not conservative.

25.



From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate

$\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(1 + x \cos y)$. Thus $\mathbf{F}$ is conservative, by Theorem 6.

27. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

29. $D = \{(x, y) \mid x > 0, y > 0\} =$ the first quadrant (excluding the axes).

(a) D is open because around every point in D we can put a disk that lies in D.

(b) D is connected because the straight line segment joining any two points in D lies in D.

(c) D is simply-connected because it's connected and has no holes.

31. $D = \{(x, y) \mid 1 < x^2 + y^2 < 4\} =$ the annular region between the circles with center $(0, 0)$ and radii 1 and 2.

(a) D is open.

(b) D is connected.

(c) D is not simply-connected. For example, $x^2 + y^2 = (1.5)^2$ is simple and closed and lies within D but encloses points that are not in D. (Or we can say, D has a hole, so is not simply-connected.)

33. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^{\pi} dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

17.4 Green's Theorem ET 16.4

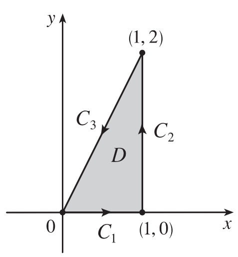
1. (a) Parametric equations for C are $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Then

$$\oint_C (x - y)\, dx + (x + y)\, dy = \int_0^{2\pi} [(2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)]\, dt$$
$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\, dt = \int_0^{2\pi} 4\, dt = 4t\Big]_0^{2\pi} = 8\pi$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$$\oint_C (x - y)\, dx + (x + y)\, dy = \iint_D \left[\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x - y)\right] dA = \iint_D [1 - (-1)]\, dA = 2\iint_D dA$$
$$= 2A(D) = 2\pi(2)^2 = 8\pi$$

3. (a)



C_1: $x = t \;\Rightarrow\; dx = dt$, $y = 0 \;\Rightarrow\; dy = 0\, dt$, $0 \le t \le 1$.

C_2: $x = 1 \;\Rightarrow\; dx = 0\, dt$, $y = t \;\Rightarrow\; dy = dt$, $0 \le t \le 2$.

C_3: $x = 1 - t \;\Rightarrow\; dx = -dt$, $y = 2 - 2t \;\Rightarrow\; dy = -2\, dt$, $0 \le t \le 1$.

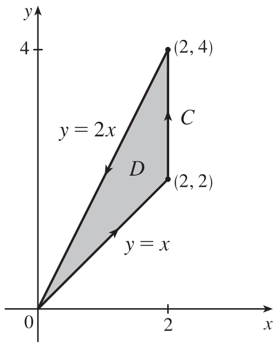
Thus

$$\oint_C xy\, dx + x^2 y^3\, dy = \oint_{C_1 + C_2 + C_3} xy\, dx + x^2 y^3\, dy$$
$$= \int_0^1 0\, dt + \int_0^2 t^3\, dt + \int_0^1 \left[-(1 - t)(2 - 2t) - 2(1 - t)^2 (2 - 2t)^3\right] dt$$
$$= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \left[\tfrac{2}{3}(1 - t)^3 + \tfrac{8}{3}(1 - t)^6\right]_0^1 = 4 - \tfrac{10}{3} = \tfrac{2}{3}$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}$$

5.



The region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, x \le y \le 2x\}$, so

$$\begin{aligned}\int_C xy^2\,dx + 2x^2y\,dy &= \iint_D \left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA \\ &= \int_0^2 \int_x^{2x} (4xy - 2xy)\,dy\,dx \\ &= \int_0^2 \left[xy^2\right]_{y=x}^{y=2x} dx \\ &= \int_0^2 3x^3\,dx = \tfrac{3}{4}x^4\Big]_0^2 = 12\end{aligned}$$

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1)\,dx\,dy = \int_0^1 (y^{1/2} - y^2)\,dy = \tfrac{1}{3}$$

9. $\int_C y^3\,dx - x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 (-3r^2)\,r\,dr\,d\theta$

$$= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3(2\pi)(4) = -24\pi$$

11. $\mathbf{F}(x,y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y}\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} \left(\sqrt{x} + y^3\right) dx + \left(x^2 + \sqrt{y}\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}\left(\sqrt{x} + y^3\right)\right] dA \\ &= -\int_0^\pi \int_0^{\sin x} (2x - 3y^2)\,dy\,dx = -\int_0^\pi \left[2xy - y^3\right]_{y=0}^{y=\sin x} dx \\ &= -\int_0^\pi (2x\sin x - \sin^3 x)\,dx = -\int_0^\pi (2x\sin x - (1 - \cos^2 x)\sin x)\,dx \\ &= -\left[2\sin x - 2x\cos x + \cos x - \tfrac{1}{3}\cos^3 x\right]_0^\pi \qquad \text{[integrate by parts in the first term]} \\ &= -\left(2\pi - 2 + \tfrac{2}{3}\right) = \tfrac{4}{3} - 2\pi\end{aligned}$$

13. $\mathbf{F}(x,y) = \left\langle e^x + x^2y, e^y - xy^2\right\rangle$ and the region D enclosed by C is the disk $x^2 + y^2 \le 25$.

C is traversed clockwise, so $-C$ gives the positive orientation.

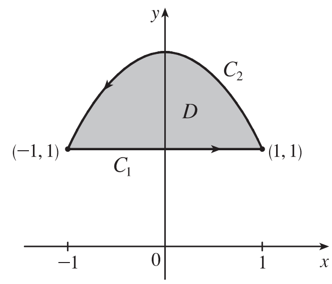
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\int_{-C} (e^x + x^2y)\,dx + (e^y - xy^2)\,dy = -\iint_D \left[\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y)\right] dA \\ &= -\iint_D (-y^2 - x^2)\,dA = \iint_D (x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^5 (r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^5 r^3\,dr = 2\pi\left[\tfrac{1}{4}r^4\right]_0^5 = \tfrac{625}{2}\pi\end{aligned}$$

15. Here $C = C_1 + C_2$ where

C_1 can be parametrized as $x = t,\ \ y = 1,\ \ -1 \le t \le 1$, and

C_2 is given by $x = -t,\ \ y = 2 - t^2,\ \ -1 \le t \le 1$.

Then the line integral is



$$\oint_{C_1+C_2} y^2e^x\,dx + x^2e^y\,dy = \int_{-1}^{1}[1\cdot e^t + t^2e\cdot 0]\,dt + \int_{-1}^{1}[(2-t^2)^2e^{-t}(-1) + (-t)^2e^{2-t^2}(-2t)]\,dt$$

$$= \int_{-1}^{1}[e^t - (2-t^2)^2e^{-t} - 2t^3e^{2-t^2}]\,dt = -8e + 48e^{-1}$$

according to a CAS. The double integral is

$$\iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \int_{-1}^{1}\int_{1}^{2-x^2}(2xe^y - 2ye^x)\,dy\,dx = -8e + 48e^{-1},$$ verifying Green's Theorem in this case.

17. By Green's Theorem, $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dy\,dx$ where C is the path described in the question and D is the triangle bounded by C. So

$$W = \int_0^1\int_0^{1-x}(y^2-x)\,dy\,dx = \int_0^1\left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x}dx = \int_0^1\left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right)dx$$

$$= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2}+\tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}$$

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi,0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi,0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt)$$

$$= \int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$$

21. (a) Using Equation 17.2.8 [ET 16.2.8], we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$$

$$= \int_0^1 (x_1(y_2-y_1) - y_1(x_2-x_1) + t[(y_2-y_1)(x_2-x_1) - (x_2-x_1)(y_2-y_1)])\,dt$$

$$= \int_0^1 (x_1y_2 - x_2y_1)\,dt = x_1y_2 - x_2y_1$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon} = A(D) = \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx$$

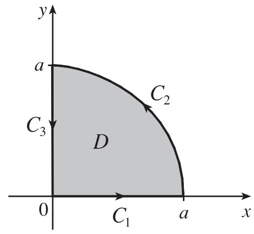
$$= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$$

To evaluate these integrals we use the formula from (a) to get

$A(D) = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$.

(c) $A = \frac{1}{2}[(0 \cdot 1 - 2 \cdot 0) + (2 \cdot 3 - 1 \cdot 1) + (1 \cdot 2 - 0 \cdot 3) + (0 \cdot 1 - (-1) \cdot 2) + (-1 \cdot 0 - 0 \cdot 1)]$
$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2} \displaystyle\oint_C x^2\, dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2} \displaystyle\oint_C y^2 dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t, \quad y = 0, \quad 0 \le t \le a$;

C_2: $x = a\cos t, \quad y = a \sin t, \quad 0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0, y = a - t, 0 \le t \le a$. Then

$$\oint_C x^2\, dy = \int_{C_1} x^2\, dy + \int_{C_2} x^2\, dy + \int_{C_3} x^2\, dy = \int_0^a 0\, dt + \int_0^{\pi/2} (a\cos t)^2 (a \cos t)\, dt + \int_0^a 0\, dt$$
$$= \int_0^{\pi/2} a^3 \cos^3 t\, dt = a^3 \int_0^{\pi/2} (1 - \sin^2 t) \cos t\, dt = a^3 \left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3$$

so $\overline{x} = \dfrac{1}{\pi a^2/2} \displaystyle\oint_C x^2\, dy = \dfrac{4a}{3\pi}$.

$$\oint_C y^2 dx = \int_{C_1} y^2\, dx + \int_{C_2} y^2\, dx + \int_{C_3} y^2\, dx = \int_0^a 0\, dt + \int_0^{\pi/2} (a\sin t)^2(-a \sin t)\, dt + \int_0^a 0\, dt$$
$$= \int_0^{\pi/2} (-a^3 \sin^3 t)\, dt = -a^3 \int_0^{\pi/2} (1 - \cos^2 t)\sin t\, dt = -a^3 \left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2} \displaystyle\oint_C y^2 dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x}, \overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho \oint_C y^3\, dx = -\frac{1}{3}\rho \iint_D (-3y^2)\, dA = \iint_D y^2 \rho\, dA = I_x$ and

$\frac{1}{3}\rho \oint_C x^3\, dy = \frac{1}{3}\rho \iint_D (3x^2)\, dA = \iint_D x^2 \rho\, dA = I_y$.

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2 + y^2)$ and $Q = x/(x^2 + y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 0\, dA = 0$.

29. Using the first part of (5), we have that $\iint_R dx\, dy = A(R) = \int_{\partial R} x\, dy$. But $x = g(u, v)$, and $dy = \dfrac{\partial h}{\partial u}\, du + \dfrac{\partial h}{\partial v}\, dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\int_{\partial R} x\, dy = \int_{\partial S} g(u, v)\left(\frac{\partial h}{\partial u}\, du + \frac{\partial h}{\partial v}\, dv\right) = \int_{\partial S} g(u, v)\, \frac{\partial h}{\partial u}\, du + g(u, v)\, \frac{\partial h}{\partial v}\, dv$$
$$= \pm \iint_S \left[\tfrac{\partial}{\partial u}\left(g(u, v)\, \tfrac{\partial h}{\partial v}\right) - \tfrac{\partial}{\partial v}\left(g(u, v)\, \tfrac{\partial h}{\partial u}\right)\right] dA \qquad \text{[using Green's Theorem in the } uv\text{-plane]}$$
$$= \pm \iint_S \left(\tfrac{\partial g}{\partial u}\tfrac{\partial h}{\partial v} + g(u, v)\, \tfrac{\partial^2 h}{\partial u\, \partial v} - \tfrac{\partial g}{\partial v}\tfrac{\partial h}{\partial u} - g(u, v)\, \tfrac{\partial^2 h}{\partial v\, \partial u}\right) dA \qquad \text{[using the Chain Rule]}$$
$$= \pm \iint_S \left(\tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial v} - \tfrac{\partial x}{\partial v}\tfrac{\partial y}{\partial u}\right) dA \quad \text{[by the equality of mixed partials]} \quad = \pm \iint_S \tfrac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial\,(x, y)}{\partial(u, v)}$.

Therefore $A(R) = \displaystyle\iint_R dx\, dy = \iint_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$.

17.5 Curl and Divergence ET 16.5

1. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 0 & -x^2y \end{vmatrix} = (-x^2 - 0)\,\mathbf{i} - (-2xy - xy)\,\mathbf{j} + (0 - xz)\,\mathbf{k}$

$= -x^2\,\mathbf{i} + 3xy\,\mathbf{j} - xz\,\mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(xyz) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(-x^2y) = yz + 0 + 0 = yz$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \sqrt{z} \end{vmatrix} = (x - y)\,\mathbf{i} - (y - 0)\,\mathbf{j} + (1 - 0)\,\mathbf{k}$

$= (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(1) + \dfrac{\partial}{\partial y}(x + yz) + \dfrac{\partial}{\partial z}\left(xy - \sqrt{z}\right) = z - \dfrac{1}{2\sqrt{z}}$

5. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$

$= \dfrac{1}{(x^2+y^2+z^2)^{3/2}}\left[(-yz + yz)\,\mathbf{i} - (-xz + xz)\,\mathbf{j} + (-xy + xy)\,\mathbf{k}\right] = \mathbf{0}$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$= \dfrac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \dfrac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \dfrac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}} = \dfrac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix} = \left(\dfrac{xz}{xyz} - 0\right)\mathbf{i} - \left(\dfrac{yz}{xyz} - 0\right)\mathbf{j} + \left(\dfrac{y}{xy} - 0\right)\mathbf{k} = \left\langle \dfrac{1}{y}, -\dfrac{1}{x}, \dfrac{1}{x} \right\rangle$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(\ln x) + \dfrac{\partial}{\partial y}(\ln(xy)) + \dfrac{\partial}{\partial z}(\ln(xyz)) = \dfrac{1}{x} + \dfrac{x}{xy} + \dfrac{xy}{xyz} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0 - 0)\,\mathbf{i} + (0 - 0)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = \mathbf{0}$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial y} = \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. P increases as y increases, so $\dfrac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = -\dfrac{\partial P}{\partial y}\mathbf{k}$

Since $\dfrac{\partial P}{\partial y} > 0$, $-\dfrac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

13. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k} = \mathbf{0}$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x,y,z) = y^2z^3$ implies $f(x,y,z) = xy^2z^3 + g(y,z)$ and $f_y(x,y,z) = 2xyz^3 + g_y(y,z)$. But $f_y(x,y,z) = 2xyz^3$, so $g(y,z) = h(z)$ and $f(x,y,z) = xy^2z^3 + h(z)$. Thus $f_z(x,y,z) = 3xy^2z^2 + h'(z)$ but $f_z(x,y,z) = 3xy^2z^2$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xy^2z^3 + K$.

15. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 + 2yz & y^2 \end{vmatrix} = (2y - 2y)\,\mathbf{i} - (0-0)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$,

and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = 2xy$ implies $f(x,y,z) = x^2y + g(y,z)$ and $f_y(x,y,z) = x^2 + g_y(y,z)$. But $f_y(x,y,z) = x^2 + 2yz$, so $g(y,z) = y^2z + h(z)$ and $f(x,y,z) = x^2y + y^2z + h(z)$. Thus $f_z(x,y,z) = y^2 + h'(z)$ but $f_z(x,y,z) = y^2$ so $h(z) = K$ and $f(x,y,z) = x^2y + y^2z + K$.

17. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-e^{-x} - e^{-x})\,\mathbf{k} = -2e^{-x}\,\mathbf{k} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \dfrac{\partial}{\partial x}(x\sin y) + \dfrac{\partial}{\partial y}(\cos y) + \dfrac{\partial}{\partial z}(z - xy) = \sin y - \sin y + 1 \neq 0$, which contradicts Theorem 11.

21. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$. Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$

is irrotational.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

23. $$\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\langle P_1+P_2, Q_1+Q_2, R_1+R_2\rangle = \frac{\partial(P_1+P_2)}{\partial x} + \frac{\partial(Q_1+Q_2)}{\partial y} + \frac{\partial(R_1+R_2)}{\partial z}$$
$$= \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_1}{\partial z} + \frac{\partial R_2}{\partial z} = \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)$$
$$= \operatorname{div}\langle P_1, Q_1, R_1\rangle + \operatorname{div}\langle P_2, Q_2, R_2\rangle = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$$

25. $$\operatorname{div}(f\mathbf{F}) = \operatorname{div}(f\,\langle P_1, Q_1, R_1\rangle) = \operatorname{div}\langle fP_1, fQ_1, fR_1\rangle = \frac{\partial(fP_1)}{\partial x} + \frac{\partial(fQ_1)}{\partial y} + \frac{\partial(fR_1)}{\partial z}$$
$$= \left(f\,\frac{\partial P_1}{\partial x} + P_1\,\frac{\partial f}{\partial x}\right) + \left(f\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial f}{\partial y}\right) + \left(f\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial f}{\partial z}\right)$$
$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

27. $$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \frac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \frac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$$
$$= \left[Q_1\,\frac{\partial R_2}{\partial x} + R_2\,\frac{\partial Q_1}{\partial x} - Q_2\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial Q_2}{\partial x}\right] - \left[P_1\,\frac{\partial R_2}{\partial y} + R_2\,\frac{\partial P_1}{\partial y} - P_2\,\frac{\partial R_1}{\partial y} - R_1\,\frac{\partial P_2}{\partial y}\right]$$
$$+ \left[P_1\,\frac{\partial Q_2}{\partial z} + Q_2\,\frac{\partial P_1}{\partial z} - P_2\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial P_2}{\partial z}\right]$$
$$= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$
$$- \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$
$$= \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

29. $$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla\times(\nabla\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$$
$$= \left(\frac{\partial^2 Q_1}{\partial y\partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z\partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x\partial y}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x\partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y\partial z}\right)\mathbf{k}$$

Now let's consider $\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$ and compare with the above.

(Note that $\nabla^2\mathbf{F}$ is defined on page 1102 [ET 1066].)

$$\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x\partial y} + \frac{\partial^2 R_1}{\partial x\partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z\partial x} + \frac{\partial^2 Q_1}{\partial z\partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$= \left(\frac{\partial^2 Q_1}{\partial x\partial y} + \frac{\partial^2 R_1}{\partial x\partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y\partial x} + \frac{\partial^2 R_1}{\partial y\partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z\partial x} + \frac{\partial^2 Q_1}{\partial z\partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

31. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{r}$

(b) $\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left[\dfrac{\partial}{\partial y}(z) - \dfrac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\dfrac{\partial}{\partial z}(x) - \dfrac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\dfrac{\partial}{\partial x}(y) - \dfrac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$

(c) $\nabla\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla\ln r = \nabla\ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\nabla\ln(x^2+y^2+z^2)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

33. By (13), $\oint_C f(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$. Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

35. Let $f(x,y)=1$. Then $\nabla f = \mathbf{0}$ and Green's first identity (see Exercise 33) says $\iint_D \nabla^2 g\,dA = \oint_C (\nabla g)\cdot\mathbf{n}\,ds - \iint_D \mathbf{0}\cdot\nabla g\,dA \;\Rightarrow\; \iint_D \nabla^2 g\,dA = \oint_C \nabla g\cdot\mathbf{n}\,ds$. But g is harmonic on D, so $\nabla^2 g = 0 \;\Rightarrow\; \oint_C \nabla g\cdot\mathbf{n}\,ds = 0$ and $\oint_C D_{\mathbf{n}}g\,ds = \oint_C(\nabla g\cdot\mathbf{n})\,ds = 0$.

37. (a) We know that $\omega = v/d$, and from the diagram $\sin\theta = d/r \;\Rightarrow\; v = d\omega = (\sin\theta)r\omega = |\mathbf{w}\times\mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0 \cdot z - \omega y)\,\mathbf{i} + (\omega x - 0 \cdot z)\,\mathbf{j} + (0 \cdot y - x \cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

(c) $\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k}$$

$$= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}$$

39. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$ where $g(x, y, z) = \int_0^x f(t, y, z)\,dt$.

Then $\operatorname{div} \mathbf{G} = \dfrac{\partial}{\partial x}(g(x, y, z)) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(0) = \dfrac{\partial}{\partial x} \int_0^x f(t, y, z)\,dt = f(x, y, z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

17.6 Parametric Surfaces and Their Areas ET 16.6

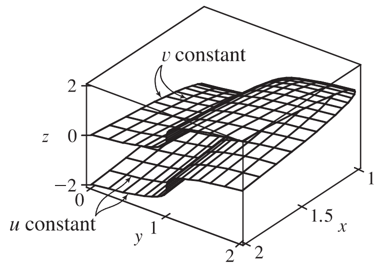
1. $P(7, 10, 4)$ lies on the parametric surface $\mathbf{r}(u, v) = \langle 2u + 3v, 1 + 5u - v, 2 + u + v \rangle$ if and only if there are values for u and v where $2u + 3v = 7$, $1 + 5u - v = 10$, and $2 + u + v = 4$. But solving the first two equations simultaneously gives $u = 2$, $v = 1$ and these values do not satisfy the third equation, so P does not lie on the surface.

$Q(5, 22, 5)$ lies on the surface if $2u + 3v = 5$, $1 + 5u - v = 22$, and $2 + u + v = 5$ for some values of u and v. Solving the first two equations simultaneously gives $u = 4$, $v = -1$ and these values satisfy the third equation, so Q lies on the surface.

3. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k} = \langle 0, 3, 1 \rangle + u\,\langle 1, 0, 4 \rangle + v\,\langle 1, -1, 5 \rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0, 3, 1)$ and containing vectors $\mathbf{a} = \langle 1, 0, 4 \rangle$ and $\mathbf{b} = \langle 1, -1, 5 \rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x - 0) - (y - 3) - (z - 1) = 0$ or $4x - y - z = -4$.

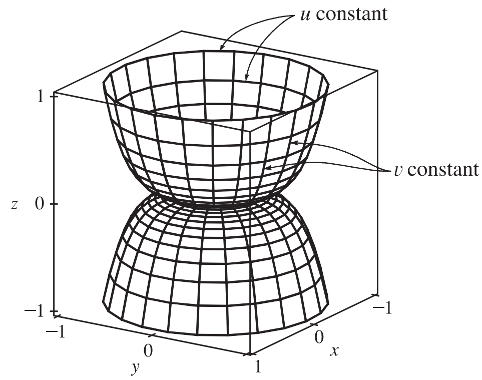
5. $\mathbf{r}(s, t) = \left\langle s, t, t^2 - s^2 \right\rangle$, so the corresponding parametric equations for the surface are $x = s$, $y = t$, $z = t^2 - s^2$. For any point (x, y, z) on the surface, we have $z = y^2 - x^2$. With no restrictions on the parameters, the surface is $z = y^2 - x^2$, which we recognize as a hyperbolic paraboloid.

7. $\mathbf{r}(u,v) = \langle u^2+1, v^3+1, u+v \rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.



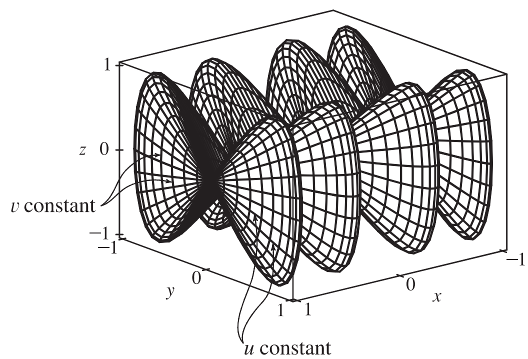
The surface has parametric equations $x = u^2+1$, $y = v^3+1$, $z = u+v$, $-1 \le u \le 1$, $-1 \le v \le 1$. In Maple, the surface can be graphed by entering `plot3d([u^2+1,v^3+1,u+v],u=-1..1,v=-1..1);`. In Mathematica we use the `ParametricPlot3D` command. If we keep u constant at u_0, $x = u_0^2+1$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^3+1$, a constant, so these grid curves are the curves parallel to the xz-plane.

9. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^5 \rangle$.



The surface has parametric equations $x = u\cos v$, $y = u\sin v$, $z = u^5$, $-1 \le u \le 1$, $0 \le v \le 2\pi$. Note that if $u = u_0$ is constant then $z = u_0^5$ is constant and $x = u_0\cos v$, $y = u_0\sin v$ describe a circle in x, y of radius $|u_0|$, so the corresponding grid curves are circles parallel to the xy-plane. If $v = v_0$, a constant, the parametric equations become $x = u\cos v_0$, $y = u\sin v_0$, $z = u^5$. Then $y = (\tan v_0)x$, so these are the grid curves we see that lie in vertical planes $y = kx$ through the z-axis.

11. $x = \sin v$, $y = \cos u \sin 4v$, $z = \sin 2u \sin 4v$, $0 \le u \le 2\pi$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.



Note that if $v = v_0$ is constant, then $x = \sin v_0$ is constant, so the corresponding grid curves must be parallel to the yz-plane. These are the vertically oriented grid curves we see, each shaped like a "figure-eight." When $u = u_0$ is held constant, the parametric equations become $x = \sin v$, $y = \cos u_0 \sin 4v$, $z = \sin 2u_0 \sin 4v$. Since z is a constant multiple of y, the corresponding grid curves are the curves contained in planes $z = ky$ that pass through the x-axis.

13. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$. The parametric equations for the surface are $x = u\cos v$, $y = u\sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph I.

15. $\mathbf{r}(u,v) = \sin v\,\mathbf{i} + \cos u\,\sin 2v\,\mathbf{j} + \sin u\,\sin 2v\,\mathbf{k}$. Parametric equations for the surface are $x = \sin v$, $y = \cos u\,\sin 2v$, $z = \sin u\,\sin 2v$. If $v = v_0$ is fixed, then $x = \sin v_0$ is constant, and $y = (\sin 2v_0)\cos u$ and $z = (\sin 2v_0)\sin u$ describe a circle of radius $|\sin 2v_0|$, so each corresponding grid curve is a circle contained in the vertical plane $x = \sin v_0$ parallel to the

yz-plane. The only possible surface is graph II. The grid curves we see running lengthwise along the surface correspond to holding u constant, in which case $y = (\cos u_0)\sin 2v$, $z = (\sin u_0)\sin 2v \Rightarrow z = (\tan u_0)y$, so each grid curve lies in a plane $z = ky$ that includes the x-axis.

17. $x = \cos^3 u \cos^3 v$, $y = \sin^3 u \cos^3 v$, $z = \sin^3 v$. If $v = v_0$ is held constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curve lies in a horizontal plane. Several of the graphs exhibit horizontal grid curves, but the curves for this surface are neither circles nor straight lines, so graph III is the only possibility. (In fact, the horizontal grid curves here are members of the family $x = a\cos^3 u$, $y = a\sin^3 u$ and are called astroids.) The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, as then we have $x = \cos^3 u_0 \cos^3 v$, $y = \sin^3 u_0 \cos^3 v$ so the corresponding grid curve lies in the vertical plane $y = (\tan^3 u_0)x$ through the z-axis.

19. From Example 3, parametric equations for the plane through the point $(1, 2, -3)$ that contains the vectors $\mathbf{a} = \langle 1, 1, -1\rangle$ and $\mathbf{b} = \langle 1, -1, 1\rangle$ are $x = 1 + u(1) + v(1) = 1 + u + v$, $y = 2 + u(1) + v(-1) = 2 + u - v$, $z = -3 + u(-1) + v(1) = -3 - u + v$.

21. Solving the equation for y gives $y^2 = 1 - x^2 + z^2 \Rightarrow y = \sqrt{1 - x^2 + z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $y \geq 0$.) If we let x and z be the parameters, parametric equations are $x = x$, $z = z$, $y = \sqrt{1 - x^2 + z^2}$.

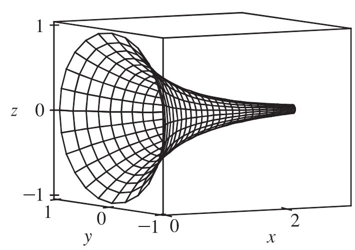
23. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

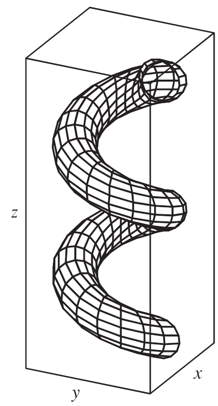
25. Parametric equations are $x = x$, $y = 4\cos\theta$, $z = 4\sin\theta$, $0 \leq x \leq 5$, $0 \leq \theta \leq 2\pi$.

27. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \leq x \leq 5$, $y \leq 0$ to obtain the portion shown. To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter domain $0 \leq u \leq 5$, $\frac{\pi}{2} \leq v \leq \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \leq x \leq 5$ and $-3 \leq z \leq 3$.

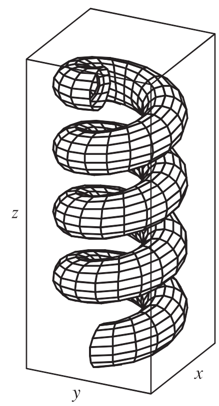
29. Using Equations 3, we have the parametrization $x = x$, $y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $0 \leq x \leq 3$, $0 \leq \theta \leq 2\pi$.



31. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



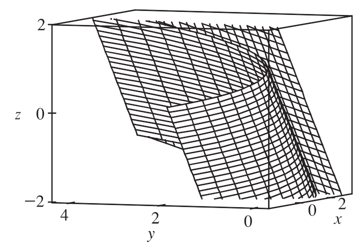
(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.



33. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$.

Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.



35. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}(1, 0) = (1, 0, 1)$.

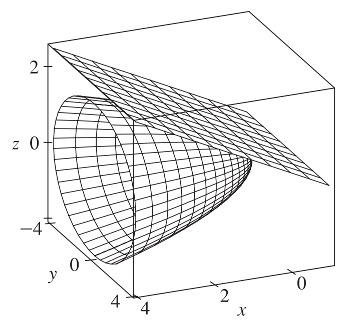
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1, 0, 1)$ is

$\mathbf{r}_u(1, 0) \times \mathbf{r}_v(1, 0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1, 0, 1)$ is

$-2(x - 1) + 0(y - 0) + 4(z - 1) = 0$ or $-x + 2z = 1$.



37. The surface S is given by $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}. \end{aligned}$$

39. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (\sqrt{x})^2 + (\sqrt{y})^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx \\ &= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3} \int_0^1 \left[(x + 2)^{3/2} - (x + 1)^{3/2}\right] dx \\ &= \tfrac{2}{3}\left[\tfrac{2}{5}(x + 2)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

41. $z = f(x, y) = xy$ with $0 \le x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}\left(r^2 + 1\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right) \end{aligned}$$

43. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr \\ &= \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

45. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \le x \le 1$, $0 \le z \le 1$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x, z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$ and $A\,(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dA$. Then

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{17 + 4z^2}\, dx\, dz = \int_0^1 \sqrt{17 + 4z^2}\, dz \\ &= \tfrac{1}{2}\left(z\sqrt{17 + 4z^2} + \tfrac{17}{2} \ln\left|2z + \sqrt{4z^2 + 17}\,\right|\right)\Big]_0^1 = \tfrac{\sqrt{21}}{2} + \tfrac{17}{4}\left[\ln\left(2 + \sqrt{21}\right) - \ln\sqrt{17}\,\right] \end{aligned}$$

47. $\mathbf{r}_u = \langle 2u, v, 0\rangle$, $\mathbf{r}_v = \langle 0, u, v\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \left\langle v^2, -2uv, 2u^2\right\rangle$. Then

$$\begin{aligned} A(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\, dv\, du = \int_0^1 \int_0^2 \sqrt{(v^2 + 2u^2)^2}\, dv\, du \\ &= \int_0^1 \int_0^2 (v^2 + 2u^2)\, dv\, du = \int_0^1 \left[\tfrac{1}{3}v^3 + 2u^2 v\right]_{v=0}^{v=2} du = \int_0^1 \left(\tfrac{8}{3} + 4u^2\right) du = \left[\tfrac{8}{3}u + \tfrac{4}{3}u^3\right]_0^1 = 4 \end{aligned}$$

49. $z = f(x, y) = e^{-x^2 - y^2}$ with $x^2 + y^2 \le 4$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(-2xe^{-x^2 - y^2}\right)^2 + \left(-2ye^{-x^2 - y^2}\right)^2}\, dA = \iint_D \sqrt{1 + 4(x^2 + y^2)e^{-2(x^2 + y^2)}}\, dA \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2 e^{-2r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{1 + 4r^2 e^{-2r^2}}\, dr = 2\pi \int_0^2 r\sqrt{1 + 4r^2 e^{-2r^2}}\, dr \approx 13.9783 \end{aligned}$$

51. (a) $A(S) = \iint_D \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\, dA = \int_0^6 \int_0^4 \sqrt{1 + \dfrac{4x^2 + 4y^2}{(1 + x^2 + y^2)^4}}\, dy\, dx.$

Using the Midpoint Rule with $f(x, y) = \sqrt{1 + \dfrac{4x^2 + 4y^2}{(1 + x^2 + y^2)^4}}$, $m = 3$, $n = 2$ we have

$$A(S) \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = 4\left[f(1, 1) + f(1, 3) + f(3, 1) + f(3, 3) + f(5, 1) + f(5, 3)\right] \approx 24.2055$$

(b) Using a CAS we have $A(S) = \int_0^6 \int_0^4 \sqrt{1 + \dfrac{4x^2 + 4y^2}{(1 + x^2 + y^2)^4}}\, dy\, dx \approx 24.2476$. This agrees with the estimate in part (a) to the first decimal place.

53. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3 + 8y)^2}\, dy\, dx = \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx.$$

Using a CAS, we have

$\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx = \frac{45}{8}\sqrt{14} + \frac{15}{16} \ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \frac{15}{16} \ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$

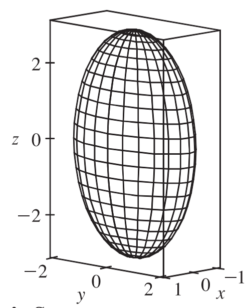
or $\frac{45}{8}\sqrt{14} + \frac{15}{16} \ln \frac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

55. (a) $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u \quad \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\sin u \cos v)^2 + (\sin u \sin v)^2 + (\cos u)^2 = \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

(b)



(c) From the parametric equations (with $a = 1$, $b = 2$, and $c = 3$), we calculate $\mathbf{r}_u = \cos u \cos v\, \mathbf{i} + 2 \cos u \sin v\, \mathbf{j} - 3 \sin u\, \mathbf{k}$ and $\mathbf{r}_v = -\sin u \sin v\, \mathbf{i} + 2 \sin u \cos v\, \mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6 \sin^2 u \cos v\, \mathbf{i} + 3 \sin^2 u \sin v\, \mathbf{j} + 2 \sin u \cos u\, \mathbf{k}$, and the surface area is given by

$$A(S) = \int_0^{2\pi} \int_0^{\pi} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \int_0^{2\pi} \int_0^{\pi} \sqrt{36 \sin^4 u \cos^2 v + 9 \sin^4 u \sin^2 v + 4 \cos^2 u \sin^2 u}\, du\, dv$$

57. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z - 2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z - 2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3 - 2)^2 = 4$, that is, $D = \left\{(x, y) \mid x^2 + y^2 \le 3\right\}$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta \\ &= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi \end{aligned}$$

59. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or $\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$. Hence $D = \left\{(\phi,\theta) \mid 0 \le \phi \le \frac{\pi}{2}, \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi} a^2\sin\phi\,d\theta\,d\phi = a^2\int_0^{\pi/2}(\pi - 2\phi)\sin\phi\,d\phi \\ &= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$.

Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$\begin{aligned} A(S_1) &= \iint\limits_{0 \le (x-(a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} -a(a^2 - r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} a^2[1 - (1 - \cos^2\theta)^{1/2}]\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\,d\theta = 2a^2\int_0^{\pi/2}(1 - \sin\theta)\,d\theta = 2a^2\left(\tfrac{\pi}{2} - 1\right) \end{aligned}$$

Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

17.7 Surface Integrals — ET 16.7

1. The faces of the box in the planes $x = 0$ and $x = 2$ have surface area 24 and centers $(0,2,3)$, $(2,2,3)$. The faces in $y = 0$ and $y = 4$ have surface area 12 and centers $(1,0,3)$, $(1,4,3)$, and the faces in $z = 0$ and $z = 6$ have area 8 and centers $(1,2,0)$, $(1,2,6)$. For each face we take the point P_{ij}^* to be the center of the face and $f(x,y,z) = e^{-0.1(x+y+z)}$, so by Definition 1,

$$\begin{aligned} \iint_S f(x,y,z)\,dS &\approx [f(0,2,3)](24) + [f(2,2,3)](24) + [f(1,0,3)](12) \\ &\quad + [f(1,4,3)](12) + [f(1,2,0)](8) + [f(1,2,6)](8) \\ &= 24(e^{-0.5} + e^{-0.7}) + 12(e^{-0.4} + e^{-0.8}) + 8(e^{-0.3} + e^{-0.9}) \approx 49.09 \end{aligned}$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm 3, \pm 4, 5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\begin{aligned}\textstyle\iint_H f(x,y,z)\,dS &\approx f(3,4,5)\,\Delta S + f(3,-4,5)\,\Delta S + f(-3,4,5)\,\Delta S + f(-3,-4,5)\,\Delta S \\ &= (7+8+9+12)(25\pi) = 900\pi \approx 2827\end{aligned}$$

5. $z = 1 + 2x + 3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\begin{aligned}\textstyle\iint_S x^2yz\,dS &= \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA = \textstyle\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dy\,dx \\ &= \textstyle\sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3 \left[\frac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2} dx \\ &= \textstyle\sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx = \sqrt{14}\left[\frac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}\end{aligned}$$

7. S is the part of the plane $z = 1 - x - y$ over the region $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Thus

$$\begin{aligned}\textstyle\iint_S yz\,dS &= \textstyle\iint_D y(1-x-y)\sqrt{(-1)^2 + (-1)^2 + 1}\,dA = \sqrt{3}\int_0^1\int_0^{1-x}\left(y - xy - y^2\right)dy\,dx \\ &= \textstyle\sqrt{3}\int_0^1 \left[\frac{1}{2}y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3\right]_{y=0}^{y=1-x} dx = \sqrt{3}\int_0^1 \frac{1}{6}(1-x)^3\,dx = -\frac{\sqrt{3}}{24}(1-x)^4\Big]_0^1 = \frac{\sqrt{3}}{24}\end{aligned}$$

9. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$, so

$\mathbf{r}_u \times \mathbf{r}_v = (2u\,\mathbf{i} + \sin v\,\mathbf{j} + \cos v\,\mathbf{k}) \times (u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}) = -u\,\mathbf{i} + 2u^2\sin v\,\mathbf{j} + 2u^2\cos v\,\mathbf{k}$ and

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v} = \sqrt{u^2 + 4u^4(\sin^2 v + \cos^2 v)} = u\sqrt{1+4u^2}$ (since $u \ge 0$). Then by Formula 2,

$$\begin{aligned}\textstyle\iint_S yz\,dS &= \textstyle\iint_D (u\sin v)(u\cos v)\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^{\pi/2}\int_0^1 (u\sin v)(u\cos v)\cdot u\sqrt{1+4u^2}\,du\,dv \\ &= \textstyle\int_0^1 u^3\sqrt{1+4u^2}\,du\int_0^{\pi/2}\sin v\cos v\,dv \qquad \left[\text{let } t = 1 + 4u^2 \;\Rightarrow\; u^2 = \frac{1}{4}(t-1) \text{ and } \frac{1}{8}\,dt = u\,du\right] \\ &= \textstyle\int_1^5 \frac{1}{8}\cdot\frac{1}{4}(t-1)\sqrt{t}\,dt\int_0^{\pi/2}\sin v\,\cos v\,dv = \frac{1}{32}\int_1^5\left(t^{3/2} - \sqrt{t}\right)dt\int_0^{\pi/2}\sin v\,\cos v\,dv \\ &= \textstyle\frac{1}{32}\left[\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2}\right]_1^5\left[\frac{1}{2}\sin^2 v\right]_0^{\pi/2} = \frac{1}{32}\left(\frac{2}{5}(5)^{5/2} - \frac{2}{3}(5)^{3/2} - \frac{2}{5} + \frac{2}{3}\right)\cdot\frac{1}{2}(1-0) = \frac{5}{48}\sqrt{5} + \frac{1}{240}\end{aligned}$$

11. S is the portion of the cone $z^2 = x^2 + y^2$ for $1 \le z \le 3$, or equivalently, S is the part of the surface $z = \sqrt{x^2+y^2}$ over the region $D = \left\{(x,y) \mid 1 \le x^2 + y^2 \le 9\right\}$. Thus

$$\begin{aligned}\iint_S x^2z^2\,dS &= \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\,dA \\ &= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA = \sqrt{2}\int_0^{2\pi}\int_1^3 (r\cos\theta)^2(r^2)\,r\,dr\,d\theta \\ &= \textstyle\sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta\int_1^3 r^5\,dr = \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{6}r^6\right]_1^3 = \sqrt{2}\,(\pi)\cdot\frac{1}{6}(3^6 - 1) = \dfrac{364\sqrt{2}}{3}\,\pi\end{aligned}$$

13. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + z\,\mathbf{k}$, $x^2+z^2 \le 4$. Then

$\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 2x\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1+4z^2} = \sqrt{1+4(x^2+z^2)}$. Thus

$$\iint_S y\,dS = \iint\limits_{x^2+z^2\le 4} (x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$= 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr \qquad \left[\text{let } u = 1+4r^2 \;\Rightarrow\; r^2 = \tfrac{1}{4}(u-1) \text{ and } \tfrac{1}{8}du = r\,dr\right]$$

$$= 2\pi\int_1^{17} \tfrac{1}{4}(u-1)\sqrt{u}\cdot\tfrac{1}{8}du = \tfrac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})\,du$$

$$= \tfrac{1}{16}\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{17} = \tfrac{1}{16}\pi\left[\tfrac{2}{5}(17)^{5/2} - \tfrac{2}{3}(17)^{3/2} - \tfrac{2}{5} + \tfrac{2}{3}\right] = \frac{\pi}{60}\left(391\sqrt{17}+1\right)$$

15. Using spherical coordinates and Example 17.6.10 [ET 16.6.10] we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$. Then $\iint_S (x^2z + y^2z)\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 16\pi\sin^4\phi\big]_0^{\pi/2} = 16\pi$.

17. S is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + \cos v\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le \pi/2$. Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i} \times (-\sin v\,\mathbf{j} + \cos v\,\mathbf{k}) = -\cos v\,\mathbf{j} - \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\iint_S (z + x^2y)\,dS = \int_0^{\pi/2}\int_0^3 (\sin v + u^2\cos v)(1)\,du\,dv = \int_0^{\pi/2}(3\sin v + 9\cos v)\,dv$$

$$= [-3\cos v + 9\sin v]_0^{\pi/2} = 0 + 9 + 3 - 0 = 12$$

19. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1 [2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx$$

$$= \int_0^1 \left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}$$

21. $\mathbf{F}(x,y,z) = xze^y\,\mathbf{i} - xze^y\,\mathbf{j} + z\,\mathbf{k}$, $z = g(x,y) = 1 - x - y$, and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1 - x\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D [-xze^y(-1) - (-xze^y)(-1) + z]\,dA = -\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$= -\int_0^1 \left(\tfrac{1}{2}x^2 - x + \tfrac{1}{2}\right)dx = -\tfrac{1}{6}$$

23. $\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\left\{(x,y) \,\middle|\, 0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\right\}$. S has downward orientation, so by Formula 10,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right]dA$$

$$= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right)dA$$

$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\cos^2\theta\,d\theta\int_0^2 r^3(4-r^2)^{-1/2}\,dr \qquad \left[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr\right]$$

$$= -\int_0^{\pi/2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta\int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du$$

$$= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0 = -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16 + \tfrac{16}{3}\right) = -\tfrac{4}{3}\pi$$

25. Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

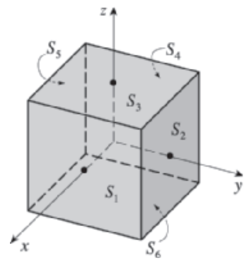
$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{x^2+z^2 \le 1} [-(x^2+z^2) - 2z^2]\, dA = -\int_0^{2\pi}\int_0^1 (r^2 + 2r^2\cos^2\theta)\, r\, dr\, d\theta$$

$$= -\int_0^{2\pi} \tfrac{1}{4}(1 + 2\cos^2\theta)\, d\theta = -\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) = -\pi$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{x^2+z^2 \le 1} (1)\, dA = \pi$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = -\pi + \pi = 0$.

27. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\, dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i} + 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 2\, dx\, dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i} - 2\,\mathbf{j} + 3z\,\mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} 3\, dx\, dy = 12$.

Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot d\mathbf{S} = 48$.

29. Here S consists of four surfaces: S_1, the top surface (a portion of the circular cylinder $y^2 + z^2 = 1$); S_2, the bottom surface (a portion of the xy-plane); S_3, the front half-disk in the plane $x = 2$, and S_4, the back half-disk in the plane $x = 0$.

On S_1: The surface is $z = \sqrt{1-y^2}$ for $0 \le x \le 2$, $-1 \le y \le 1$ with upward orientation, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_{-1}^{1} \left[-x^2\,(0) - y^2\left(-\frac{y}{\sqrt{1-y^2}}\right) + z^2\right] dy\, dx = \int_0^2 \int_{-1}^{1} \left(\frac{y^3}{\sqrt{1-y^2}} + 1 - y^2\right) dy\, dx$$

$$= \int_0^2 \left[-\sqrt{1-y^2} + \tfrac{1}{3}(1-y^2)^{3/2} + y - \tfrac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_0^2 \tfrac{4}{3}\, dx = \tfrac{8}{3}$$

On S_2: The surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_{-1}^{1} (-z^2)\, dy\, dx = \int_0^2 \int_{-1}^{1} (0)\, dy\, dx = 0$$

On S_3: The surface is $x = 2$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1-y^2}$, oriented in the positive x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} x^2\, dz\, dy = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} 4\, dz\, dy = 4A\,(S_3) = 2\pi$$

On S_4: The surface is $x = 0$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1-y^2}$, oriented in the negative x-direction. Regarding y and z as parameters, we use $-\,(\mathbf{r}_y \times \mathbf{r}_z) = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} x^2\, dz\, dy = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} (0)\; dz\, dy = 0$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{8}{3} + 0 + 2\pi + 0 = 2\pi + \frac{8}{3}$.

31. $z = xy \quad \Rightarrow \quad \partial z/\partial x = y$, $\partial z/\partial y = x$, so by Formula 4, a CAS gives

$\iint_S xyz\, dS = \int_0^1 \int_0^1 xy(xy)\sqrt{y^2 + x^2 + 1}\, dx\, dy \approx 0.1642$.

33. We use Formula 4 with $z = 3 - 2x^2 - y^2 \quad \Rightarrow \quad \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \geq 0$ are $-\sqrt{\frac{3}{2}} \leq x \leq \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \leq y \leq \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation may take a long time) to calculate

$$\iint_S x^2y^2z^2\, dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2y^2(3 - 2x^2 - y^2)^2\sqrt{16x^2 + 4y^2 + 1}\, dy\, dx \approx 3.4895$$

35. If S is given by $y = h(x, z)$, then S is also the level surface $f(x, y, z) = y - h(x, z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (10), using Formula 4 to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\, \frac{\dfrac{\partial h}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\,\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\, dA$$

where D is the projection of S onto the xz-plane. Therefore $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P\,\frac{\partial h}{\partial x} - Q + R\,\frac{\partial h}{\partial z}\right) dA$.

37. $m = \iint_S K\, dS = K \cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zK\, dS = K \int_0^{2\pi} \int_0^{\pi/2} (a \cos\phi)(a^2 \sin\phi)\, d\phi\, d\theta = 2\pi K a^3 \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi K a^3$.

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{2}a\right)$.

39. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\, dS$

(b) $I_z = \iint_S (x^2 + y^2)\left(10 - \sqrt{x^2 + y^2}\right) dS = \displaystyle\iint_{1 \leq x^2 + y^2 \leq 16} (x^2 + y^2)\left(10 - \sqrt{x^2 + y^2}\right)\sqrt{2}\, dA$

$= \int_0^{2\pi} \int_1^4 \sqrt{2}\,(10r^3 - r^4)\, dr\, d\theta = 2\sqrt{2}\,\pi\left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\,\pi$

41. The rate of flow through the cylinder is the flux $\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho\mathbf{v} \cdot d\mathbf{S}$. We use the parametric representation $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\,\mathbf{k}$ for S, where $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$, so $\mathbf{r}_u = -2\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j}$, $\mathbf{r}_v = \mathbf{k}$, and the outward orientation is given by $\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}$. Then

$$\begin{aligned}
\iint_S \rho\mathbf{v} \cdot d\mathbf{S} &= \rho \int_0^{2\pi} \int_0^1 \left(v\,\mathbf{i} + 4\sin^2 u\,\mathbf{j} + 4\cos^2 u\,\mathbf{k}\right) \cdot (2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j})\, dv\, du \\
&= \rho \int_0^{2\pi} \int_0^1 \left(2v\cos u + 8\sin^3 u\right) dv\, du = \rho \int_0^{2\pi} \left(\cos u + 8\sin^3 u\right) du \\
&= \rho\left[\sin u + 8\left(-\tfrac{1}{3}\right)(2 + \sin^2 u)\cos u\right]_0^{2\pi} = 0 \text{ kg/s}
\end{aligned}$$

43. S consists of the hemisphere S_1 given by $z = \sqrt{a^2 - x^2 - y^2}$ and the disk S_2 given by $0 \le x^2 + y^2 \le a^2$, $z = 0$.

On S_1: $\mathbf{E} = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$,

$\mathbf{T}_\phi \times \mathbf{T}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. Thus

$$\begin{aligned}\iint_{S_1} \mathbf{E}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi + 2a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi + a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta = (2\pi)a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3\end{aligned}$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = 0$. Hence the total charge is $q = \varepsilon_0\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\varepsilon_0$.

45. $K\nabla u = 6.5(4y\,\mathbf{j} + 4z\,\mathbf{k})$. S is given by $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\,\cos\theta\,\mathbf{j} + \sqrt{6}\,\sin\theta\,\mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6}\,\cos\theta\,\mathbf{j} - \sqrt{6}\,\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by

$\iint_S(-K\,\nabla u)\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^4 -(6.5)(-24)\,dx\,d\theta = (2\pi)(156)(4) = 1248\pi$.

47. Let S be a sphere of radius a centered at the origin. Then $|\mathbf{r}| = a$ and $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = \left(c/a^3\right)(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$. A parametric representation for S is $\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. The flux of $\mathbf{F}$ across S is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\,d\mathbf{S} &= \int_0^\pi\int_0^{2\pi}\frac{c}{a^3}\,(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}) \\ &\qquad\qquad\cdot\left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi \\ &= \frac{c}{a^3}\int_0^\pi\int_0^{2\pi}a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right)d\theta\,d\phi = c\int_0^\pi\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = 4\pi c\end{aligned}$$

Thus the flux does not depend on the radius a.

17.8 Stokes' Theorem ET 16.8

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2 + y^2 = 4$, $z = 0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_P \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ (where C is the boundary curve).

3. The paraboloid $z = x^2 + y^2$ intersects the cylinder $x^2 + y^2 = 4$ in the circle $x^2 + y^2 = 4$, $z = 4$. This boundary curve C should be oriented in the counterclockwise direction when viewed from above, so a vector equation of C is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$,

$$\mathbf{F}(\mathbf{r}(t)) = (4\cos^2 t)(16)\,\mathbf{i} + (4\sin^2 t)(16)\,\mathbf{j} + (2\cos t)(2\sin t)(4)\,\mathbf{k} = 64\cos^2 t\,\mathbf{i} + 64\sin^2 t\,\mathbf{j} + 16\sin t\,\cos t\,\mathbf{k},$$

and by Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-128\cos^2 t\,\sin t + 128\sin^2 t\,\cos t + 0)\,dt \\ &= 128\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0\end{aligned}$$

5. C is the square in the plane $z=-1$. By (3), $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl} \mathbf{F} = x^2 z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus $\iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} (x+y)\,dx\,dy = 0$ so $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

7. $\operatorname{curl} \mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x+y+z=1$ over $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 17.7.10 [ET 16.7.10], we have $z = g(x,y) = 1 - x - y$, $P = -2z$, $Q = -2x$, $R = -2y$, and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\,dA \\ &= \int_0^1 \int_0^{1-x} (-2)\,dy\,dx = -2\int_0^1 (1-x)\,dx = -1\end{aligned}$$

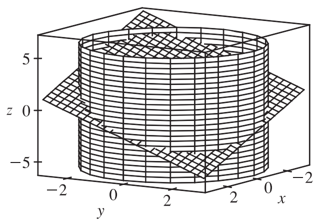
9. $\operatorname{curl} \mathbf{F} = (xe^{xy} - 2x)\,\mathbf{i} - (ye^{xy} - y)\,\mathbf{j} + (2z - z)\,\mathbf{k}$ and we take S to be the disk $x^2 + y^2 \le 16$, $z = 5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\mathbf{n} = \mathbf{k}$ and $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 2z - z$ on S, where $z = 5$. Thus

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (2z - z)\;dS = \iint_S (10-5)\,dS = 5(\text{area of } S) = 5(\pi \cdot 4^2) = 80\pi$$

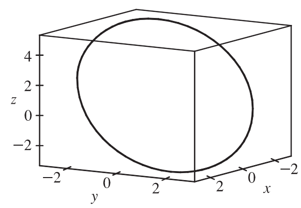
11. (a) The curve of intersection is an ellipse in the plane $x+y+z=1$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\operatorname{curl} \mathbf{F} = x^2\,\mathbf{j} + y^2\,\mathbf{k}$, and $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \tfrac{1}{\sqrt{3}}(x^2+y^2)\,dS = \iint_{x^2+y^2 \le 9} (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81\pi}{2}$$

(b)



(c) One possible parametrization is $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3\cos t - 3\sin t$, $0 \le t \le 2\pi$.



13. The boundary curve C is the circle $x^2 + y^2 = 1$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = \sin^2 t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \cos^2 t - \sin^3 t$, and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos^2 t - \sin^3 t)\,dt = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2t)\,dt - \int_0^{2\pi} (1 - \cos^2 t)\sin t\,dt \\ &= \tfrac{1}{2}\left[t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} - \left[-\cos t + \tfrac{1}{3}\cos^3 t\right]_0^{2\pi} = \pi\end{aligned}$$

Now $\operatorname{curl} \mathbf{F} = (1 - 2y)\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 1$, so by Equation 17.7.10 [ET 16.7.10] with $z = g(x,y) = x^2 + y^2$ we have

$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 - 2y)\,dA = \int_0^{2\pi}\int_0^1 (1 - 2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \left(\frac{1}{2} - \frac{2}{3}\sin\theta\right)d\theta = \pi$.

15. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y-axis. Then C can be described by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$. Thus

$\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\cos^2 t$, and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -\cos^2 t\,dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t\Big]_0^{2\pi} = -\pi$.

Now $\operatorname{curl}\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 17.6.10 [ET 16.6.10]) by

$\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \le \theta \le \pi$, $0 \le \phi \le \pi$. Then

$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$ and

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1} \operatorname{curl}\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^\pi \int_0^\pi (-\sin^2\phi\,\cos\theta - \sin^2\phi\,\sin\theta - \sin\phi\,\cos\phi)\,d\theta\,d\phi \\ &= \int_0^\pi (-2\sin^2\phi - \pi\sin\phi\,\cos\phi)\,d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi\end{aligned}$$

17. It is easier to use Stokes' Theorem than to compute the work directly. Let S be the planar region enclosed by the path of the particle, so S is the portion of the plane $z = \frac{1}{2}y$ for $0 \le x \le 1$, $0 \le y \le 2$, with upward orientation.

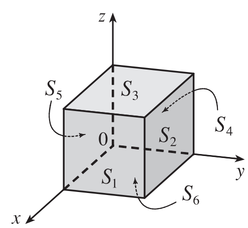
$\operatorname{curl}\mathbf{F} = 8y\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ and

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-8y\,(0) - 2z\left(\tfrac{1}{2}\right) + 2y\right] dA = \int_0^1 \int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\,dx \\ &= \int_0^1 \int_0^2 \tfrac{3}{2}y\,dy\,dx = \int_0^1 \left[\tfrac{3}{4}y^2\right]_{y=0}^{y=2} dx = \int_0^1 3\,dx = 3\end{aligned}$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

17.9 The Divergence Theorem ET 16.9

1. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so



$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1 \int_0^1 \int_0^1 (3x + 3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, on

S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$. Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{9}{2}$.

3. $\operatorname{div}\mathbf{F} = x + y + z$, so

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \int_0^{2\pi} \int_0^1 \int_0^1 (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \left(r^2\cos\theta + r^2\sin\theta + \tfrac{1}{2}r\right) dr\,d\theta \\ &= \int_0^{2\pi} \left(\tfrac{1}{3}\cos\theta + \tfrac{1}{3}\sin\theta + \tfrac{1}{4}\right) d\theta = \tfrac{1}{4}(2\pi) = \tfrac{\pi}{2}\end{aligned}$$

Let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. On S_1, $z = 1$, $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, so

$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} x\, dS = \int_0^{2\pi} \int_0^1 (r\cos\theta)\, r\, dr\, d\theta = \left[\sin\theta\right]_0^{2\pi} \left[\frac{1}{3}r^3\right]_0^1 = 0.$

On S_2, $z = 0$, $\mathbf{n} = -\mathbf{k}$, and $\mathbf{F} = xy\,\mathbf{i}$ so $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} 0\, dS = 0$.

S_3 is given by $\mathbf{r}(\theta, z) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$. Then $\mathbf{r}_\theta \times \mathbf{r}_z = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z)\, dA = \int_0^{2\pi} \int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)\, dz\, d\theta \\ &= \int_0^{2\pi} \left(\cos^2\theta\sin\theta + \tfrac{1}{2}\sin^2\theta\right) d\theta = \left[-\tfrac{1}{3}\cos^3\theta + \tfrac{1}{4}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \tfrac{\pi}{2}\end{aligned}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

5. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(yz^2) = e^x\sin y - e^x \sin y + 2yz = 2yz$, so by the Divergence Theorem,

$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^2 2yz\, dz\, dy\, dx = 2\int_0^1 dx \int_0^1 y\, dy \int_0^1 z\, dz = 2\left[x\right]_0^1 \left[\frac{1}{2}y^2\right]_0^1 \left[\frac{1}{2}z^2\right]_0^2 = 2.$

7. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (3y^2 + 3z^2)\, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\, r\, dx\, dr\, d\theta \\ &= 3\int_0^{2\pi} d\theta \int_0^1 r^3\, dr \int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}\end{aligned}$$

9. $\operatorname{div}\mathbf{F} = y\sin z + 0 - y\sin z = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0\, dV = 0$.

11. $\operatorname{div}\mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$ so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cdot r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3(4 - r^2)\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 (4r^3 - r^5)\, dr = 2\pi\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 = \tfrac{32}{3}\pi\end{aligned}$$

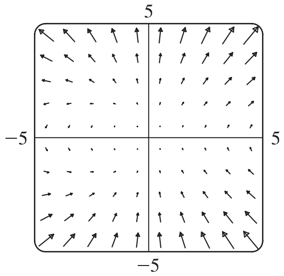
13. $\operatorname{div}\mathbf{F} = 12x^2z + 12y^2z + 12z^3$ so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E 12z(x^2 + y^2 + z^2)\, dV = \int_0^{2\pi} \int_0^\pi \int_0^R 12(\rho\cos\phi)(\rho^2)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= 12\int_0^{2\pi} d\theta \int_0^\pi \sin\phi\, \cos\phi\, d\phi \int_0^R \rho^5\, d\rho = 12(2\pi)\left[\tfrac{1}{2}\sin^2\phi\right]_0^\pi \left[\tfrac{1}{6}\rho^6\right]_0^R = 0\end{aligned}$$

15. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \sqrt{3 - x^2}\, dV = \int_{-1}^1 \int_{-1}^1 \int_0^{2 - x^4 - y^4} \sqrt{3 - x^2}\, dz\, dy\, dx = \frac{341}{60}\sqrt{2} + \frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

17. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot (-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (-y^2)\, dA = -\int_0^{2\pi} \int_0^1 r^2 (\sin^2\theta)\, r\, dr\, d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2 x) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan z\right) + \frac{\partial}{\partial z}(x^2 z + y^2) = z^2 + y^2 + x^2$, we use spherical coordinates to get $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{2}{5}\pi$. Finally $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

19. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

21. 

From the graph it appears that for points above the x-axis, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the x-axis, where divergence is negative.
$\mathbf{F}(x,y) = \langle xy, x+y^2\rangle \quad\Rightarrow\quad \operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(x+y^2) = y + 2y = 3y.$
Thus $\operatorname{div}\mathbf{F} > 0$ for $y > 0$, and $\operatorname{div}\mathbf{F} < 0$ for $y < 0$.

23. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at $(0,0,0)$ where it is undefined.

25. $\iint_S \mathbf{a}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{a}\,dV = 0$ since $\operatorname{div}\mathbf{a} = 0$.

27. $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0$ by Theorem 17.5.11 [ET 16.5.11].

29. $\iint_S (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}(f\nabla g)\,dV = \iiint_E (f\nabla^2 g + \nabla g\cdot\nabla f)\,dV$ by Exercise 17.5.25 [ET 16.5.25].

31. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then $\operatorname{div}\mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x}c_1 + \dfrac{\partial f}{\partial y}c_2 + \dfrac{\partial f}{\partial z}c_3 = \nabla f\cdot\mathbf{c}$ and the Divergence Theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV \quad\Rightarrow$ $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{c}\,dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{i}\,dV \quad\Rightarrow$ $\displaystyle\iint_S fn_1\,dS = \iiint_E \frac{\partial f}{\partial x}\,dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S fn_2\,dS = \iiint_E \frac{\partial f}{\partial y}\,dV$, and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S fn_3\,dS = \iiint_E \frac{\partial f}{\partial z}\,dV$. Then

$$\begin{aligned}
\iint_S f\mathbf{n}\,dS &= \left(\iint_S fn_1\,dS\right)\mathbf{i} + \left(\iint_S fn_2\,dS\right)\mathbf{j} + \left(\iint_S fn_3\,dS\right)\mathbf{k} \\
&= \left(\iiint_E \frac{\partial f}{\partial x}\,dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\,dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\,dV\right)\mathbf{k} = \iiint_E \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)dV \\
&= \iiint_E \nabla f\,dV \quad \text{as desired.}
\end{aligned}$$

17 Review ET 16

CONCEPT CHECK

1. See Definitions 1 and 2 in Section 17.1 [ET 16.1]. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

(b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 17.2.2 [ET 16.2.2].

(b) We normally evaluate the line integral using Formula 17.2.3 [ET 16.2.3].

(c) The mass is $m = \int_C \rho(x, y)\, ds$, and the center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds$.

(d) See (5) and (6) in Section 17.2 [ET 16.2] for plane curves; we have similar definitions when C is a space curve (see the equation preceding (10) in Section 17.2 [ET 16.2]).

(e) For plane curves, see Equations 17.2.7 [ET 16.2.7]. We have similar results for space curves (see the equation preceding (10) in Section 17.2 [ET 16.2]).

4. (a) See Definition 17.2.13 [ET 16.2.13].

(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

(c) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 17.3.2 [ET 16.3.2].

6. (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

(b) See Theorem 17.3.4 [ET 16.3.4].

7. See the statement of Green's Theorem on page 1091 [ET 1055].

8. See Equations 17.4.5 [ET 16.4.5].

9. (a) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$

(b) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$

(c) For curl $\mathbf{F}$, see the discussion accompanying Figure 1 on page 1100 [ET 1064] as well as Figure 6 and the accompanying discussion on page 1132 [ET 1096]. For div $\mathbf{F}$, see the discussion following Example 5 on page 1102 [ET 1066] as well as the discussion preceding (8) on page 1139 [ET 1103].

10. See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

11. (a) See (1) and (2) and the accompanying discussion in Section 17.6 [ET 16.6] ; See Figure 4 and the accompanying discussion on page 1107 [ET 1071] .

(b) See Definition 17.6.6 [ET 16.6.6].

(c) See Equation 17.6.9 [ET 16.6.9].

12. (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.2 [ET 16.7.2].

(c) See Formula 17.7.4 [ET 16.7.4].

(d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 6 and 7 and the accompanying discussion in Section 17.7 [ET 16.7]. A Möbius strip is a nonorientable surface; see Figures 4 and 5 and the accompanying discussion on page 1121 [ET 1085].

(b) See Definition 17.7.8 [ET 16.7.8].

(c) See Formula 17.7.9 [ET 16.7.9].

(d) See Formula 17.7.10 [ET 16.7.10].

14. See the statement of Stokes' Theorem on page 1129 [ET 1093.].

15. See the statement of the Divergence Theorem on page 1135 [ET 1099].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

3. True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

5. False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if D is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

7. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

(b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}\,(P)$ is positive.

3. $$\begin{aligned}\int_C yz\cos x\, ds &= \int_0^\pi (3\cos t)\,(3\sin t)\cos t\,\sqrt{(1)^2 + (-3\sin t)^2 + (3\cos t)^2}\, dt = \int_0^\pi (9\cos^2 t\,\sin t)\sqrt{10}\, dt \\ &= 9\sqrt{10}\left(-\tfrac{1}{3}\cos^3 t\right)\Big]_0^\pi = -3\sqrt{10}\,(-2) = 6\sqrt{10}\end{aligned}$$

5. $$\begin{aligned}\int_C y^3\, dx + x^2\, dy &= \int_{-1}^1 \left[y^3(-2y) + (1-y^2)^2\right] dy = \int_{-1}^1 (-y^4 - 2y^2 + 1)\, dy \\ &= \left[-\tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 + y\right]_{-1}^1 = -\tfrac{1}{5} - \tfrac{2}{3} + 1 - \tfrac{1}{5} - \tfrac{2}{3} + 1 = \tfrac{4}{15}\end{aligned}$$

7. C: $x = 1 + 2t \;\Rightarrow\; dx = 2\,dt,\ y = 4t \;\Rightarrow\; dy = 4\,dt,\ z = -1 + 3t \;\Rightarrow\; dz = 3\,dt,\ 0 \le t \le 1.$

$$\begin{aligned}\int_C xy\, dx + y^2\, dy + yz\, dz &= \int_0^1 [(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)]\, dt \\ &= \int_0^1 (116t^2 - 4t)\, dt = \left[\tfrac{116}{3}t^3 - 2t^2\right]_0^1 = \tfrac{116}{3} - 2 = \tfrac{110}{3}\end{aligned}$$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2 + t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

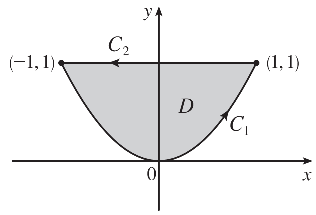
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2te^{-t} - 3t^5 - (t^2 + t^3))\, dt = \left[-2te^{-t} - 2e^{-t} - \tfrac{1}{2}t^6 - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4\right]_0^1 = \tfrac{11}{12} - \tfrac{4}{e}.$$

11. $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \Rightarrow g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

13. Since $\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x,y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0,1)$ and $t = 1$ corresponds to $(1,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(0,1) = 1 - 1 = 0$.

15. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.



Then

$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1}(t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt$$
$$= \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA = \iint_D (-2xy - 2xy)\,dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx$$
$$= \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1}(2x^5 - 2x)\,dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint\limits_{x^2+y^2\le 4}\left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint\limits_{x^2+y^2\le 4}(-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx + g(y)\,dy = \iint_D\left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right] dA = \iint_D 0\,dA = 0$.

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D\left[\frac{\partial}{\partial x}(-f_x) - \frac{\partial}{\partial y}(f_y)\right] dA = -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

25. $z = f(x,y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\,dA = \int_0^1\int_0^{2x}\sqrt{5 + 4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5 + 4x^2}\,dx = \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\,dS = \iint_{x^2+y^2\le 4} (x^2+y^2)\sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r^3\sqrt{1+4r^2}\,dr\,d\theta = \tfrac{1}{60}\pi\big(391\sqrt{17}+1\big)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (z-2)\,dV = \iiint_E z\,dV - 2\iiint_E dV = m\overline{z} - 2\big(\frac{4}{3}\pi 2^3\big) = -\frac{64}{3}\pi.$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\,\cos\theta\,\cos\phi\,\mathbf{i} - 4\sin\phi\,\sin\theta\,\mathbf{j} + 6\sin\phi\,\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\,\cos\theta\,\mathbf{i} + 4\sin^2\phi\,\sin\theta\,\mathbf{j} + 4\sin\phi\,\cos\phi\,\mathbf{k}$, and

$\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 16\sin^3\phi\,\cos^2\theta\,\cos\phi - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta$. Then

$$\begin{aligned}\iint_S F\cdot dS &= \int_0^{2\pi}\int_0^{\pi}(16\sin^3\phi\,\cos\phi\,\cos^2\theta - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi\end{aligned}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S(\operatorname{curl}\mathbf{F})\cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-\cos^2 t\,\sin t + \sin^2 t\,\cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0.$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D(-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dA = \iint_D(-1)\,dA = -(\text{area of } D) = -\frac{1}{2}.$

35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_{x^2+y^2+z^2\le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = \sin^3\phi\,\cos^2\theta + \sin^3\phi\,\sin^2\theta + \sin\phi\,\cos^2\phi = \sin\phi$ and

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi.$

37. Because $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, and if $f(x,y,z) = x^3yz - 3xy + z^2$, then $\nabla f = \mathbf{F}$. Hence

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(0,3,0) - f(0,0,2) = 0 - 4 = -4.$

39. By the Divergence Theorem, $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = 3(\text{volume of } E) = 3(8-1) = 21.$

41. Let $\mathbf{F} = \mathbf{a}\times\mathbf{r} = \langle a_1,a_2,a_3\rangle \times \langle x,y,z\rangle = \langle a_2z - a_3y, a_3x - a_1z, a_1y - a_2x\rangle$. Then $\operatorname{curl}\mathbf{F} = \langle 2a_1, 2a_2, 2a_3\rangle = 2\mathbf{a}$, and $\iint_S 2\mathbf{a}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C(\mathbf{a}\times\mathbf{r})\cdot d\mathbf{r}$ by Stokes' Theorem.

PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV$.

But

$$\begin{aligned}\nabla\cdot\frac{\mathbf{r}}{r^3} &= \left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}},\frac{y}{(x^2+y^2+z^2)^{3/2}},\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle\\ &= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}}=0\end{aligned}$$

$\Rightarrow\ \iint_{\partial S_1}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iiint_{S_1}0\,dV=0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n}=0\ \Rightarrow$

$0=\iint_{\partial S_1}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S_L}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS+\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS\quad\Rightarrow$

$\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=-\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r=a\ \Rightarrow\ \mathbf{n}=-\frac{\mathbf{r}}{r}=-\frac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r}=r^2=a^2$, so that $-\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS=\iint_{S(a)}\frac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS=\iint_{S(a)}\frac{a^2}{a^4}\,dS=\frac{1}{a^2}\iint_{S(a)}dS=\frac{\text{area of }S(a)}{a^2}=|\Omega(S)|$.

Therefore $|\Omega(S)|=\iint_S\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

3. The given line integral $\frac{1}{2}\int_C(bz-cy)\,dx+(cx-az)\,dy+(ay-bx)\,dz$ can be expressed as $\int_C\mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z)=P\,\mathbf{i}+Q\,\mathbf{j}+R\,\mathbf{k}=\frac{1}{2}(bz-cy)\,\mathbf{i}+\frac{1}{2}(cx-az)\,\mathbf{j}+\frac{1}{2}(ay-bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$.

Now

$$\begin{aligned}\operatorname{curl}\mathbf{F}&=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\\ &=\left(\tfrac{1}{2}a+\tfrac{1}{2}a\right)\mathbf{i}+\left(\tfrac{1}{2}b+\tfrac{1}{2}b\right)\mathbf{j}+\left(\tfrac{1}{2}c+\tfrac{1}{2}c\right)\mathbf{k}=a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}=\mathbf{n}\end{aligned}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\mathbf{n}\cdot\mathbf{n}=|\mathbf{n}|^2=1$, hence $\iint_S\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_S dS$ which is simply the surface area of S. Thus, $\int_C\mathbf{F}\cdot d\mathbf{r}=\frac{1}{2}\int_C(bz-cy)\,dx+(cx-az)\,dy+(ay-bx)\,dz$ is the plane area enclosed by C.

5. $(\mathbf{F}\cdot\nabla)\,\mathbf{G}=\left(P_1\dfrac{\partial}{\partial x}+Q_1\dfrac{\partial}{\partial y}+R_1\dfrac{\partial}{\partial z}\right)(P_2\,\mathbf{i}+Q_2\,\mathbf{j}+R_2\,\mathbf{k})$

$$=\left(P_1\frac{\partial P_2}{\partial x}+Q_1\frac{\partial P_2}{\partial y}+R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i}+\left(P_1\frac{\partial Q_2}{\partial x}+Q_1\frac{\partial Q_2}{\partial y}+R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}$$

$$+\left(P_1\frac{\partial R_2}{\partial x}+Q_1\frac{\partial R_2}{\partial y}+R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

$$=(\mathbf{F}\cdot\nabla P_2)\,\mathbf{i}+(\mathbf{F}\cdot\nabla Q_2)\,\mathbf{j}+(\mathbf{F}\cdot\nabla R_2)\,\mathbf{k}.$$

Similarly, $(\mathbf{G}\cdot\nabla)\,\mathbf{F}=(\mathbf{G}\cdot\nabla P_1)\,\mathbf{i}+(\mathbf{G}\cdot\nabla Q_1)\,\mathbf{j}+(\mathbf{G}\cdot\nabla R_1)\,\mathbf{k}$. Then

$$\mathbf{F}\times\operatorname{curl}\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ P_1 & Q_1 & R_1\\ \partial R_2/\partial y-\partial Q_2/\partial z & \partial P_2/\partial z-\partial R_2/\partial x & \partial Q_2/\partial x-\partial P_2/\partial y\end{vmatrix}$$

$$=\left(Q_1\frac{\partial Q_2}{\partial x}-Q_1\frac{\partial P_2}{\partial y}-R_1\frac{\partial P_2}{\partial z}+R_1\frac{\partial R_2}{\partial x}\right)\mathbf{i}+\left(R_1\frac{\partial R_2}{\partial y}-R_1\frac{\partial Q_2}{\partial z}-P_1\frac{\partial Q_2}{\partial x}+P_1\frac{\partial P_2}{\partial y}\right)\mathbf{j}$$

$$+\left(P_1\frac{\partial P_2}{\partial z}-P_1\frac{\partial R_2}{\partial x}-Q_1\frac{\partial R_2}{\partial y}+Q_1\frac{\partial Q_2}{\partial z}\right)\mathbf{k}$$

and

$$\mathbf{G}\times\operatorname{curl}\mathbf{F}=\left(Q_2\frac{\partial Q_1}{\partial x}-Q_2\frac{\partial P_1}{\partial y}-R_2\frac{\partial P_1}{\partial z}+R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i}+\left(R_2\frac{\partial R_1}{\partial y}-R_2\frac{\partial Q_1}{\partial z}-P_2\frac{\partial Q_1}{\partial x}+P_2\frac{\partial P_1}{\partial y}\right)\mathbf{j}$$

$$+\left(P_2\frac{\partial P_1}{\partial z}-P_2\frac{\partial R_1}{\partial x}-Q_2\frac{\partial R_1}{\partial y}+Q_2\frac{\partial Q_1}{\partial z}\right)\mathbf{k}.$$

Then

$$(\mathbf{F}\cdot\nabla)\,\mathbf{G}+\mathbf{F}\times\operatorname{curl}\mathbf{G}=\left(P_1\frac{\partial P_2}{\partial x}+Q_1\frac{\partial Q_2}{\partial x}+R_1\frac{\partial R_2}{\partial x}\right)\mathbf{i}+\left(P_1\frac{\partial P_2}{\partial y}+Q_1\frac{\partial Q_2}{\partial y}+R_1\frac{\partial R_2}{\partial y}\right)\mathbf{j}$$

$$+\left(P_1\frac{\partial P_2}{\partial z}+Q_1\frac{\partial Q_2}{\partial z}+R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

and

$$(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{G}\times\operatorname{curl}\mathbf{F}=\left(P_2\frac{\partial P_1}{\partial x}+Q_2\frac{\partial Q_1}{\partial x}+R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i}+\left(P_2\frac{\partial P_1}{\partial y}+Q_2\frac{\partial Q_1}{\partial y}+R_2\frac{\partial R_1}{\partial y}\right)\mathbf{j}$$

$$+\left(P_2\frac{\partial P_1}{\partial z}+Q_2\frac{\partial Q_1}{\partial z}+R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}.$$

Hence

$(\mathbf{F}\cdot\nabla)\,\mathbf{G}+\mathbf{F}\times\operatorname{curl}\mathbf{G}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}+\mathbf{G}\times\operatorname{curl}\mathbf{F}$

$$=\left[\left(P_1\frac{\partial P_2}{\partial x}+P_2\frac{\partial P_1}{\partial x}\right)+\left(Q_1\frac{\partial Q_2}{\partial x}+Q_2\frac{\partial Q_1}{\partial y}\right)+\left(R_1\frac{\partial R_2}{\partial x}+R_2\frac{\partial R_1}{\partial x}\right)\right]\mathbf{i}$$

$$+\left[\left(P_1\frac{\partial P_2}{\partial y}+P_2\frac{\partial P_1}{\partial y}\right)+\left(Q_1\frac{\partial Q_2}{\partial y}+Q_2\frac{\partial Q_1}{\partial y}\right)+\left(R_1\frac{\partial R_2}{\partial y}+R_2\frac{\partial R_1}{\partial y}\right)\right]\mathbf{j}$$

$$+\left[\left(P_1\frac{\partial P_2}{\partial z}+P_2\frac{\partial P_1}{\partial z}\right)+\left(Q_1\frac{\partial Q_2}{\partial z}+Q_2\frac{\partial Q_1}{\partial z}\right)+\left(R_1\frac{\partial R_2}{\partial z}+R_2\frac{\partial R_1}{\partial z}\right)\right]\mathbf{k}$$

$$=\nabla(P_1P_2+Q_1Q_2+R_1R_2)=\nabla(\mathbf{F}\cdot\mathbf{G}).$$

18 □ SECOND-ORDER DIFFERENTIAL EQUATIONS □ ET 17

18.1 Second-Order Linear Equations ET 17.1

1. The auxiliary equation is $r^2 - r - 6 = 0 \quad \Rightarrow \quad (r-3)(r+2) = 0 \quad \Rightarrow \quad r = 3, r = -2$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-2x}$.

3. The auxiliary equation is $r^2 + 16 = 0 \quad \Rightarrow \quad r = \pm 4i$. Then by (11) the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$.

5. The auxiliary equation is $9r^2 - 12r + 4 = 0 \quad \Rightarrow \quad (3r-2)^2 = 0 \quad \Rightarrow \quad r = \frac{2}{3}$. Then by (10), the general solution is $y = c_1 e^{2x/3} + c_2 x e^{2x/3}$.

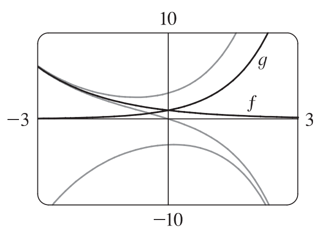
7. The auxiliary equation is $2r^2 - r = r(2r-1) = 0 \quad \Rightarrow \quad r = 0, r = \frac{1}{2}$, so $y = c_1 e^{0x} + c_2 e^{x/2} = c_1 + c_2 e^{x/2}$.

9. The auxiliary equation is $r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$, so $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$.

11. The auxiliary equation is $2r^2 + 2r - 1 = 0 \quad \Rightarrow \quad r = \dfrac{-2 \pm \sqrt{12}}{4} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}$, so $y = c_1 e^{\left(-1/2 + \sqrt{3}/2\right)t} + c_2 e^{\left(-1/2 - \sqrt{3}/2\right)t}$.

13. The auxiliary equation is $100r^2 + 200r + 101 = 0 \quad \Rightarrow \quad r = \dfrac{-200 \pm \sqrt{-400}}{200} = -1 \pm \frac{1}{10}i$, so $P = e^{-t}\left[c_1 \cos\left(\frac{1}{10}t\right) + c_2 \sin\left(\frac{1}{10}t\right)\right]$.

15. The auxiliary equation is $5r^2 - 2r - 3 = (5r+3)(r-1) = 0 \quad \Rightarrow \quad r = -\frac{3}{5}$, $r = 1$, so the general solution is $y = c_1 e^{-3x/5} + c_2 e^{x}$. We graph the basic solutions $f(x) = e^{-3x/5}$, $g(x) = e^x$ as well as $y = e^{-3x/5} + 2e^x$, $y = e^{-3x/5} - e^x$, and $y = -2e^{-3x/5} - e^x$. Each solution consists of a single continuous curve that approaches either 0 or $\pm\infty$ as $x \to \pm\infty$.



17. $2r^2 + 5r + 3 = (2r+3)(r+1) = 0$, so $r = -\frac{3}{2}$, $r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \quad \Rightarrow \quad c_1 + c_2 = 3$ and $y'(0) = -4 \quad \Rightarrow \quad -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

19. $4r^2 - 4r + 1 = (2r-1)^2 = 0 \quad \Rightarrow \quad r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \quad \Rightarrow \quad c_1 = 1$ and $y'(0) = -1.5 \quad \Rightarrow \quad \frac{1}{2}c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2xe^{x/2}$.

21. $r^2 + 16 = 0 \quad \Rightarrow \quad r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$. Then $y\left(\frac{\pi}{4}\right) = -3 \quad \Rightarrow \quad -c_1 = -3 \quad \Rightarrow \quad c_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 4 \quad \Rightarrow \quad -4c_2 = 4 \quad \Rightarrow \quad c_2 = -1$, so the solution to the initial-value problem is $y = 3\cos 4x - \sin 4x$.

23. $r^2 + 2r + 2 = 0 \quad \Rightarrow \quad r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \quad \Rightarrow \quad c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2\cos x + 3\sin x)$.

25. $4r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm\frac{1}{2}i$ and the general solution is $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right)$.

27. $r^2 - 3r + 2 = (r-2)(r-1) = 0 \quad \Rightarrow \quad r = 1, r = 2$ and the general solution is $y = c_1 e^x + c_2 e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1 e^3 + c_2 e^6$ so $c_2 = 1/(1 - e^3)$ and $c_1 = e^3/(e^3 - 1)$. The solution of the boundary-value problem is $y = \dfrac{e^{x+3}}{e^3 - 1} + \dfrac{e^{2x}}{1 - e^3}$.

29. $r^2 - 6r + 25 = 0 \quad \Rightarrow \quad r = 3 \pm 4i$ and the general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1 e^{3\pi} \quad \Rightarrow \quad c_1 = 2/e^{3\pi}$, so there is no solution.

31. $r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i$ and the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$. But $2 = y(0) = c_1$ and $1 = y\left(\frac{\pi}{2}\right) = e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y = e^{-2x}(2\cos 3x - e^{\pi}\sin 3x)$.

33. (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\Rightarrow$ $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ $(*)$ and $0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$.

Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\,L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,L = n\pi$ where n is an integer $\quad \Rightarrow \quad \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.

18.2 Nonhomogeneous Linear Equations ET 17.2

1. The auxiliary equation is $r^2 + 3r + 2 = (r+2)(r+1) = 0$, so the complementary solution is $y_c(x) = c_1 e^{-2x} + c_2 e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

3. The auxiliary equation is $r^2 - 2r = r(r-2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2 e^{2x}$. Try the particular solution $y_p(x) = A\cos 4x + B\sin 4x$, so $y_p' = -4A\sin 4x + 4B\cos 4x$ and $y_p'' = -16A\cos 4x - 16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x - 16B\sin 4x) - 2(-4A\sin 4x + 4B\cos 4x) = \sin 4x \;\Rightarrow$ $(-16A - 8B)\cos 4x + (8A - 16B)\sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \;\Rightarrow\; A = \frac{1}{40}$ and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$.

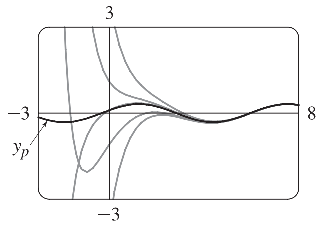
5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1\cos x + c_2\sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \;\Rightarrow\; 10Ae^{-x} = e^{-x} \;\Rightarrow\; A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$.

7. The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is $y_c(x) = c_1\cos x + c_2\sin x$. For $y'' + y = e^x$ try $y_{p_1}(x) = Ae^x$. Then $y_{p_1}' = y_{p_1}'' = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \;\Rightarrow\; A = \frac{1}{2}$, so $y_{p_1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$. Then $y_{p_2}' = 3Ax^2 + 2Bx + C$ and $y_{p_2}'' = 6Ax + 2B$. Substituting, we have $6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \;\Rightarrow\; C = -6$, and $2B + D = 0 \;\Rightarrow\; D = 0$. Thus $y_{p_2}(x) = x^3 - 6x$ and the general solution is $y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = c_1\cos x + c_2\sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \;\Rightarrow$ $c_1 = \frac{3}{2}$ and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \;\Rightarrow\; c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is $y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$.

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2 e^x$. Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then $y_p' = (Ax^2 + (2A + B)x + B)e^x$ and $y_p'' = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \;\Rightarrow\; (2Ax + (2A + B))e^x = xe^x \;\Rightarrow$ $A = \frac{1}{2}$, $B = -1$. Thus $y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x$ and the general solution is $y(x) = c_1 + c_2 e^x + \left(\frac{1}{2}x^2 - x\right)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right)$.

11. The auxiliary equation is $r^2 + 3r + 2 = (r+1)(r+2) = 0$, so $r = -1$, $r = -2$ and $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$.

Try $y_p = A\cos x + B\sin x \Rightarrow y_p' = -A\sin x + B\cos x$, $y_p'' = -A\cos x - B\sin x$. Substituting into the differential equation gives $(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \cos x$ or $(A + 3B)\cos x + (-3A + B)\sin x = \cos x$. Then solving the equations $A + 3B = 1$, $-3A + B = 0$ gives $A = \frac{1}{10}$, $B = \frac{3}{10}$ and the general solution is $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$. The graph shows y_p and several other solutions. Notice that all solutions are asymptotic to y_p as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.



13. Here $y_c(x) = c_1\cos 3x + c_2\sin 3x$. For $y'' + 9y = e^{2x}$ try $y_{p_1}(x) = Ae^{2x}$ and for $y'' + 9y = x^2\sin x$ try $y_{p_2}(x) = (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$. Thus a trial solution is

$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$.

15. Here $y_c(x) = c_1 + c_2 e^{-9x}$. For $y'' + 9y' = 1$ try $y_{p_1}(x) = Ax$ (since $y = A$ is a solution to the complementary equation) and for $y'' + 9y' = xe^{9x}$ try $y_{p_2}(x) = (Bx + C)e^{9x}$.

17. Since $y_c(x) = e^{-x}(c_1\cos 3x + c_2\sin 3x)$ we try $y_p(x) = x(Ax^2 + Bx + C)e^{-x}\cos 3x + x(Dx^2 + Ex + F)e^{-x}\sin 3x$ (so that no term of y_p is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1' = -\frac{Gy_2}{a\,(y_1y_2' - y_2y_1')} \qquad \text{and} \qquad u_2' = \frac{Gy_1}{a\,(y_1y_2' - y_2y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) Here $4r^2 + 1 = 0 \Rightarrow r = \pm\frac{1}{2}i$ and $y_c(x) = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right)$. We try a particular solution of the form $y_p(x) = A\cos x + B\sin x \Rightarrow y_p' = -A\sin x + B\cos x$ and $y_p'' = -A\cos x - B\sin x$. Then the equation $4y'' + y = \cos x$ becomes $4(-A\cos x - B\sin x) + (A\cos x + B\sin x) = \cos x$ or $-3A\cos x - 3B\sin x = \cos x \Rightarrow A = -\frac{1}{3}$, $B = 0$. Thus, $y_p(x) = -\frac{1}{3}\cos x$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos\left(\frac{1}{2}x\right) + c_2\sin\left(\frac{1}{2}x\right) - \frac{1}{3}\cos x$.

(b) From (a) we know that $y_c(x) = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}$. Setting $y_1 = \cos\frac{x}{2}$, $y_2 = \sin\frac{x}{2}$, we have

$y_1y_2' - y_2y_1' = \frac{1}{2}\cos^2\frac{x}{2} + \frac{1}{2}\sin^2\frac{x}{2} = \frac{1}{2}$. Thus $u_1' = -\dfrac{\cos x\,\sin\frac{x}{2}}{4\cdot\frac{1}{2}} = -\frac{1}{2}\cos\left(2\cdot\frac{x}{2}\right)\sin\frac{x}{2} = -\frac{1}{2}\left(2\cos^2\frac{x}{2} - 1\right)\sin\frac{x}{2}$

and $u_2' = \dfrac{\cos x\,\cos\frac{x}{2}}{4\cdot\frac{1}{2}} = \frac{1}{2}\cos\left(2\cdot\frac{x}{2}\right)\cos\frac{x}{2} = \frac{1}{2}\left(1 - 2\sin^2\frac{x}{2}\right)\cos\frac{x}{2}$. Then

$u_1(x) = \int\left(\frac{1}{2}\sin\frac{x}{2} - \cos^2\frac{x}{2}\,\sin\frac{x}{2}\right)dx = -\cos\frac{x}{2} + \frac{2}{3}\cos^3\frac{x}{2}$ and

$u_2(x) = \int \left(\frac{1}{2}\cos\frac{x}{2} - \sin^2\frac{x}{2}\cos\frac{x}{2}\right) dx = \sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2}$. Thus

$$\begin{aligned} y_p(x) &= \left(-\cos\tfrac{x}{2} + \tfrac{2}{3}\cos^3\tfrac{x}{2}\right)\cos\tfrac{x}{2} + \left(\sin\tfrac{x}{2} - \tfrac{2}{3}\sin^3\tfrac{x}{2}\right)\sin\tfrac{x}{2} = -\left(\cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2}\right) + \tfrac{2}{3}\left(\cos^4\tfrac{x}{2} - \sin^4\tfrac{x}{2}\right) \\ &= -\cos\left(2\cdot\tfrac{x}{2}\right) + \tfrac{2}{3}\left(\cos^2\tfrac{x}{2} + \sin^2\tfrac{x}{2}\right)\left(\cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2}\right) = -\cos x + \tfrac{2}{3}\cos x = -\tfrac{1}{3}\cos x \end{aligned}$$

and the general solution is $y(x) = y_c(x) + y_p(x) = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2} - \frac{1}{3}\cos x$.

21. (a) $r^2 - 2r + 1 = (r-1)^2 = 0 \quad\Rightarrow\quad r = 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 x e^x$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \quad\Rightarrow\quad Ae^{2x} = e^{2x} \quad\Rightarrow\quad A = 1 \quad\Rightarrow\quad y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}$.

(b) From (a), $y_c(x) = c_1 e^x + c_2 x e^x$, so set $y_1 = e^x$, $y_2 = xe^x$. Then, $y_1 y_2' - y_2 y_1' = e^{2x}(1+x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^x \quad\Rightarrow\quad u_1(x) = -\int xe^x\,dx = -(x-1)e^x$ [by parts] and $u_2' = e^x \quad\Rightarrow\quad u_2(x) = \int e^x\,dx = e^x$. Hence $y_p(x) = (1-x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}$.

23. As in Example 5, $y_c(x) = c_1\sin x + c_2\cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$,

so $u_1' = -\dfrac{\sec^2 x\,\cos x}{-1} = \sec x \quad\Rightarrow\quad u_1(x) = \int \sec x\,dx = \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$,

and $u_2' = \dfrac{\sec^2 x\,\sin x}{-1} = -\sec x\,\tan x \quad\Rightarrow\quad u_2(x) = -\sec x$. Hence

$y_p(x) = \ln(\sec x + \tan x)\cdot\sin x - \sec x\cdot\cos x = \sin x\ln(\sec x + \tan x) - 1$ and the general solution is

$y(x) = c_1\sin x + c_2\cos x + \sin x\,\ln(\sec x + \tan x) - 1$.

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1+e^{-x}}$ and

$u_1(x) = \displaystyle\int -\frac{e^{-x}}{1+e^{-x}}\,dx = \ln(1+e^{-x})$. $u_2' = \dfrac{e^x}{(1+e^{-x})e^{3x}} = \dfrac{e^x}{e^{3x}+e^{2x}}$ so

$u_2(x) = \displaystyle\int \frac{e^x}{e^{3x}+e^{2x}}\,dx = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}$. Hence

$y_p(x) = e^x\ln(1+e^{-x}) + e^{2x}[\ln(1+e^{-x}) - e^{-x}]$ and the general solution is

$y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}$.

27. $r^2 - 2r + 1 = (r-1)^2 = 0 \quad\Rightarrow\quad r = 1$ so $y_c(x) = c_1 e^x + c_2 x e^x$. Thus $y_1 = e^x$, $y_2 = xe^x$ and

$y_1 y_2' - y_2 y_1' = e^x(x+1)e^x - xe^x e^x = e^{2x}$. So $u_1' = -\dfrac{xe^x\cdot e^x/(1+x^2)}{e^{2x}} = -\dfrac{x}{1+x^2} \quad\Rightarrow$

$u_1 = -\displaystyle\int \frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln\left(1+x^2\right)$, $u_2' = \dfrac{e^x\cdot e^x/(1+x^2)}{e^{2x}} = \dfrac{1}{1+x^2} \quad\Rightarrow\quad u_2 = \displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x$ and

$y_p(x) = -\frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1}x$. Hence the general solution is $y(x) = e^x\left[c_1 + c_2 x - \frac{1}{2}\ln(1+x^2) + x\tan^{-1}x\right]$.

18.3 Applications of Second-Order Differential Equations ET 17.3

1. By Hooke's Law $k(0.25) = 25$ so $k = 100$ is the spring constant and the differential equation is $5x'' + 100x = 0$.

The auxiliary equation is $5r^2 + 100 = 0$ with roots $r = \pm 2\sqrt{5}\,i$, so the general solution to the differential equation is $x(t) = c_1 \cos\left(2\sqrt{5}\,t\right) + c_2 \sin\left(2\sqrt{5}\,t\right)$. We are given that $x(0) = 0.35 \quad\Rightarrow\quad c_1 = 0.35$ and $x'(0) = 0 \quad\Rightarrow$ $2\sqrt{5}\,c_2 = 0 \quad\Rightarrow\quad c_2 = 0$, so the position of the mass after t seconds is $x(t) = 0.35\cos\left(2\sqrt{5}\,t\right)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\dfrac{d^2x}{dt^2} + c\,\dfrac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$.

If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}\,i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1\cos\left(5\sqrt{3}\,t\right) + c_2\sin\left(5\sqrt{3}\,t\right)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}\,c_2 - 5c_1 \quad\Rightarrow\quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos\left(5\sqrt{3}\,t\right) - \frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}\,t\right)\right]$.

If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1\cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x\,(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}\,c_1 \quad\Rightarrow\quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$.

For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2 t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \quad\Rightarrow\quad c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$.

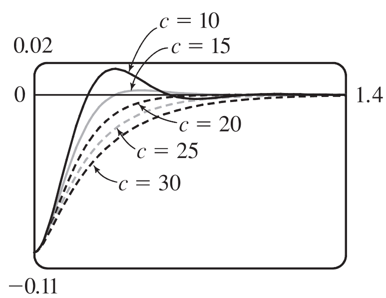
If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1 e^{-5t} + c_2 e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \quad\Rightarrow\quad c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$, so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$.

If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1 e^{\left(-15 + 5\sqrt{5}\right)t} + c_2 e^{\left(-15 - 5\sqrt{5}\right)t}$. Then $-0.1 = x(0) = c_1 + c_2$ and

$$0 = x'(0) = \left(-15 + 5\sqrt{5}\right)c_1 + \left(-15 - 5\sqrt{5}\right)c_2 \quad\Rightarrow$$

$c_1 = \frac{-5 - 3\sqrt{5}}{100}$ and $c_2 = \frac{-5 + 3\sqrt{5}}{100}$, so

$$x = \left(\frac{-5 - 3\sqrt{5}}{100}\right)e^{\left(-15 + 5\sqrt{5}\right)t} + \left(\frac{-5 + 3\sqrt{5}}{100}\right)e^{\left(-15 - 5\sqrt{5}\right)t}.$$



9. The differential equation is $mx'' + kx = F_0 \cos \omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\,i = \pm\omega i$ so $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t$. Then we need $(m)\left(-\omega_0^2\right)(A \cos \omega_0 t + B \sin \omega_0 t) + k(A \cos \omega_0 t + B \sin \omega_0 t) = F_0 \cos \omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and $B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \dfrac{F_0}{k - m\omega_0^2} = \dfrac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \dfrac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$.

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $g(t) = \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$. Then f is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \quad\Rightarrow\quad a = \frac{b\omega}{\omega_0}$ where a and b are non-zero integers. Then

$x\left(t + a \cdot \frac{2\pi}{\omega}\right) = f\left(t + a \cdot \frac{2\pi}{\omega}\right) + g\left(t + a \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + \frac{b\omega}{\omega_0} \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + b \cdot \frac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t)$

so $x(t)$ is periodic.

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then

$Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ and try $Q_p(t) = A \quad\Rightarrow\quad 500A = 12$ or $A = \frac{3}{125}$.

The general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250}e^{-10t}(6 \cos 20t + 3 \sin 20t) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right)\sin 20t$.

15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try

$Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives

$(-100A + 200B + 500A)\cos 10t + (-100B - 200A + 500B)\sin 10t = 12 \sin 10t \quad\Rightarrow$

$400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is

$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$.

Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2)\cos 20t + (-10c_2 - 20c_1)\sin 20t]$ and

$0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by

$Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$.

17. $x(t) = A \cos(\omega t + \delta) \quad\Leftrightarrow\quad x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \quad\Leftrightarrow\quad x(t) = A\left(\dfrac{c_1}{A} \cos \omega t + \dfrac{c_2}{A} \sin \omega t\right)$ where $\cos \delta = c_1/A$ and $\sin \delta = -c_2/A \quad\Leftrightarrow\quad x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. [Note that $\cos^2 \delta + \sin^2 \delta = 1 \quad\Rightarrow\quad c_1^2 + c_2^2 = A^2$.]

18.4 Series Solutions ET 17.4

1. Let $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n$. Then $y'(x)=\sum\limits_{n=1}^{\infty}nc_nx^{n-1}$ and the given equation, $y'-y=0$, becomes

$\sum\limits_{n=1}^{\infty}nc_nx^{n-1}-\sum\limits_{n=0}^{\infty}c_nx^n=0$. Replacing n by $n+1$ in the first sum gives $\sum\limits_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum\limits_{n=0}^{\infty}c_nx^n=0$, so

$\sum\limits_{n=0}^{\infty}[(n+1)c_{n+1}-c_n]x^n=0$. Equating coefficients gives $(n+1)c_{n+1}-c_n=0$, so the recursion relation is

$c_{n+1}=\dfrac{c_n}{n+1}$, $n=0,1,2,\ldots$. Then $c_1=c_0$, $c_2=\frac{1}{2}c_1=\frac{c_0}{2}$, $c_3=\frac{1}{3}c_2=\frac{1}{3}\cdot\frac{1}{2}c_0=\frac{c_0}{3!}$, $c_4=\frac{1}{4}c_3=\frac{c_0}{4!}$, and

in general, $c_n=\dfrac{c_0}{n!}$. Thus, the solution is $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n=\sum\limits_{n=0}^{\infty}\dfrac{c_0}{n!}x^n=c_0\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}=c_0e^x$.

3. Assuming $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n$, we have $y'(x)=\sum\limits_{n=1}^{\infty}nc_nx^{n-1}=\sum\limits_{n=0}^{\infty}(n+1)c_{n+1}x^n$ and

$-x^2y=-\sum\limits_{n=0}^{\infty}c_nx^{n+2}=-\sum\limits_{n=2}^{\infty}c_{n-2}x^n$. Hence, the equation $y'=x^2y$ becomes $\sum\limits_{n=0}^{\infty}(n+1)c_{n+1}x^n-\sum\limits_{n=2}^{\infty}c_{n-2}x^n=0$

or $c_1+2c_2x+\sum\limits_{n=2}^{\infty}[(n+1)c_{n+1}-c_{n-2}]\,x^n=0$. Equating coefficients gives $c_1=c_2=0$ and $c_{n+1}=\dfrac{c_{n-2}}{n+1}$

for $n=2,3,\ldots$. But $c_1=0$, so $c_4=0$ and $c_7=0$ and in general $c_{3n+1}=0$. Similarly $c_2=0$ so $c_{3n+2}=0$. Finally

$c_3=\dfrac{c_0}{3}$, $c_6=\dfrac{c_3}{6}=\dfrac{c_0}{6\cdot 3}=\dfrac{c_0}{3^2\cdot 2!}$, $c_9=\dfrac{c_6}{9}=\dfrac{c_0}{9\cdot 6\cdot 3}=\dfrac{c_0}{3^3\cdot 3!}$, …, and $c_{3n}=\dfrac{c_0}{3^n\cdot n!}$. Thus, the solution

is $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n=\sum\limits_{n=0}^{\infty}c_{3n}x^{3n}=\sum\limits_{n=0}^{\infty}\dfrac{c_0}{3^n\cdot n!}x^{3n}=c_0\sum\limits_{n=0}^{\infty}\dfrac{x^{3n}}{3^nn!}=c_0\sum\limits_{n=0}^{\infty}\dfrac{\left(x^3/3\right)^n}{n!}=c_0e^{x^3/3}$.

5. Let $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n \;\Rightarrow\; y'(x)=\sum\limits_{n=1}^{\infty}nc_nx^{n-1}$ and $y''(x)=\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. The differential equation

becomes $\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n+x\sum\limits_{n=1}^{\infty}nc_nx^{n-1}+\sum\limits_{n=0}^{\infty}c_nx^n=0$ or $\sum\limits_{n=0}^{\infty}[(n+2)(n+1)c_{n+2}+nc_n+c_n]x^n=0$

$\left[\text{since }\sum\limits_{n=1}^{\infty}nc_nx^n=\sum\limits_{n=0}^{\infty}nc_nx^n\right]$. Equating coefficients gives $(n+2)(n+1)c_{n+2}+(n+1)c_n=0$, thus the

recursion relation is $c_{n+2}=\dfrac{-(n+1)c_n}{(n+2)(n+1)}=-\dfrac{c_n}{n+2}$, $n=0,1,2,\ldots$. Then the even

coefficients are given by $c_2=-\dfrac{c_0}{2}$, $c_4=-\dfrac{c_2}{4}=\dfrac{c_0}{2\cdot 4}$, $c_6=-\dfrac{c_4}{6}=-\dfrac{c_0}{2\cdot 4\cdot 6}$, and in general,

$c_{2n}=(-1)^n\dfrac{c_0}{2\cdot 4\cdot\cdots\cdot 2n}=\dfrac{(-1)^nc_0}{2^n\,n!}$. The odd coefficients are $c_3=-\dfrac{c_1}{3}$, $c_5=-\dfrac{c_3}{5}=\dfrac{c_1}{3\cdot 5}$, $c_7=-\dfrac{c_5}{7}=-\dfrac{c_1}{3\cdot 5\cdot 7}$,

and in general, $c_{2n+1}=(-1)^n\dfrac{c_1}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}=\dfrac{(-2)^n\,n!\,c_1}{(2n+1)!}$. The solution is

$y(x)=c_0\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2^n\,n!}x^{2n}+c_1\sum\limits_{n=0}^{\infty}\dfrac{(-2)^n\,n!}{(2n+1)!}x^{2n+1}$.

7. Let $y(x)=\sum\limits_{n=0}^{\infty}c_nx^n \;\Rightarrow\; y'(x)=\sum\limits_{n=1}^{\infty}nc_nx^{n-1}=\sum\limits_{n=0}^{\infty}(n+1)c_{n+1}x^n$ and $y''(x)=\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$. Then

$(x-1)y''(x)=\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n+1}-\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n=\sum\limits_{n=1}^{\infty}n(n+1)c_{n+1}x^n-\sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$.

Since $\sum\limits_{n=1}^{\infty} n(n+1)c_{n+1}x^n = \sum\limits_{n=0}^{\infty} n(n+1)c_{n+1}x^n$, the differential equation becomes

$$\sum_{n=0}^{\infty} n(n+1)c_{n+1}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n = 0 \quad \Rightarrow$$

$$\sum_{n=0}^{\infty}[n(n+1)c_{n+1} - (n+2)(n+1)c_{n+2} + (n+1)c_{n+1}]x^n = 0 \text{ or } \sum_{n=0}^{\infty}[(n+1)^2 c_{n+1} - (n+2)(n+1)c_{n+2}]x^n = 0.$$

Equating coefficients gives $(n+1)^2 c_{n+1} - (n+2)(n+1)c_{n+2} = 0$ for $n = 0, 1, 2, \ldots$. Then the recursion relation is

$c_{n+2} = \dfrac{(n+1)^2}{(n+2)(n+1)}c_{n+1} = \dfrac{n+1}{n+2}c_{n+1}$, so given c_0 and c_1, we have $c_2 = \frac{1}{2}c_1$, $c_3 = \frac{2}{3}c_2 = \frac{1}{3}c_1$, $c_4 = \frac{3}{4}c_3 = \frac{1}{4}c_1$, and in general $c_n = \dfrac{c_1}{n}$, $n = 1, 2, 3, \ldots$. Thus the solution is $y(x) = c_0 + c_1 \sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$. Note that the solution can be expressed as $c_0 - c_1 \ln(1-x)$ for $|x| < 1$.

9. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum\limits_{n=1}^{\infty} nc_n x^{n-1} = -\sum\limits_{n=1}^{\infty} nc_n x^n = -\sum\limits_{n=0}^{\infty} nc_n x^n$,

$y''(x) = \sum\limits_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$, and the equation $y'' - xy' - y = 0$

becomes $\sum\limits_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{nc_n + c_n}{(n+2)(n+1)} = \dfrac{c_n(n+1)}{(n+2)(n+1)} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. One of the given conditions is $y(0) = 1$. But

$y(0) = \sum\limits_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0$, so $c_0 = 1$. Hence, $c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2\cdot 4}$, $c_6 = \dfrac{c_4}{6} = \dfrac{1}{2\cdot 4\cdot 6}$, $\ldots$,

$c_{2n} = \dfrac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But $y'(0) = \sum\limits_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1$, so $c_1 = 0$.

By the recursion relation, $c_3 = \dfrac{c_1}{3} = 0$, $c_5 = 0, \ldots, c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus, the solution to the initial-value

problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_{2n}x^{2n} = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{2^n n!} = \sum\limits_{n=0}^{\infty} \dfrac{(x^2/2)^n}{n!} = e^{x^2/2}$.

11. Assuming that $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, we have $xy = x\sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_n x^{n+1}$, $x^2 y' = x^2 \sum\limits_{n=1}^{\infty} nc_n x^{n-1} = \sum\limits_{n=0}^{\infty} nc_n x^{n+1}$,

$$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-1}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1} \qquad \text{[replace } n \text{ with } n+3] \\ &= 2c_2 + \sum_{n=0}^{\infty}(n+3)(n+2)c_{n+3}x^{n+1}, \end{aligned}$$

and the equation $y'' + x^2y' + xy = 0$ becomes $2c_2 + \sum\limits_{n=0}^{\infty}\left[(n+3)(n+2)c_{n+3} + nc_n + c_n\right]x^{n+1} = 0$. So $c_2 = 0$ and the

recursion relation is $c_{n+3} = \dfrac{-nc_n - c_n}{(n+3)(n+2)} = -\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0 = c_2$ and by the

recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$. Also, $c_1 = y'(0) = 1$, so $c_4 = -\dfrac{2c_1}{4\cdot 3} = -\dfrac{2}{4\cdot 3}$,

$c_7 = -\dfrac{5c_4}{7\cdot 6} = (-1)^2 \dfrac{2\cdot 5}{7\cdot 6\cdot 4\cdot 3} = (-1)^2\dfrac{2^2 5^2}{7!}, \ldots, c_{3n+1} = (-1)^n \dfrac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty}\left[(-1)^n \frac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2 x^{3n+1}}{(3n+1)!}\right].$$

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CONCEPT CHECK

1. (a) $ay'' + by' + cy = 0$ where a, b, and c are constants.

(b) $ar^2 + br + c = 0$

(c) If the auxiliary equation has two distinct real roots r_1 and r_2, the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. If the roots are real and equal, the solution is $y = c_1 e^{rx} + c_2 x e^{rx}$ where r is the common root. If the roots are complex, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and the solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

2. (a) An initial-value problem consists of finding a solution y of a second-order differential equation that also satisfies given conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are constants.

(b) A boundary-value problem consists of finding a solution y of a second-order differential equation that also satisfies given boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

3. (a) $ay'' + by' + cy = G(x)$ where a, b, and c are constants and G is a continuous function.

(b) The complementary equation is the related homogeneous equation $ay'' + by' + cy = 0$. If we find the general solution y_c of the complementary equation and y_p is any particular solution of the original differential equation, then the general solution of the original differential equation is $y(x) = y_p(x) + y_c(x)$.

(c) See Examples 1–5 and the associated discussion in Section 18.2 [ET 17.2].

(d) See the discussion on pages 1158–1160 [ET 1122–1124].

4. Second-order linear differential equations can be used to describe the motion of a vibrating spring or to analyze an electric circuit; see the discussion in Section 18.3 [ET 17.3].

5. See Example 1 and the preceding discussion in Section 18.4 [ET 17.4].

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TRUE-FALSE QUIZ

1. True. See Theorem 18.1.3 [ET 17.1.3].

3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.

EXERCISES

1. The auxiliary equation is $r^2 - 2r - 15 = 0 \quad \Rightarrow \quad (r-5)(r+3) = 0 \quad \Rightarrow \quad r = 5, r = -3$. Then the general solution is $y = c_1 e^{5x} + c_2 e^{-3x}$.

3. The auxiliary equation is $r^2 + 3 = 0 \quad \Rightarrow \quad r = \pm\sqrt{3}\, i$. Then the general solution is $y = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)$.

5. $r^2 - 4r + 5 = 0 \quad \Rightarrow \quad r = 2 \pm i$, so $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{2x} \quad \Rightarrow \quad y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \quad \Rightarrow \quad A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.

7. $r^2 - 2r + 1 = 0 \quad \Rightarrow \quad r = 1$ and $y_c(x) = c_1 e^x + c_2 x e^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \quad \Rightarrow$ $y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \quad \Rightarrow \quad A = 0, B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2}\cos x - \frac{1}{2}(x + 1)\sin x$.

9. $r^2 - r - 6 = 0 \quad \Rightarrow \quad r = -2, r = 3$ and $y_c(x) = c_1 e^{-2x} + c_2 e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p_1}(x) = A$. Then $y_{p_1}'(x) = y_{p_1}''(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p_2}(x) = Bxe^{-2x}$ [since $y = Be^{-2x}$ satisfies the complementary equation]. Then $y_{p_2}' = (B - 2Bx)e^{-2x}$ and $y_{p_2}'' = (4Bx - 4B)e^{-2x}$, and substitution gives $-5Be^{-2x} = e^{-2x} \quad \Rightarrow \quad B = -\frac{1}{5}$. The general solution then is $y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_{p_1}(x) + y_{p_2}(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5}xe^{-2x}$.

11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k^2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.

13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.

15. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum\limits_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and the differential equation becomes $\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \ldots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \ldots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\dfrac{1}{3}$, $c_5 = \dfrac{(-1)^2}{3 \cdot 5}$, $c_7 = \dfrac{(-1)^3}{3 \cdot 5 \cdot 7} = \dfrac{(-1)^3 2^3 3!}{7!}, \ldots, c_{2n+1} = \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!}$ for $n = 0, 1, 2, \ldots$. Thus the solution to the initial-value problem is $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\, 2^n\, n!}{(2n+1)!} x^{2n+1}$.

17. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$. Thus the general solution is $Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}$. But $0.01 = Q'(0) = c_1 + 0.03$ and $0 = Q''(0) = -10c_1 + 10c_2$, so $c_1 = -0.02 = c_2$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

19. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows: $\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center, then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3}r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion,

$m\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3}\,y$, so $y''(t) = -k^2 y(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so

$g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 18.1 [ET 17.1], not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 18.1 [ET 17.1], the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R\cos kt$ and $y'(t) = -kR\sin kt$. This is simple harmonic motion (see Section 18.3 [ET 17.3]) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32\ \text{ft/s}^2$, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3}\ \text{s}^{-1}$ and $T = 2\pi/k \approx 5079$ s ≈ 85 min.

(d) $y(t) = 0 \quad \Leftrightarrow \quad \cos kt = 0 \quad \Leftrightarrow \quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad \Rightarrow \quad y'(t) = -kR\sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17{,}600$ mi/h.